TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Operations Research and Control Systems

Markets for Emission Permits with Free Endowment: a Vintage Capital Analysis

Thierry Bréchet, Tsvetomir Tsachev, Vladimir M. Veliov

Research Report 2010-03

March, 2010

Operations Research and Control Systems
Institute of Mathematical Methods in Economics
Vienna University of Technology

Research Unit ORCOS
Argentinierstraße 8/E105-4,
1040 Vienna, Austria
E-mail: orcos@eos.tuwien.ac.at

Markets for Emission Permits with Free Endowment: a Vintage Capital Analysis*

Thierry Bréchet[†] Tsvetomir Tsachev[‡] Vladimir M. Veliov[§]

Abstract

In this paper we develop a vintage capital model for a firm involved in a market for tradable emission permits. We analyze both the firm's optimal investment plans and the market equilibrium. This allows us to scrutinize how firms use permits free endowment, and to highlight the implications of non-optimal uses both at the firm and at the market level. We provide a new rationale for the market of tradable permits not to be cost-efficient. The novel technical points in this context are the use a distributed (vintage) optimal control model of the firm, the use of optimality conditions for non-smooth problems, and the involvement of a nonlinear Fredholm integral equation of the first kind for the description of the equilibrium price of permits, and its practical meaning for market regularization.

JEL CC: C61, Q58, Q52, Q55
Key Words: environmental economics, emission permits, optimal control, distributed systems

*This research was partly financed by the Austrian Science Foundation (FWF) under grant No P18161-N13 and by the Belgian Science Policy under the CLIMNEG project (SD/CP/05A).

[†]CORE and Louvain School of Management, Chair Lhoist Berghmans in Environmental Economics and Management, Université Catholique de Louvain, Voie du Roman Pays, 34, B-1348 Louvain-la-Neuve, Belgium, e-mail: thierry.brechet@uclouvain.be

[‡]Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev str., block 8, 1113 Sofia, Bulgaria, e-mail: tsachev@math.bas.bg

[§]**Corresponding author.** ORCOS, Institute of Mathematical Methods in Economics, Vienna University of Technology, Argentinierstrasse 8/105-4, A-1040 Vienna, Austria, e-mail: veliov@tuwien.ac.at. Phone: +43 1 58801 11923

1 Introduction

Markets for tradable emission permits have become a key policy instrument in many environmental areas today. One may naturally have in mind the US acid rain program, that launched in 1995 a market for SO_2 permits covering the whole US power industry (see Ellerman *et al.*, 2000). Even broader, the Kyoto protocol enforced a world market under the United Nations Framework Convention on Climate Change (UNFCCC, 1997; see UNFCCC, 2005, for a survey). In 2005 the European Emission Trading Scheme (EU-ETS) entered into force, covering about 14,000 industrial installations in the EU-27 (see Ellerman *et al.*, 2007). In the US, the Regional Greenhouse Gas Initiative also makes use of tradable emission permits.[1] Many more local markets have been implemented recently, see Ellerman (2005) or Tietenberg and Thoman (2006) for comprehensive presentations of all these experiments.

The idea of tradable emission permits originates to Dales (1968). Montgomery (1972) formally showed that trading is cost-efficient under perfect competition. Another key result of Montgomery is that the way permits are distributed among firms does not alter the efficiency of the market for tradable permits. This result offers a major advantage to tradable emission permits: the authority responsible for issuing the permits needs not be aware of the incumbents' abatement costs. In this paper we will question the result that markets for tradable permits are cost-efficient.[2]

Another crucial issue about tradable emission permits is to know whether permits should be given for free (*i.e.* grand-fathered) or auctioned. The economic literature is quite clear on that point: giving permits for free is not socially optimal, while full auctioning is (see *e.g.* Cramton and Kerrn, 2002, Jouvet *et al.*, 2005). Nevertheless, as argued by Stavins (1998), giving some permits for free enhances political acceptability because it is less costly for the firms (see Goulder, 2002, for a CGE numerical appraisal and Bréchet *et al.*, 2010, for a critical discussion in dynamic general equilibrium.). The policy maker has thus some good reasons to issue some permits for free, the remainder being auctioned. In this paper we highlight some potential adverse effects of giving permits for free both at the firm and at the market level.

To our knowledge, there is no paper in the literature that explicitly considers how permits are managed *inside* the firm, and no paper that analyzes the related impacts on market equilibrium. How do firms use emission permits? Do they consider differently free endowments and auctioned permits? How does it change their investment plans? In this paper we answer these questions. To this purpose we develop a vintage capital model of the firm. In all of the works that analyze

[1]The Regional Greenhouse Gas Initiative (RGGI) is the first mandatory, market-based effort in the United States to reduce greenhouse gas emissions. Ten Northeastern and Mid-Atlantic states haved capped and will reduce CO_2 emissions from the power sector 10% by 2018. See *http://www.rggi.org.*

[2]We will do it under perfect competition. Hahn (1984) elaborated on that point under imperfect competition by showing that the firms' permit endowments could be used to control for market power. A large literature is also devoted to revelation mechanisms under information asymmetry. See *e.g.* Moledina *et al.* (2003). This is beyond the purpose of our paper.

tradable emission permits, it is assumed that capital is homogeneous.[3] We will be able to analyze how free endowments of tradable emission permits are used inside the firm by distinguishing capital vintages with respect to their productivity and emission intensity. This heterogeneity enables the firm's managers to decide which machines to scrap and which machines to buy, when confronted with emission restrictions. In particular, our model allows for investing in and scrapping technologies of any vintage.

Formally, our firm-level model consists of a continual family of controlled ordinary differential equations (thus has a distributed dynamics) describing the evolution of the firm vintage-wise. From a technical point of view the vintage differentiation does not cause problems at all, since the AK-framework that we employ, does not include vintage-externalities and evidently allows to decompose the firm's optimization problem to independent subproblems corresponding to different vintages. The main trouble with the firm-level problem is caused by the possibility of endogenous scrapping (which is necessary to include as a policy instrument when the firm faces emission costs of any kind).[4] The resulting firm's optimization problem involves a non-differentiable term in the objective functions, which makes the problem intractable by the standard dynamic optimization techniques based on the Pontryagin maximum principle (Pontryagin et al., 1962). We apply an optimality condition for non-smooth optimal control problems (Loewen, 1993) and perform the relevant analysis.

Another technically interesting point is the equation for the equilibrium price of emissions that we derive based on the obtained optimal behaviour of the firms facing costs of emission. The market equation can be classified as a nonlinear Fredholm integral equation of the first kind, which reduces to a classical linear one under additional assumptions. This already says a lot: the equilibrium price of emission permits may be very sensitive, non-uniquely determined or even not having economic meaning (being negative). Therefore we address the question of *regularization* of this equation in a way that is practically implementable. The main idea is that the emission restriction should be posed period-wise, with not too short *commitment periods*. This sort of regularization is consistent with the discussion about so-called commitment period under the Kyoto protocol (2008-2012), or the so-called "phases" under the EU-ETS market (2005-2008, then 2009-2013). We show that a long enough commitment period is essential for firms to implement efficient investment plans.

Our main results can be summarized as follows. We start by considering the benchmark model of a firm that uses machines of different vintages (in other words, different technologies). Each technology has its own productivity and emission intensity. We naturally assume that older machines are more polluting but less productive. At any time the firm can invest in machines of any vintage and switch-off any machines. The firm faces an emission tax. It maximizes its discounted net revenue. The optimal investment decision is described. This first model allows us to understand part of the

[3]The only paper dealing with CO_2 emissions abatement with heterogenous capital stocks is Jaccard and Rivers (2007). This paper explores the relationship between the natural turn-over for different society's capital stocks (infrastructures, building and machines) and the pace of emission abatement, but it does not consider the role of policy instruments.

[4]We refer to Boucekkine et al. (1998) for "creative destruction" of capital in a different context and with a different technique applied.

firm behavior with emission permits, because in both cases (emission tax or permits), the price of emissions plays the same role. It represents the opportunity cost of pollution for the firm. Then, we introduce tradable emission permits. For the sake of generality we assume that the regulator issues some permits for free, and that some are auctioned. We consider that there exists a top manager in the firm who distributes the free endowment of permits among the different technologies. By doing this, she decentralizes the investment and production decisions at the level of the operating manager. The firm is supposed to be price taker in the market for tradable permits. We show that there exists a set of optimal allocation rules of permits, that is, ones that maximize the firm's intertemporal revenue. In other words, there is not one single optimal allocation rule among machines. This result is important as it shows that there exists room for flexibility within the firm about how to manage the free endowment.

In order to study market equilibrium, the issue of *regularization* needs to be tackled. As mentioned just above, we show that the market equilibrium may not be well-defined if the commitment periods are too short. In economic terms this means that the length of the commitment period must fit the firms' planning horizon. The longer the firm's planning horizon(like in the steel, cement, lime, power industry), the longer the commitment period should be. We provide numerical analysis for this question. If the commitment period is not long enough, then the firm cannot adapt optimally to the regulation, which leads to market failures.

Another unanswered question in the literature is the following. Will a firm, facing a forthcoming emission restrictions at some later time period, change its investment plan in advance? Put differently, may one expect some anticipation effect? We show that there exists an anticipation effect and that, due to this effect, firm's emissions decrease even before the mandatory time period. This provides a rationale why the EU-ETS was actually non-binding during its first phase. By simulation we show that the anticipation effect is associated with an overshoot in the market price, which is followed by a sharp drop. There is a wide debate about the first phase of the EU-ETS, because the emission price collapsed to almost zero after some time. This was interpreted by some authors as an evidence for over-allocation of permits (see Ellerman *et al.*, 2007). We claim that, even if it is not the only cause, the price collapse partially happened because of the anticipation effect.

Maybe even more important are the implications of a non-optimal use of the free endowment of permits inside the firm. We analyze the implications of the fact that a firm may not follow an optimal allocation rule. First, it may lead to a higher emission level for that firm, for some time periods, which is costly. Second, it may also have severe implications at the market level. In equilibrium the marker price for tradable permits may increase *w.r.t.* the efficient price, which is detrimental to all firms in the market. The firms that behave optimally will face a higher permit price, and thus will abate more. The firms that behave non-optimally will suffer twice.

The paper is organized as follows. In Section 2 we solve the firm's optimal investment under emission tax. In Section 3 we present and investigate a reasonable scenario of *non-optimal* utilization of free endowments by a firm facing exogenous price of emissions. Section 4 is devoted to the equation for the equilibrium price of emission permits, reproducing in short the investigation of a companion conference paper (Bréchet *et al.*, 2009). The anticipation effect is presented in Section 5, and the paper is concluded by Section 6, where we analyze how the non-optimal use of free endowments by some firms influences the equilibrium price of emission permit and impacts the other firms in the market.

2 The firm's problem with an emission tax

The model of the firm presented below is a version of the PDE-vintage models introduced by Barucci and Gozzi (1998), Barucci and Gozzi (2001), and further investigated and applied in a large number of contributions (see Feichtinger *et al.* (2006), Feichtinger *et al.* (2008) and the literature therein). The formal difference of our model with the abovementioned ones is technical and not essential from a methodological point of view.

First we describe the basic model of a firm that is composed of machines of different vintages (technologies) τ: $K(\tau, s)$ will denote the capital stock of vintage τ and of age s. That is, $K(\tau, t-\tau)$ is the stock of vintage τ that exists at time $t \geq \tau$. The maximal life-time of machines of each technology will be denoted by ω, and the depreciation rate of each technology – by δ. Both are assumed independent of the vintage just for notational convenience. At any time $t > 0$ the firm may invest with intensity $I(\tau, s)$ in machines of vintage τ that are of age s at time t (so that $s = t - \tau$).

The planning horizon of the firm is $[0, \infty)$, therefore the stock of machines of vintage $\tau \in [-\omega, 0]$ which are present in the firm at time $t = 0$ is considered as exogenous and will be denoted by $K_0(\tau)$. These machines have age $-\tau$ at time $t = 0$ and may be in use until they reach age ω. Machines of vintage $\tau > 0$ may be in use for ages $s \in [0, \omega]$, and their stock at age zero equals zero. Therefore $K_0(\tau)$ will be defined as zero for $\tau > 0$. The ages of possible use of any vintage can be written as $[s_0(\tau), \omega]$, where $s_0(\tau) = \max\{0, -\tau\}$.

Summarizing, the dynamics of each vintage $\tau \in [-\omega, \infty)$ is given by the equation

$$\dot{K}(\tau, s) = -\delta K(\tau, s) + I(\tau, s), \qquad K(\tau, s_0(\tau)) = K_0(\tau), \qquad s \in [s_0(\tau), \omega], \tag{1}$$

where "dot" means everywhere the derivative with respect to s (the argument representing the age).

The productivity of machines of vintage τ is denoted by $f(\tau)$, while $g(\tau)$ denotes the emission per machine of vintage τ. The firm faces costs due to emissions at an exogenous price $v(t)$ per unit of emission (this price will be endogenized later on). Due to this cost the firm may decide to (possibly temporarily) switch off a part of the machines. Let us denote by $W(\tau, s) \in [0, 1]$ the fraction of the machines of vintage τ that operate at age s.

The cost of investment I in s years old machines of any vintage will be denoted by $C(s, I)$.

The present value (at time $t = 0$) of the total production of machines of vintage τ, discounted with a rate r, is

$$e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} f(\tau) K(\tau, s) W(\tau, s) \,\mathrm{d}s,$$

the cost of emission is

$$e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} v(\tau + s) g(\tau) K(\tau, s) W(\tau, s) \,\mathrm{d}s,$$

and the investment costs are

$$e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} C(s, I(\tau, s)) \,\mathrm{d}s.$$

The firm maximizes the aggregated over time discounted net revenue, that is, solves the problem

$$\max_{I \geq 0,\, W \in [0,1]} \int_{-\omega}^{\infty} e^{-r\tau} \int_{s_0(\tau)}^{\omega} e^{-rs} \left[(f(\tau) - v(\tau + s) g(\tau)) K(\tau, s) W(\tau, s) - C(s, I(\tau, s)) \right] \mathrm{d}s \,\mathrm{d}\tau \qquad (2)$$

subject to (1).

The emission of the firm at time $t > 0$ is given by the expression

$$E(t) = \int_{t-\omega}^{t} g(\tau) K(\tau, t - \tau) W(\tau, t - \tau) \,\mathrm{d}\tau. \qquad (3)$$

Remark 1 Due to the specific form of the problem there is an easy way to define optimality even though the optimal value in (2) may be infinite. Namely (I^*, W^*, K^*) is a solution of (2), (1) if for any $T > 0$ the restriction of these functions to $D_T = \{(\tau, s) : \ \tau \in [-\omega, T], \ s \in [s_0(\tau), \omega]\}$ is an optimal solution of the problem in which the integration in (2) is carried out on D_T.

Problem (2), (1) fits to the general framework of heterogeneous optimal control systems developed in Veliov (2008). However, the present problem can be treated also by the classical Pontryagin optimality conditions for ODEs since it decomposes along vintages: every technology vintage $\tau \in [-\omega, \infty)$ is managed separately by solving the problem

$$\max_{i \geq 0,\, w \in [0,1]} \int_{s_0(\tau)}^{\omega} e^{-rs} \left[(f(\tau) - v(\tau + s) g(\tau)) k(s) w(s) - C(s, i(s)) \right] \mathrm{d}s \qquad (4)$$

subject to

$$\dot{k}(s) = -\delta k(s) + i(s), \qquad k(s_0(\tau)) = K_0(\tau), \qquad s \in [s_0(\tau), \omega]. \qquad (5)$$

If for any fixed $\tau \in [-\omega, \infty)$ the triple $(i(\cdot), w(\cdot), k(\cdot)) = (I(\tau, \cdot), W(\tau, \cdot), K(\tau, \cdot))$ is a solution of this problem, then (I, W, K) is a solution of (2), (1) and vice versa.[5]

[5] This is not a self-evident fact, but can easily be proven in natural space settings for the two problems and on the assumptions made below.

The following is assumed throughout the paper.

(A1) The exogenous data K_0, f, g are non-negative and continuous, f and g are continuously differentiable, $g > 0$, $f' > 0$, $g' \leq 0$, $r \geq 0$, $\delta \geq 0$;
(A2) The cost function $C(s,i)$ is two times differentiable in i, the derivatives C'_i and C''_{ii} are continuous in (s,i), $C(s,0) = 0$, $C'_i(s,0) \geq 0$, $C''_{ii} \geq \varepsilon_C > 0$;
(A3) The price of emission, $v(\cdot)$, is a measurable bounded function[6].

Under these conditions problem (2), (1) has an unique solution $(I^*[v], W^*[v], K^*[v])$. The corresponding emission given by (3) will be denoted by $E^*[v]$.

Since W enters only in the objective function, clearly we have

$$W^*[v](\tau,s) = \begin{cases} 0 & \text{if } f(\tau) - v(\tau+s)g(\tau) \leq 0, \\ 1 & \text{if } f(\tau) - v(\tau+s)g(\tau) > 0. \end{cases} \tag{6}$$

The optimal control i^* of problem (4), (5) for a fixed τ and $v(\cdot)$ is easy to obtain by applying the Pontryagin maximum principle (Pontryagin *et al.*, 1962). Namely,

$$i^*(s) = \begin{cases} 0 & \text{if } \xi^*(s) < C'_i(s,0), \\ (C'_i)^{-1}(s, \xi^*(s)) & \text{if } \xi^*(s) \geq C'_i(s,0), \end{cases} \tag{7}$$

where $\xi \longrightarrow (C'_i)^{-1}(s,\xi)$ is the inverse of the function $i \longrightarrow C'_i(s,i)$ and $\xi^*(s)$ is the unique solution of the *adjoint equation*

$$\dot{\xi}(s) = (r+\delta)\xi(s) - (f(\tau) - v(\tau+s)g(\tau))w^*(s), \qquad \xi(\omega) = 0. \tag{8}$$

According to (6) the solution ξ is nonnegative, hence $(C'_i)^{-1}(s, \xi^*(s))$ is well defined due to (A2).

3 The firm's problem with tradable emission permits

In this section we introduce a model of a firm that participates to a market of tradable emission permits, a so-called *cap-and-trade* system. Each permit allows the firm to emit one unit of pollution, and the firm is legally bound to cover all its polluting emissions by the corresponding number of permits. We assume a competitive market. Each firm is thus price-taker and the permit price is exogenous to it. The emission permits can be issued by the regulator in two ways: some amount of permits is given for free to the firms, another amount is auctioned. The confrontation between the permits supply (decided by the regulator) and demand (coming from the firms which need permits for their polluting activity) constitutes the *primary permits market*. The total amount of permits issued corresponds to the global emission cap in the economy. In such a market, the regulator controls the total amount of pollution, but not the price of permits, which will be determined by market equilibrium.

[6] For reasons that will become clear later we formally allow for negative values of $v(t)$.

Because some firms also receive free endowments of emission permits, the question about how it will use this endowment is asked. This is the question addressed in this section. Before entering this analysis, the functioning of the secondary market has to be explained. The secondary market is the one in which firms trade emission permits between themselves, the total amount of permits being given. If, at a given price, a firm has too many permits because of the free endowment it received, it can sell them on the secondary market. If the firm has is lacking permits, then it will be a buyer. When buying permits on the secondary market, the firm behaves exactly like during the auction in the primary market or when facing an emission tax.

Let us now turn back to the free endowment of emission permits received by the firm. We shall consider that this endowment is distributed among the different technologies used by the firm. This distribution is the responsibility of the top manager. Associated to each technology, there exists an operating manager whose job is to operate the machine efficiently. The operating manager is now allowed to sell the permits, this is the responsibility of the top manager. This situation reflects the dichotomy observed in reality between the operating staff in charge of the production services of the firm, and the financial staff (here represented by our top manager) in charge of buying or selling of the emission permits. By distributing the free endowment at the machine level, the top manager simply delegates the operating of the technology, as it is done with an emission tax, for example. The issue of how emission permits are used inside the firm, at the technology level, has never been addressed in the literature because it requires a non-homogeneous capital stock.

We will see that the way the free endowment of permits is distributed among the different technologies may have severe implications for the firm. Technically, in our model the possibly "non-optimal" use (to be defined here below) of the free endowment comes from an internal mechanism of distribution of the permits among the different production units, represented by the vintage of the corresponding technologies.

We proceed with the formal description of the above two-level decision scheme practiced by many large firms. The central management distributes the received free endowment $D(t)$ among the different technologies (*i.e.* production units) in order to decentralize the investment and production decisions at the level of the operating manager. Let $d(\tau, s)$ be the endowment that technology τ obtains at time $\tau + s$ (when this technology is s years old), so that

$$\int_{t-\omega}^{t} d(\tau, t-\tau)\, \mathrm{d}\tau = D(t). \tag{9}$$

The above expression for the free endowment comes from the fact that, at time t, only machines of vintages $\tau \in [t-\omega, t]$ exist, and the machines of vintage τ are of age $s = t - \tau$.

The operating managers may require from the top manager additional permits (should this be profitable, given the market price) that the firm will buy in the secondary market at price $v(t)$.

The problem (4), (5) of technology vintage τ transforms in the case of a free endowment $d(\tau, s)$ to

$$\max_{i \geq 0,\, w \in [0,1]} \int_{s_0(\tau)}^{\omega} e^{-rs} \left[f(\tau)k(s)w(s) - v(\tau+s) \max\{0,\, g(\tau)k(s)w(s) - d(\tau,s)\} - C(s, i(s)) \right] \mathrm{d}s, \tag{10}$$

subject to (5). This objective function takes into account that the firm buys additional permits for the technology τ at age s only if its emission exceeds the free endowment $d(\tau, s)$.

As before, (I^*, W^*, K^*) will be the solution of the firm-level problem (2), (1), or equivalently, $(i^*(\cdot), w^*(\cdot), k^*(\cdot)) = (I^*(\tau, \cdot), W^*(\tau, \cdot), K^*(\tau, \cdot))$ will be the solution of (4), (5) for any fixed τ. Note that the dependence of the solution (also of the resulting emission E^*) on the price v is suppressed in the notations used in this section since v is considered as given.

Below in this section we assume the following.

(A4) The exogenous price v is non-negative and continuous from the right.[7]

The next proposition claims that the top management of the firm can always choose the part of the free endowment necessary to meet the firm's needs and the distribution of this endowment among the technologies in such a way that the decentralized optimal decisions of the different technology managers coincide with the optimal solution at the firm level, (I^*, W^*, K^*). We call such a distribution d of the free endowment an *optimal distribution*. Sufficient conditions for a distribution to be optimal are given in the next proposition, that later will turn out to be also necessary if a mild additional assumption is fulfilled.

Proposition 1 *Let (A1), (A2) and (A4) hold. Then*
(i) If $d(\tau, s) \geq 0$ is a measurable function that satisfies the inequality

$$d(\tau, s) \leq g(\tau) K^*(\tau, s) W^*(\tau, s) \qquad \forall \tau \in [-\omega, \infty), \ \ \forall s \in [s_0(\tau), \omega], \tag{11}$$

then $(I^*(\tau, \cdot), W^*(\tau, \cdot), K^*(\tau, \cdot))$ is the unique optimal solution of problem (10), (5) for technology of vintage τ.
(ii) For every measurable function $D(t)$ that satisfies the inequality $D(t) \leq E^*(t)$ there exists a measurable $d(\tau, s)$ such that both (9) and (11) are fulfilled.

The inequality $D(t) \leq E^*(t)$ means that if the firm has more free endowments than what is required at the optimal emission level (the one with no free endowment) at the given price $v(\cdot)$, then the permits in excess should be sold on the secondary market. Inequality (11) means that the central manager of the firm should not give to any technology and at any time more free endowment than the optimal emission level of this technology for the given $v(\cdot)$. We mention also that a function d for which claim (ii) holds is explicitly given in the proof, but in general the choice of d is not unique.

Proof. First we prove the second claim. A function d that is claimed to exist can be defined (in a non-unique way, in general) as

$$d(\tau, s) = \frac{D(\tau + s)}{E^*(\tau + s)} g(\tau) K^*(\tau, s) W^*(\tau, s)$$

[7] The latter assumption is made for technical convenience. It means that $\hat{v}$ may have discontinuities, but $\lim_{s \to t, \, s > t} \hat{v}(s) = \hat{v}(t)$.

(in the degenerate case $E^*(\tau+s)=0$ one can merely take $d(\tau,s)=0$). Due to (3) and $D(\tau+s)\leq E^*(\tau+s)$ it is straightforward that (9) and (11) hold true. The measurability of d follows from the measurability of W^*, which in its turn follows from (6).[8]

To prove the first claim we take any d satisfying (11) and notice that $(i^*(\cdot),w^*(\cdot),k^*(\cdot)) = (I^*(\tau,\cdot),W^*(\tau,\cdot),K^*(\tau,\cdot))$ is the solution also of the problem

$$\max_{i\geq 0,\, w\in[0,1]} \int_{s_0(\tau)}^{\omega} e^{-rs}\left[(f(\tau)-v(\tau+s)g(\tau))k(s)w(s)+v(\tau+s)d(\tau,s)-C(s,i(s))\right]\mathrm{d}s \tag{12}$$

subject to (5), since $v(\tau+s)d(\tau,s)$ is just an additive exogenous term. The optimal objective value of (10), (5) cannot be higher than that in the last problem above, since the integrand in the objective function (10) is majorized by that in (12).

On the other hand (i^*,w^*,k^*) is an admissible solution of (10), (5), and due to (11) the maximum in the integral in (10) is achieved at the second argument of the "max", $g(\tau)k^*(s)w^*(s)-d(\tau,s)$. Then the corresponding objective value is the same as in (12), which implies that (i^*,w^*,k^*) is an optimal solution to (10), (5). The uniqueness can be proven from the strict concavity of the objective function as a function of the control i. Q.E.D.

What would happen if the free endowment is not used optimally? We can now turn to the investigation of the consequences of a distribution d of the free endowment $D(t)$ for which condition (11) is not satisfied. These consequences are summarized in the following proposition.[9]

Proposition 2 *Let (A1), (A2) and (A4) hold. Then for every $\tau\geq 0$ it holds that:*
(i) Whatever is the measurable function $d(\tau,s)\geq 0$, the solution (k^d,i^d,w^d) of (10), (5) (for the fixed τ) satisfies

$$k^d(s)\geq k^*(s),\quad i^d(s)\geq i^*(s),\quad w^d(s)\geq w^*(s). \tag{13}$$

(ii) Assume that for all $\tau\geq 0$ the function $d(\tau,\cdot)$ is non-negative, piece-wise continuous and one-sided continuous[10] and that $i^*(s)>0$ at least for all sufficiently small s. If the inequality (11) is violated for some $\bar{s}\in(s_0(\tau),\omega]$ for which $w^*(\bar{s})=1$ and $v(\tau+s)>0$ for s in a neighbourhood of $\bar{s}$, then the first two of the above inequalities are strict for the fixed τ and all s from a non-degenerate subinterval of $[s_0(\tau),\omega]$.

We mention that the assumption that $i^*(s)>0$ for some s eliminates the unusual possibility that the technology τ is not efficient even as brand new. The assumption $w^*(\bar{s})=1$ means that (11) is violated for some machines that would be in use if there were no free endowments.

[8] The second fact follows from the measurability of the set $\{(\tau,s):\ f(\tau)-v(\tau+s)g(\tau)>0\}$, which is implied by the assumptions.

[9] Though the obtained result is not spectacular (since it is rather intuitive), the formal analysis that we perform allows to strictly define the assumptions that are necessary for its validity.

[10] This means that at its points of discontinuity the function $d(\tau,\cdot)$ is continuous either from the left or from the right.

Proof. The optimal control i^* is characterized by equations (7) and (8). Due to (6) the adjoint equation takes the form

$$\dot{\xi}(s) = (r+\delta)\xi(s) - \chi(f(\tau) - v(\tau+s)g(\tau)), \qquad \xi(\omega) = 0, \tag{14}$$

with $\chi(x) = x$ for $x > 0$ and $\chi(x) = 0$ for $x \le 0$.

The integrand in (10) is non-differentiable with respect to the state variable k, which does not allow to apply the classical Pontryagin maximum principle to this problem. Therefore, we apply a non-smooth version of the maximum principle involving the subdifferential of the integrand (see e.g. Theorem 6E.1 in Loewen, 1993).

Denote

$$L(s,k,w) = f(\tau)kw - v(\tau+s)\max\{0,\, g(\tau)kw - d(\tau,s)\}, \qquad \hat{L}(s,k) = \max_{w\in[0,1]} L(s,k,w).$$

Then problem (10) can be equivalently reformulated as

$$\max_{i\ge 0} \int_{s_0(\tau)}^{\omega} e^{-rs}\left(\hat{L}(s,k(s)) - C(s,i(s))\right) \mathrm{d}s,$$

subject to (5). As a consequence of Theorem 6E.1 in Loewen, 1993 (the assumptions of this theorem are easy to check in our case), there exists an absolutely continuous function $\xi^d : [s_0(\tau),\omega] \to \mathbf{R}$ such that

$$\dot{\xi}^d(s) \in (r+\delta)\xi^d(s) - \partial_k \hat{L}(s,k(s)), \qquad \xi^d(\omega) = 0, \tag{15}$$

and the optimal control $i^*(s)$ in the problem (10), (5) is given by (7) with ξ^d substituted for ξ^*. The subdifferential $\partial_k \hat{L}$ of $\hat{L}$ with respect to k can be understood in the sense of the convex analysis, since $\hat{L}$ is a concave function. In order to calculate $\partial_k L$ we denote $\sigma(s) = f(\tau) - v(\tau+s)g(\tau)$ and calculate the maximizer $\hat{w}(s,k)$ in the definition of $\hat{L}$:

$$\hat{w}(s,k) = \begin{cases} 1 & \text{if } k \in [0, d(\tau,s)/g(\tau)] \\ 1 & \text{if } k > d(\tau,s)/g(\tau) \ \text{ and } \ \sigma(s) \ge 0, \\ d(\tau,s)/kg(\tau) & \text{if } k > d(\tau,s)/g(\tau) \ \text{ and } \ \sigma(s) < 0. \end{cases}$$

From here we immediately see that $\hat{w}(s,k) \ge w^*(s)$ for every s and k, therefore the last inequality in (13) holds true.

Using the above expression in $\hat{L}(s,k) = L(s,k,\hat{w}(s,k))$ we represent

$$\hat{L}(s,k) = \begin{cases} f(\tau)k & \text{if } k \in [0, d(\tau,s)/g(\tau)] \\ f(\tau)d(\tau,s)/g(\tau) & \text{if } k > d(\tau,s)/g(\tau) \ \text{ and } \ \sigma(s) < 0, \\ \sigma(s)k + v(\tau+s)d(\tau,s) & \text{if } k > d(\tau,s)/g(\tau) \ \text{ and } \ \sigma(s) \ge 0. \end{cases}$$

Then we obtain the following formula for the subdifferential depending on the sign of $\sigma(s)$:

$$\text{if } \ \sigma(s) < 0 \ \text{ then } \ \partial_k L(s,k) = \begin{cases} f(\tau) & \text{for } \ k \in [0, d(\tau,s)/g(\tau)), \\ [0, f(\tau)] & \text{for } \ k = d(\tau,s)/g(\tau), \\ 0 & \text{for } k > d(\tau,s)/g(\tau), \end{cases} \tag{16}$$

$$\text{if } \sigma(s) \ge 0 \text{ then } \partial_k L(s,k) = \begin{cases} f(\tau) & \text{for } k \in [0, d(\tau,s)/g(\tau)), \\ [\sigma(s), f(\tau)] & \text{for } k = d(\tau,s)/g(\tau), \\ \sigma(s) & \text{for } k > d(\tau,s)/g(\tau), \end{cases} \tag{17}$$

Now we can compare the two adjoint equations (14) and (15). We shall check that for any value of k and any element $l \in \partial_k \hat{L}(s,k)$ it holds that

$$l \ge \chi(\sigma(s)). \tag{18}$$

Indeed, if $\chi(\sigma(s)) = 0$, then we have $l \ge 0$ according to the first formula for $\partial_k \hat{L}(s,k)$. Notice that the inequality is strict if $k \in [0, d(\tau,s)/g(\tau))$.

If $\chi(\sigma(s)) > 0$ then from the formula for $\partial_k L(s,k)$ in this case we verify again (18). Again, the inequality is strict if $k \in [0, d(\tau,s)/g(\tau))$ and $v(\tau+s) > 0$.

The relation (18) implies that at points s at which $\xi^*(s) = \xi^d(s)$ and the two derivatives exist, the inequality $\dot{\xi}^*(s) \ge \dot{\xi}^d(s)$ holds true. Due to the end point conditions, $\xi^*(\omega) = \xi^d(\omega)$. Then a standard argument implies that $\xi^d(s) \ge \xi^*(s)$ for all s. Due to (7) this inequality implies $i^d(s) \ge i^*(s)$ and the respective inequalities for the capital stock and the emissions.

To prove claim (ii) by contradiction we assume that $k^d(s) \le k^*(s)$ (hence $k^d(s) = k^*(s)$) for every s, and let $\bar{s}$ be the number assumed to exist in the formulation of the proposition. Then

$$g(\tau)k^d(\bar{s}) < d(\tau, \bar{s}),$$

and this holds in a (possibly one-sided) neighbourhood of $\bar{s}$. Then according to (16) and (17) we have $\partial_k \hat{L}(s, k^d(s)) = \{f(\tau)\}$ independently of the sign of $\sigma(s)$. Due to the assumed property of v, this implies $\dot{\xi}^d(s) < \dot{\xi}^*(s)$ at any point from a neighbourhood of $\bar{s}$ at which $\xi^d(s) = \xi^*(s)$. This implies in a standard way that $\xi^d(s') > \xi^*(s')$ for some $s' < \bar{s}$ and sufficiently close to $\bar{s}$. From (18) it follows that the same inequality holds for all $s \in [s_0(\tau), s']$. Since $i^*(s) > 0$ close to $s_0(\tau)$ we obtain from (7) and the assumptions for C that $i^d(s) > i^*(s)$ for all sufficiently small $s \ge s_0(\tau)$. Hence, the same inequality applies for $k^d(s)$ and $k^*(s)$. We obtain a contradiction with the assumption that $k^d(s) \le k^*(s)$ for all s. This implies that $k^d(s) > k^*(s)$ for some s, and necessarily $i^d(s) > i^*(s)$ for some s. Q.E.D.

The overall solution at the firm level resulting from a distribution d of the free endowment is

$$I^d(\tau,s) = i^d(s), \quad W^d(\tau,s) = w^d(s), \quad K^d(\tau,s) = k^d(s), \tag{19}$$

where (i^d, w^d, k^d) is the solution of (10), (5) for the respective τ. Then the emission of the firm at time t is given by

$$E^d(t) = \int_{t-\omega}^{t} g(\tau)K^d(\tau, t-\tau)W^d(\tau, t-\tau)\,\mathrm{d}\tau,$$

while the emission for the optimal solution (I^*, W^*, K^*) is given in a similar way by (3).

Corollary 1 *Assume that the firm has distributed the free endowments in such a way that claim (ii) of Proposition 2 holds. Then for the emission at the optimal solution, $E^d(t)$, it holds that*

$$E^d(t) \geq E^*(t) \qquad \forall t > 0,$$

and the inequality is strict in a non-degenerate time interval.

The proof follows from the established relations between the solutions and the inequality $g(\tau) > 0$.

The corollary states that, if the firm does not use its free endowment optimally (in the sense defined just before proposition 1), then its emission level will be higher, for some t, than under a tax regime (where permit price and tax coincide) or without free endowment. This result is important because it shows that giving some permits for free to the firm raises challenging problems about the way these free permits are used. It challenges the common knowledge that the way permits are distributed does not matter for efficiency.

Further, a non-optimal distribution d may affect not only the firm's emission level, but also the average age of machines. However, it is easy to argue that a non-optimal distribution d may lead to either modernization (lower average age) or to older capital stock, depending on the particular choice of d. If (11) is violated dominantly for old ages s, then the non-optimal distribution would lead to higher average age of machines. If (11) is violated dominantly for young ages s the average age would be lower than in the case of an optimal distribution. It is interesting to see that giving permits for free is not necessarily an efficient way of modernizing the capital structure. Actually, the effect can go in the two directions. It just depends on how the endowment is split among the machines.

4 The equilibrium price of tradable permits

In this section we consider an economy with a cap on global emissions and a market for tradable emission permits (cap-and-trade system). Market equilibrium will set the permit price. By doing this we now endogenize the price v. The economy consists of n firms, each of which is described by the model considered in Section 2, but with firm-specific data δ, f, g, C, K_0, indexed further by the subscript i for $i = 1, \ldots, n$. Let $\hat{E}(t)$ be a cap on global emissions in the economy for $t \geq \hat{t}$. As before, we assume that the cap enters into force at time $\hat{t} > 0$, which is known by the firms at time $t = 0$ for all the future. For technical convenience we further assume that $\hat{t} \geq \omega$, although this is not essential for the obtained results.

The global amount of emission permits $\hat{E}(t)$ is issued at time $\hat{t}$ for all the future. Like in the previous section, some permits are auctioned, some others are given for free to the firms. So there will be some trading between the firms on the secondary market, which however takes place at the same time $t = \hat{t}$ due to the perfect foresight. According to Proposition 1(i) the optimal solutions of the firms are independent of the endowment, therefore the emission of each of them for any given price function $v(t)$ is also independent of the endowment and equals

$$E_i^*[v](t) = \int_{t-\omega}^{t} g_i(\tau) K_i^*[v](\tau, t-\tau) W_i^*[v](\tau, t-\tau) \, \mathrm{d}\tau, \tag{20}$$

where $(I_i^*[v], W_i^*[v], K_i^*[v])$ is the optimal solution for the i-th firm (without free endowment) for a given v (the case of non-optimal utilization of the endowments will be investigated in Section 6). Then the equation for the market price, $v(t)$, of emissions at the combined primary and secondary market is

$$E_1^*[v](t) + \ldots + E_n^*[v](t) = \hat{E}(t), \quad t \geq \hat{t}. \tag{21}$$

The solution of this equation is sought in the space $\mathcal{V}$ of all measurable and locally bounded functions $v : [0, \infty) \mapsto \mathbf{R}$ that equal zero on $[0, \hat{t})$.

The following cases may appear, in general:

(i) equation (21) does not have a solution $v(\cdot) \in \mathcal{V}$;
(ii) a solution exists but is not unique;
(iii) a unique solution exists but is not positive for all t;
(iv) an unique strictly positive solution exists.

Only in the case (iv) equation (21) determines a market price; all other cases indicate a market failure.

In general, all of the above cases may appear. To avoid (i) one needs assumptions for the data, and these assumptions are difficult to specify due to the complexity of equation (21).

The second case obviously takes place if, for example, $\hat{E}(t) \equiv 0$, since if v is a solution of (21), then every $\tilde{v}$ with $\tilde{v}(t) > v(t)$ would also be a solution. Even if the above two possibilities could be classified as "academic", the third case could be rather realistic.

A study of the market equation (21) is presented in a companion paper in the conference proceeding (see Bréchet, Tsachev and Veliov, 2009). Below we summarize some of the results obtained there, in particular those that will be used in the rest of the present paper. Lemma 2 below is not included in Bréchet, Tsachev and Veliov (2009); therefore we supply it with a proof.

We start with two properties of the emission map

$$v(\cdot) \longrightarrow E^*[v](t) := \int_{t-\omega}^{t} g(\tau) K^*[v](\tau, t-\tau) W^*[v](\tau, t-\tau) \, \mathrm{d}\tau,$$

where $(I^*[v], W^*[v], K^*[v])$ is the optimal solution of the firm's (without endowment) corresponding to price function v.

Lemma 1 *([Lemma 1], Bréchet, Tsachev and Veliov, 2009) For every $v \in \mathcal{V}$ and $t \geq \hat{t}$ the value $E^*[v](t)$ depends only on the restriction of v to $[t-\omega, t+\omega]$.*

Lemma 2 *Let v_1, $v_2 \in \mathcal{V}$ and $v_1(t) \leq v_2(t)$ for (almost) every $t \geq \bar{t}$. Then $E^*[v_1](t) \geq E^*[v_2](t)$ for every $t \geq 0$.*

Proof. Expression (6) implies $W^*[v_1](\tau, s) \geq W^*[v_2](\tau, s)$ for almost all $(\tau, s) \in [-\omega, T] \times [s_0(\tau), \omega]$ and (8) implies for the solutions corresponding to v_1 and v_2 that $\xi[v_1](s) \geq \xi[v_2](s)$ for all $(\tau, s) \in [-\omega, T] \times [s_0(\tau), \omega]$. The last inequality together with (7) and the monotonicity of $(C_i')^{-1}$ imply that $I^*[v_1](\tau, s) \geq I^*[v_2](\tau, s)$ for all (τ, s) as above. Using (1), we obtain $K^*[v_1](\tau, s) \geq K^*[v_2](\tau, s)$. The inequalities for $W^*[v_i]$, and $K^*[v_i]$, $i = 1, 2$, together with (3) imply $E[v_1](t) \geq E[v_2](t)$ for all $t \geq 0$. Q.E.D.

To shed some more light on the market equation (21) let us consider the special case where

$$C_i(s, I) = \frac{1}{2} c_i(s) I^2,$$

assuming, in addition, that for the solution v of (21) it holds for each firms i that $f_i(\tau) - v(t) g_i(\tau) \geq 0$ for $\tau \geq 0$ and $t \in (\tau, \tau + \omega)$. The last means that the price $v(t)$ does not invoke switch-off of existing machines, hence the environmental goals can be achieved optimally only by appropriate investment policies (without premature scrapping). The last assumption is quite realistic in view of the possibility to buy old machines at low price instead of buying new machines and scrap them when the emission cap becomes stringent.

In this case the following lemma holds.

Lemma 3 *[4, Lemma 3] Then for every $t \geq \hat{t}$*

$$E_i^*[v](t) \,=\, E_i^*[0](t) \,-\, \int_{\hat{t} \vee (t-\omega)}^{t+\omega} \varphi_i(t, \alpha) v(\alpha) \,\mathrm{d}\alpha, \tag{22}$$

where

$$\varphi_i(t, \alpha) = \begin{cases} \int_{t-\omega}^{\alpha} g_i^2(\tau)\, \kappa_i(t, \tau, \alpha - \tau) \,\mathrm{d}\tau & \text{if } \alpha \in [t-\omega, t], \\ e^{-(r+\delta_i)(\alpha - t)} \int_{\alpha-\omega}^{t} g_i^2(\tau)\, \kappa_i(t, \tau, t - \tau) \,\mathrm{d}\tau & \text{if } \alpha \in [t, t+\omega], \end{cases}$$

$$\kappa_i(t, \tau, \beta) = \int_0^{\beta} \frac{1}{c_i(s)} e^{-\delta_i(t-\tau-s) - (r+\delta_i)(\beta - s)} \,\mathrm{d}s,$$

and $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$.

Notice that $E_i^*[0](t)$ is the optimal emission of the i-th firm if there is no cost of emission at all (no emission restriction).

Using this lemma we may rewrite the market equation (21) in the form

$$\int_{\hat{t}\vee(t-\omega)}^{t+\omega} \Phi(t,\alpha)\, v(\alpha)\,\mathrm{d}\alpha = E^*[0](t) - \hat{E}(t), \quad t \geq \hat{t}, \tag{23}$$

where $E^*[0](t) = \sum_{i=1}^n E_i^*[0](t)$ and $\Phi(t,\alpha) = \sum_{i=1}^n \varphi_i(t,\alpha)$. The right-hand side represents the targeted reduction of emissions, which is the difference between the total emission of the industry at its optimal operation without any environmental constraints and the emission cap. The expression in the left-hand side represents the emission reduction resulting from optimal operation of the firms facing emission price v. The market price v should balance the two quantities.

The above simplified form of the market equation is still rather complicated and deserves some additional comments. Equation (23) is a Fredholm integral equation of the first kind on the infinite interval $[\hat{t}, +\infty)$. Such equations are inherently ill-posed. To see that it is enough to add to a solution v a highly fluctuating term, such as $\sin(nt)$. Then $v(t) + \sin(nt)$ would be a solution of the same equation with the right-hand side modified by the quantity $\int_{\hat{t}\vee(t-\omega)}^{t+\omega} \Phi(t,\alpha)\sin(n\alpha)\,\mathrm{d}\alpha$, which is arbitrarily small when n is large enough. Thus an arbitrarily small disturbance in the right-hand side can lead to a finite (even arbitrarily large) change of the solution. This difficulty is caused by the smoothing effect on v of the integration with Φ: high-frequency components of v are "smoothed out". Therefore one can expect that computing v would tend to amplify any high-frequency component or irregularity of the right-hand side. In effect, the right-hand side of (23) has to be somewhat "smoother" than the solution v in order to obtain satisfactory numerical approximation (see Hansen, 1992).

A good numerical method to solve a Fredholm integral equation of the first kind should be able to somehow filter out the high-frequency components in the singular value expansion of the solution (if such exists). Different methods of regularization aimed at finding reasonable numerical approximation to the solution are known, *e.g.* Tikhonov and Arsenin (1977), Delves and Mohamed (1985) and Kress (1989). However, we stress that our aim is not just to solve the price equation (21). Our ultimate goal is to imitate the behavior of the auction market, therefore the regularization we apply for solving (21) should be implementable also in the real auction market. We argue below that a relevant regularization tool is to formulate the emission constraint (21) period-wise, rather than at any time instant. This amounts to applying *regularization by time-aggregation*, related to the Nyström's method [8, Chapter 12]. In contrast to the celebrated singular expansion/decomposition this method is directly applicable also in the general (nonlinear) case of (21) and has a clear policy implementation.

Namely, we introduce the discrete version of the emission mapping E^* as follows. Assume that an emission restriction is given period-wise: $\hat{E}_k = \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} \hat{E}(t)\,\mathrm{d}t$ is the cap on emissions over the time-period $[t_{k-1}, t_k]$. For simplicity we assume that all time periods have the same length

$h > 0$, thus $t_k = kh$, and also that $\omega = mh$, $\hat{t} = \hat{k}h$ for appropriate natural numbers m and $\hat{k}$.[11] Correspondingly, the price of emission will be constant, v_k, on each interval $[t_{k-1}, t_k)$ and zero for $k \le \hat{k}$. Thus instead of the space $\mathcal{V}$ of price functions we consider

$$\mathcal{V}^h = \{(v_1, v_2, \ldots) : \; v_k \in (-\infty, +\infty), \; v_k = 0 \text{ for } k \le \hat{k}\},$$

which can be viewed as a subset of $\mathcal{V}$ by piece-wise constant embedding of $\mathcal{V}^h$ in $\mathcal{V}$. The emission of the i-th firm resulting from $v \in \mathcal{V}^h$ becomes a vector $E_i^{*h}[v] = (E_{i,1}^{*h}[v], E_{i,2}^{*h}[v], \ldots)$, where

$$E_{i,k}^{*h}[v] = \frac{1}{h} \int_{t_{k-1}}^{t_k} \int_{t-\omega}^{t} g_i(\tau) K_i^*[v](\tau, t-\tau) W_i^*[v](\tau, t-\tau) \, \mathrm{d}\tau \, \mathrm{d}t. \tag{24}$$

Then instead of the price equation (21) we consider the equation

$$\sum_{i=1}^{n} E_{i,k}^{*h}[v] \; = \; \hat{E}_k^h, \quad k \ge \hat{k}. \tag{25}$$

for $v \in \mathcal{V}^h$.

The last equation is studied in more details in Bréchet, Tsachev and Veliov (2009),Section 4. In particular an existence result is proved and the regularizing role of the commitment period to the market equation is numerically demonstrated. The essence is, that for a sufficiently large commitment period the discrete market equation (25) may have a unique positive solution (that is, it determines a marked price of emissions) although the "continuous" market equation (21) does not have a positive solution (that is, the "continuous" market fails to determine a price). Moreover, a larger commitment period better attenuates the high oscillations of the market price that appear for small h.

For illustration we reproduce two figures from Bréchet, Tsachev and Veliov (2009) which correspond to an industry consisting of equal firms. The detailed specification are given in Bréchet, Tsachev and Veliov (2009), Section 5.

In the figures below the plotted time horizon is 80 years. In Fig. 1 and Fig. 2 the emission cap imposed at time $t = 30$ is first constant, starting from the emission level before the beginning of the emission restriction, then it decreases quadratically. The dash-dotted line in the left plot of Fig. 1 represents the unrestricted emissions, $E^*[0]$, whereas the solid line represents the emission of the firms obeying the cap.

The right plot in Fig. 1 represents the permit price. It is obtained by using short commitment periods of $h = 1$ year. The market fails in two time periods. First, this happens immediately after the introduction of the emission cap at time $\hat{t} = 30$. Then it appends again when the shape of

[11] Following the terminology used under the Kyoto protocol of the UN Framework Convention on Climate Change, we shall call h a *commitment period*. The current commitment period is five-year long, covering 2008 to 2012.

the cap is modified at time $t = 50$: the solution of the market equation (21) takes negative values. In addition, the solution is highly oscillating around these time periods. The regularization of the market by adopting longer commitment periods is displayed on Fig. 2. The left plot corresponds to commitment periods of $h = 5$ years. The market still fails immediately after the introduction of the cap, but not around the change of the shape of the cap. The right plot in Fig. 2 represents the permit price with commitment periods of 10 years. Clearly the market is now fully efficient and looks quite regular. These numerical examples shows how the market for tradable permits is sensitive to the length of the commitment periods.[12] It also provides a rational to the result of Hintermann (2010), that the allowance price in the EU-ETS was not driven by marginal abatement cost determinants (with the possible exception of gas prices) before the first round of emissions accounting in April 2006, thus violating a necessary condition for a permit market to be an efficient policy instrument.

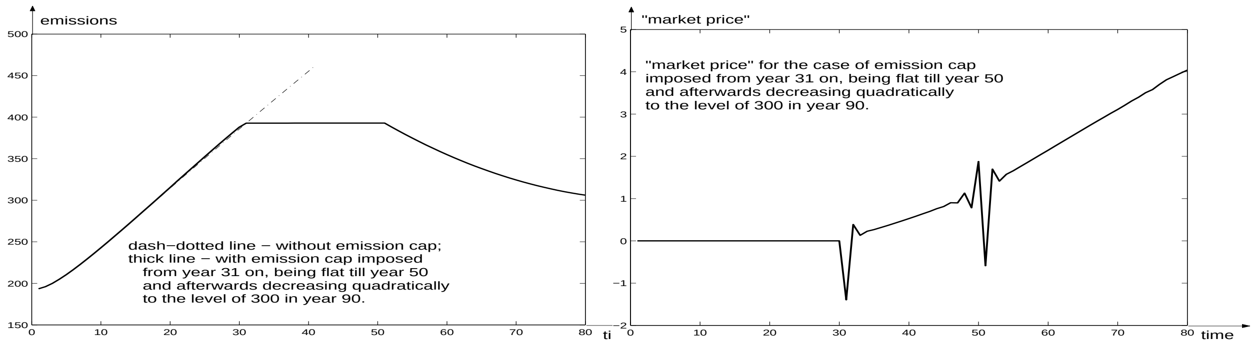

Figure 1: Emissions (left) and a solution to the price equation (21) (right) resulting from an emission restriction posed at time $t = 30$.

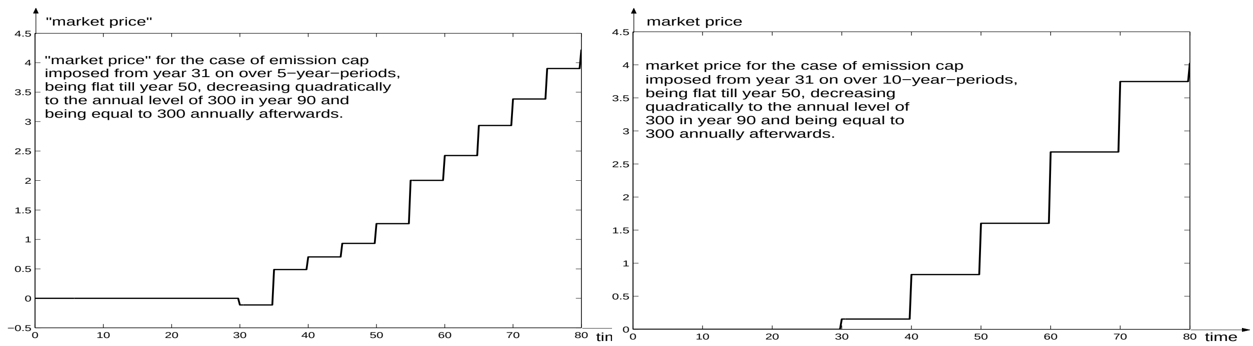

Figure 2: Auction price of emission permits for commitment periods $h = 5$ years (left) and $h = 10$ years (right).

[12]Let us recall that the Kyoto protocol established 5-year commitment periods.

5 The anticipation effect

In this section we address the following question: will the firms in the economy facing emission restrictions effective from time $\hat{t}$ change their investment decisions even before time $\hat{t}$? Two competing arguments are at stake. On one hand, early emission abatement is costly for the firm. But, on the other hand, because of adjustment costs, a sharp emission reduction (when the cap is enforced) can also be very costly. So what is the optimal mix between these two arguments? The answer to this question is that there exists an *anticipation effect*. We prove this effect and illustrate it numerically provided that the market does not fail to determine the permit price shortly after the enforcement period $\hat{t}$.

The next proposition claims that the emission cap introduced at time $\hat{t}$ decreases the investments of the firms even before time $\hat{t}$ if the firms are informed about the introduction of the cap before $\hat{t}$. This is proved under the assumption that the market equation (21) (or its period-wise version (25)) determines a positive market price $\hat{v}(t)$ for $t \geq \hat{t}$. For simplicity we assume that firms are symmetric (the subscript i in (21) is skipped). So we want to compare the solution $(I^*[\hat{v}], W^*[\hat{v}], K^*[\hat{v}])$ for the case of emission cap $\hat{E}$ with that for absence of a cap, $(I^*[0], W^*[0], K^*[0])$ (clearly zero price of emission, $v = 0$ means that the emission is unrestricted).

Proposition 3 *Assume that the market price $\hat{v}(t) > 0$ exists for $t > \hat{t}$ and that for some vintages $\tau < \hat{t}$ of positive measure there is $s_1(\tau) > s_0(\tau)$ such that $I^*[0](\tau, s) > 0$ for $s \in [s_0(\tau), s_1(\tau)]$. Then there exists a set $D \subset \{[0, \hat{t}) \times [0, \omega]\}$ of positive measure such that $I^*[\hat{v}](\tau, s) < I^*[0](\tau, s)$ for $(\tau, s) \in D$, while $I^*[\hat{v}](\tau, s) \leq I^*[0](\tau, s)$ for all (τ, s).*

Proof. The adjoint equations for a $\tau \in [\hat{t} - \omega, \hat{t}]$ for the problem with a price $\hat{v}$ of the emission and for zero price are correspondingly

$$\begin{aligned} \dot{\hat{\xi}} &= (r + \delta)\hat{\xi} - (f(\tau) - \hat{v}(\tau + s)g(\tau))\hat{w}^*(s), \quad && \hat{\xi}(\omega) = 0, \\ \dot{\xi} &= (r + \delta)\xi - f(\tau), && \xi(\omega) = 0, \end{aligned}$$

where $\hat{w}^*(s) = W^*[\hat{v}](\tau, s)$. Comparing the right-hand sides we obtain that $\xi(s) \geq \hat{\xi}(s)$ for all s, which implies the last claim of the proposition according to (7) and the monotonicity of $(C')^{-1}$ ensured by (A2). The first claim is implied by the same argument, using the additional information that the right-hand side of the equation for $\hat{\xi}$ is strictly larger than that for ξ for ages s of positive measure (and for vintages $\tau < \hat{t}$ of positive measure). The latter follows from the fact that $W^*[\hat{v}](\tau, s) = 1$ for $(\tau, s) \in [0, \hat{t}) \times [0, \min\{\hat{t} - \tau, \omega\})$, the assumption for $I^*[0]$, the fact that $I^*[0](\cdot, s)$ is nondecreasing in τ (because of the equation for ξ, the monotonicity of $f(\cdot)$ in τ and (7) together with the strict monotonicity of $(C')^{-1}$) and, again, because of (7) together with the strict monotonicity of $(C')^{-1}$. Q.E.D.

The same claim (with the same proof) applies to the period-wise market price obtained from (25).

We illustrate the anticipation effect by the following example (the details specifications are given in [4, Section 5]). Let us take $\hat{E}(t)$ constant, with value $\hat{E} = 300$, which is strictly smaller than the quantity that the firms would have emitted without regulation, $E^*[0](\hat{t})$.

On Figure 3 (left plot) we observe that the emission starts to decline downwards from the unrestricted emission much earlier than $\hat{t} = 30$ (about time $t = 15$). This is a consequence of lower investment levels even before time $\hat{t}$ (the anticipation effect).

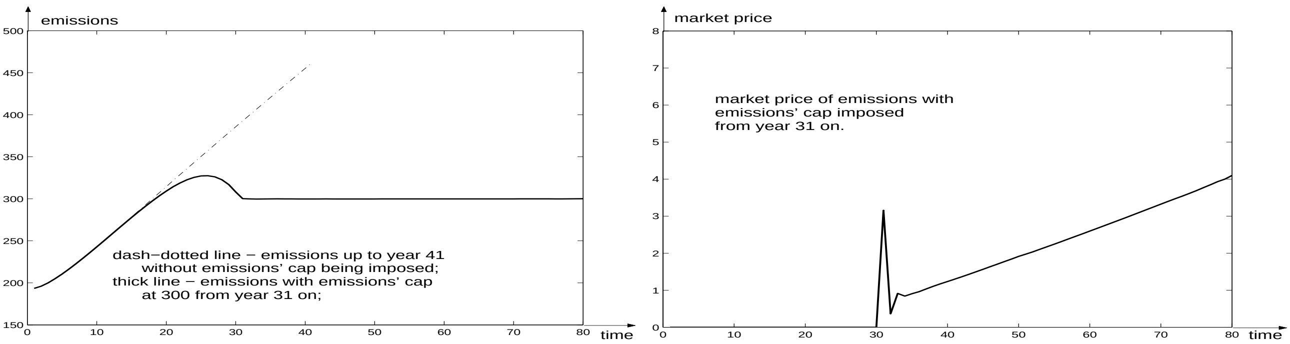

Figure 3: Emission resulting from an emission restriction posed at time $t = 30$ (left) and the corresponding market price of permits (right).

6 Equilibrium effects of the non-optimal use of the free endowments

This last section presents one of the most interesting findings in this paper. Namely, a non-optimal use of the free endowment by some firms brings disadvantages not only for this firm (in terms of cost and emissions) but also for all the other firms involved in the market for emission permits. This contamination effect goes through the permit price in equilibrium.

Again, for simplicity we consider an economy consisting of two initially identical firms. These two firms are endowed with two different permits endowments, $D_1(t)$ and $D_2(t)$, given to the first and to the second firm, respectively. It is assumed that firm 2 uses its free endowment in an optimal way. According to Proposition 1, the optimal solution $(I_2^*[v], W_2^*[v], K_2^*[v])$ for this firm is independent of its free endowment $D_2(t)$, for any market price $v(t)$. The corresponding emission level equals $E_2^*[v](t)$ given by (see (3))

$$E_2^*[v](t) = \int_{t-\omega}^{t} g_2(\tau) K_2^*[v](\tau, t-\tau) W_2^*[v](\tau, t-\tau)\, \mathrm{d}\tau.$$

On the other hand, let us assume that firm 1 distributes its free endowment among its different technologies following a non-optimal rule, as it was defined in Section 3. Then, for any price v, the solution $(I_1^d[v], W_1^d[v], K_1^d[v])$ (see (19)) results in an emission level

$$E_1^d[v](t) = \int_{t-\omega}^{t} g_1(\tau) K_1^d[v](\tau, t-\tau) W_1^d[v](\tau, t-\tau)\, \mathrm{d}\tau.$$

Then the market price equation becomes

$$E_1^d[v](t) + E_2^*[v](t) = \hat{E}(t), \quad t \geq \hat{t}.$$

(Obviously this equation coincides with (21) if firm 1 uses its free endowment optimally, since in this case $E_1^d[v] = E_1^*[v]$.) Assume that the above equation has a positive solution $v^d \in \mathcal{V}$. We rewrite it as

$$E_1^*[v](t) + E_2^*[v](t) = \hat{E}(t) - (E_1^d[v](t) - E_1^*[v](t)) =: \tilde{E}(t).$$

As a consequence of Proposition 2 (i)

$$E_1^d[v^d](t) - E_1^*[v^d](t) \geq 0.$$

Assume the first firm does not use its free endowment $D_1(t)$ optimally, so that claim (ii) of Proposition 2 holds for v^d. According to Corollary 1, the last inequality is strict on some time intervals. Thus v^* and v^d satisfy for $t \geq \hat{t}$ the equations

$$E_1^*[v^*](t) + E_2^*[v^*](t) = \hat{E}(t), \tag{26}$$
$$E_1^*[v^d](t) + E_2^*[v^d](t) = \tilde{E}(t) \tag{27}$$

and

$$\hat{E}(t) \geq \tilde{E}(t), \tag{28}$$

where the inequality is strict for some t of positive measure. That is, if one firm does not use its free endowment optimally it would create in the market for permits the same effect as if the emission cap were be lower ($\tilde{E}(t)$ instead of $\hat{E}(t)$). In other words, the non-optimal use of free endowments by some firms affects negatively the supply side of the market of permits. The basic intuition would suggest that a lower emission cap would imply a higher market price for emissions. However, in our dynamic vintage framework this is not necessarily the case. Figure 4 provides a comparison of the two market prices (right plot) resulting from an emission caps at $t = 30$ (left plot). Although one of the caps is above the other for all $t > \hat{t} = 30$, the expected inequality for the respective market prices is violated for some values of t. In other words, a more stringent emission cap can lead to a lower equilibrium price at some time periods. This effect should not be considered as typical. In the example presented by Figure 4 the "abnormal" price behavior occurs shortly after the "kink" of the emission cap and is caused by the latter circumstance rather than by the relation between the size of the caps. The next proposition shows that the intuitive argument stated above has its formal ground.

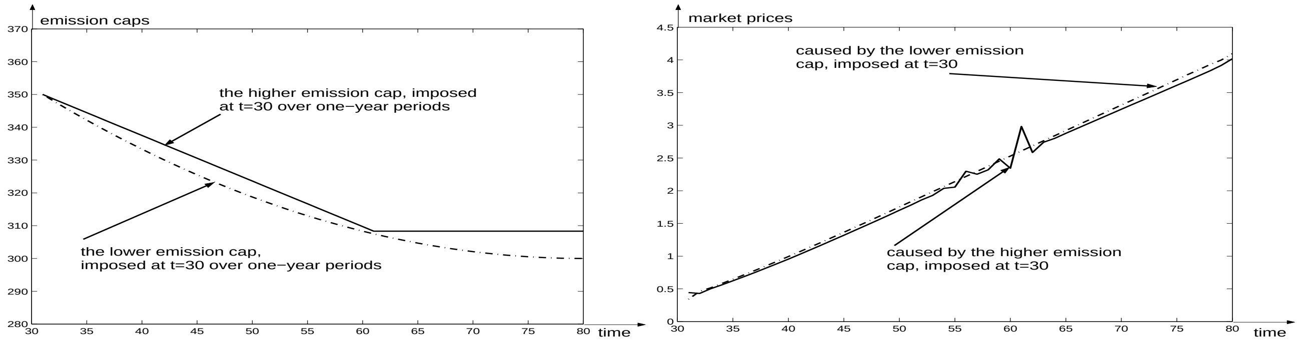

Figure 4: Two emission caps (left) and the corresponding equilibrium prices (right).

Proposition 4 *Assume that the solutions v^* and v^d of equations (26) and (27) exist on $t \in (\hat{t}, +\infty)$ (not necessarily positive). If (28) is fulfilled as strict inequality on a set of positive measure around a point $\tilde{t} > \hat{t}$, then $v^d(t) > v^*(t)$ on a subset of positive measure around $[\tilde{t} - \omega, \tilde{t} + \omega]$.*[13]

Proof. If for every $\varepsilon > 0$ we assume that $v^d(t) \le v^*(t)$ a.e. on $(\tilde{t} - \omega - \varepsilon, \tilde{t} + \omega + \varepsilon) \cap [\hat{t}, \infty)$, then $E_i^*[v^*](t) \le E_i^*[v^d]$ on $(\tilde{t} - \varepsilon, \tilde{t} + \varepsilon) \cap [\hat{t}, \infty)$ according to Lemma 1 and Lemma 2. Then (26), (27) imply $\hat{E}(t) \le \tilde{E}(t)$ on $(\tilde{t} - \varepsilon, \tilde{t} + \varepsilon) \cap [\hat{t}, \infty)$. This contradicts the assumption in the proposition. Q.E.D.

Proposition 4 shows that if one of the two firms does not use its free endowment optimally, then the other firm will suffer from a higher market price, regardless of the fact that it has allocated its free endowment in an optimal way. This latter firm will be pushed to abate more than what is optimal, just because the other firm does not use optimally its free endowments. The market for tradable emission permits thus generates a pecuniary externality among the firms. Furthermore, the firm that does not behave optimally will suffer twice. First, it pollutes more than what is optimal, which is costly. Second, because of market equilibrium, it also faces a higher equilibrium price for tradable permits that it needs, at least for some time periods.

References

[1] Barucci, E., and Gozzi, F., 1998. Investment in a vintage capital model. *Research in Economics*, **52**, 159–188.

[2] Barucci, E., and Gozzi, F., 2001. Technology adoption and accumulation in a vintage capital model. *Journal of Economics*, **74**, 1–38.

[13] A property holds on set of positive measure around a set (or point) $A \subset \mathbf{R}$ if for every neighborhood of A the property holds on a set of positive measure contained in this neighborhood.

[3] Boucekkine, R., Germain, M., Licandro, O., and Magnus, A., 1998. Creative destruction, investment volatility and the average age of capital. *Journal of Economic Growth*, **3**, 361–384.

[4] Bréchet, Th., Tsachev, Ts., and Veliov, V.M., 2009. Prices Versus Quantities in a Vintage Capital Model. *Dynamic Systems, Economic Growth, and the Environment.* Series: *Dynamic Modeling and Econometrics in Economics and Finance*, Vol. 12, (ISBN: 978-3-642-02131-2), J. Crespo Cuaresma, T. Palokangas, A. Tarasyev (Eds.), pp. 141–159, Springer.

[5] Bréchet, Th, Jouvet, P-A., Michel, Ph., Rotillon, G., 2010. Tradable pollution permits in dynamic general equilibrium: can optimality and acceptability be reconciled? CORE discussion paper, Université catholique de Louvain.

[6] Cramton, P., and Kerr, S., 2002. Tradeable carbon permit auctions. How and why to auction not grandfather. *Energy Policy*, **30**(4), 333–345.

[7] Dales, J.H., 1968. Pollution, Property & Prices: An Essay in Policy-making and Economics. Toronto: University of Toronto Press.

[8] Delves, L.M., and Mohamed, J.L., 1985. *Computational Methods for Integral Equations.* Cambridge Univ. Press.

[9] Ellerman A.D., Joskow P.L., Montero J-P., Schmalensee R., Bailey E.M., 2000. *Markets for Clean Air: The US Acid Rain Program.* Cambridge University Press, Cambridge, UK.

[10] Ellerman, D.A., 2005. A Note on Tradeable Permits. *Environmental and Resource Economics*, **31**(2), 123–131.

[11] Ellerman, D.A., Carraro, C., and Buchner, B. (Eds.), 2007. *Allocation in the European Emissions Trading Scheme - Rights, Rents and Fairness.* Cambridge: Cambridge University Press.

[12] Feichtinger, G., Hartl, R.F., Kort,P.M., and Veliov, V.M., 2006. Anticipation effects of technological progress on capital accumulation: a vintage capital approach. *Journal of Economic Theory*, **126**, 143–164.

[13] Feichtinger, G., Hartl, R.F., Kort,P.M., and Veliov, V.M., 2008. Financially constrained capital investments: the effects of disembodied and embodied technological progress. *Journal of Mathematical Economics*, **44**, 459–483.

[14] Goulder, L.H., 2002. Mitigating the Adverse Impacts of CO_2 Abatement Policies on Energy-Intensive Industries, RFF Discussion Paper 0222.

[15] Jaccard, M., and Rivers, N., 2007. Heterogenous capital stocks and the optimal timing for CO_2 abatement. *Resource and Energy Economics*, **29**, 1–16.

[16] Jouvet, P.A., Michel, Ph. and Rotillon, G., 2005. Optimal Growth with Pollution : How to Use Pollution Permits, *Journal of Economic Dynamics and Control*, **29**, 1597-1609.

[17] Hansen, P.C., 1992. Numerical methods for analysis and solution of Fredholm integral equations of the first kind. *Inverse Problems*, **8**, 849–872.

[18] Hintermann, B., 2010. Allowance price drivers in the first phase of the EU-ETS. *Journal of Environmental Economics and Management*, **59**(1), 43–56

[19] International Energy Agency, 2005. *Act Locally, Trade Globally, Emissions Trading for Climate Policy.* OECD/IEA, Paris.

[20] Kress, R., 1989. *Linear Integral Equations.* Applied Mathematical Sciences, vol. 82, Springer-Verlag.

[21] Loewen, Ph., 1993. *Optimal control via nonsmooth analysis.* CRM Proceedings & Lecture Notes, 2. American Mathematical Society, Providence, RI.

[22] Moledina, A.A., Coggins, J.S., Polasky, S., and Costello, C., 2003. Dynamic environmental policy with strategic firms: prices versus quantities. *Journal of Environmental Economics and Management* **45**, 356–376.

[23] Montgomery, D.W., 1972. Markets in licenses and efficient pollution control programs. *Journal of Economic Theory* **5**, 395–418.

[24] Pontryagin *et al.*, 1962. *The mathematical theory of optimal processes.* Wiley, NY.

[25] Stavins R.N., 1998. What Can We Learn from the Grand Policy Experiment ? Lessons from SO_2 Allowance Trading, *Journal of Economic Perspectives*, **12**(3), 69–88.

[26] Tietenberg, T., Thomas, H., 2006. *Emissions Trading: Principles and Practice.* 2nd ed., Washington, DC: Resources for the Future.

[27] Tikhonov, A.N., and Arsenin, V.Y., 1977. *Solutions of Ill-Posed Problems.* Wiley, NY.

[28] UNFCCC, 1997. Report of the Ad Hoc Group on the Berlin Mandate on the Work of its Sixth Session: Proposals for a Protocol or Another Legal Instrument, Bonn, March 1997. UNFCCC/AGBM/1997/3/Add.1, 22 1997 (http://unfccc.int/cop3/03a01-2.htm).

[29] UNFCCC, 2005. Caring for Climate - A guide to the Climate Change Convention and the Kyoto Protocol. United Nations Climate Change Secretariat, Bonn, Germany (http://unfccc.int/resource/docs/publications/caring2005en.pdf).

[30] Veliov, V.M., 2008. Optimal Control of Heterogeneous Systems: Basic Theory. *Journal of Mathematical Analysis and Applications*, **346**, 227–242.