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Social Optimality in the Constructed-Capital Model*†

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Abstract

In the constructed capital model, the steady state is derived under the assumption that each individual behaves optimally. Contrasting to this decentralized approach, in this paper we derive the first-best outcome a central planner would choose. The results show that catastrophic agglomeration is socially not optimal, irrespective of the level of trade barriers. Furthermore, the differences in the explicit solutions of both approaches are highlighted.

1 Introduction

Since the middle of the last century, many countries have been confronted with globalization and its socio-economic consequences. Fears but also opportunities arise in industrialized as well as in developing countries. Inhabitants of industrialized countries are afraid of production shifting to low

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wage countries, whereas inhabitants of developing countries fear relocation of production into industrialized economies due to better technology, better infrastructure and more high skilled workers. Relocation of industrial activities is usually strengthened by decreasing transport costs and lowering trade barriers such as tariffs and quotas.

Agglomeration tendencies are studied in theoretical economics since the seminal paper by Krugman [1]. The model - referred to as core-periphery model - assumes footloose labor, i.e. workers can migrate in response to different wage levels. Venables [2] and Krugman and Venables [3] introduced a second type of models assuming vertically linked-industries instead of labor mobility. Several other papers extend the basic models (e.g. Baldwin et al. [4] and references therein or Puga [5], who defined a more general model, where both model classes are special cases) to allow for labor mobility restrictions, capital accumulation and forward looking expectations. For extensions of these models we refer to Baldwin [6], Grafeneder-Weissteiner and Prettner [7] or again Baldwin et al. [4] and references therein.

All mentioned models derive conditions for the occurrence of so called catastrophic agglomeration. This situation is characterized by the fact that the equilibrium becomes unstable when some parameter(s), especially trade costs, cross the corresponding critical value(s). If the unstable equilibrium is slightly disturbed, a core-periphery pattern emerges, where one economy attracts all industrial production and the other economy becomes completely deindustrialized.

The constructed capital model assumes homogeneous individuals in each region who optimize their own utility, without considering externalities. Social welfare is then obtained by aggregation of all individual utilities. This decentralized market structure does not necessarily imply a first-best solution in a macroeconomic sense, where all externalities are accounted for. Instead, a first-best solution can be obtained in a model where the whole economy, i.e. both regions together, is considered and optimization is carried out by an omniscient central planner. Up to our knowledge Tafenau [8] is the only paper that deals with the central planner approach in the constructed capital model. However, that paper does not provide a complete characterization of the socially optimal outcome. It would be important to be know the first-best solution as economists could learn to implement appropriate incentives such that individuals internalize externalities and behave in the socially optimal way.

In this paper, we propose a constructed capital model and derive the first-best solution. The model is comparable to the decentralized model of Baldwin [6]. In section 2, we present the model together with a detailed discussion of the assumptions. The optimality conditions and mathematical details are contained in section 2.1. A comparison with the outcomes of Baldwin [6] is provided in section 3. Section 4 concludes.

2 The Central Planner Model

The model consists of two regions: home (H) and foreign (F). In both of them labor and capital are the only production factors. The available labor force, which is exogenously given, is immobile and divided between the investment sector, the agricultural sector and the manufacturing sector of the corresponding region. The investment sector produces new capital which cannot be shifted between regions. It uses labor as a production factor to convert individual's savings into machines according to the production function

$$I^k(t) = \frac{L_I^k(t)}{F} \tag{1}$$

of region $k = \{H, L\}$, where $I^k(t)$ denotes capital investment at time t and $L_I^k(t)$ the labor devoted to the investment sector.[1] In this sector labor is the only input with the unit input coefficient F which is assumed to be exogenous. The amount of capital determines the product variety of the manufactured goods denoted by $V_H(\cdot)$ and $V_F(\cdot)$ for region H and region F, respectively. As each capital unit is equal to one machine, the product variety equals the capital stock, i.e. $V_H(K^H(t)) = K^H(t)$ and $V_F(K^F(t)) = K^F(t)$, where $K^k(t)$ denotes the capital stock of region k at time t. Manufactured goods are assumed to be produced according to the following constant returns to scale production function

$$Y_m^k(t) = \frac{L_m^k(t)}{\beta}, \tag{2}$$

where $Y_m^k(t)$ denotes output at time t and $L_m^k(t)$ labor devoted to the manufacturing sector[2] and β is the efficiency parameter of labor. The manufactured goods can be traded between the two regions with iceberg transport costs, i.e. $\varphi \geq 1$ units have to be shipped from H to consume one unit in F and vice versa[3]. Thus the trade barriers increase in the parameter φ

[1]Within the whole article t denotes the time at which the corresponding variable is evaluated.

[2]Therefore capital is a production factor for the manufacturing sector as it determines the number of different machines (equivalent to the number of different products). The number of product varieties depends on the capital stock. However, the amount of each product variety does not depend on the capital.

[3]Trade costs are assumed to be equal for both regions. However, the model allows to generalize this assumption straightforwardly.

($\varphi = 1$ means no trade barriers, $\varphi = \infty$ means that shipment is impossible). The agricultural sector produces a homogeneous good, which can be referred to as food, according to the following constant returns to scale production function

$$Y_n^k(t) = \frac{L_n^k(t)}{\alpha}, \tag{3}$$

where labor is the only input factor, $Y_n^k(t)$ denotes output at time t, $L_n^k(t)$ refers to labor devoted to the agricultural sector and α is the efficiency parameter of labor. For both production sectors the market is in equilibrium, i.e. production equals consumption such that $Y_m^k(t) = C_m^k(t)$ and $Y_n^k(t) = C_n^k(t)$. Thus the resource constraint of both regions reads

$$L^k(t) = FI^k(t) + \alpha C_n^k(t) + \beta C_m^k(t), \tag{4}$$

where $L^k(t)$ denotes the total available labor of region k which is exogenous. As a result we can formulate the dynamics of the capital stocks which increase in capital investment and decrease due to exogenous and constant depreciation at rate δ:

$$\begin{aligned}
\dot{K}^H(t) &= \frac{1}{F}\big(L^H(t) - \alpha C_n^H(t) - \beta C_m^H(t)\big) - \delta K^H(t), K^H(0) = K_0^H, \\
\dot{K}^F(t) &= \frac{1}{F}\big(L^F(t) - \alpha C_n^F(t) - \beta C_m^F(t)\big) - \delta K^F(t), K^F(0) = K_0^F,
\end{aligned} \tag{5}$$

where total consumption of agricultural and manufactured goods in both regions are defined as

$$\begin{aligned}
C_n^H(t) &= c_n(t)N^H(t) \\
C_n^F(t) &= c_n^*(t)N^F(t) \\
C_m^H(t) &= K^H(t)\Big[c_m^H(t)N^H(t) + \varphi c_m^{*H}(t)N^F(t)\Big] \\
C_m^F(t) &= K^F(t)\Big[c_m^{*F}(t)N^F(t) + \varphi c_m^F(t)N^H(t)\Big].
\end{aligned} \tag{6}$$

The population of both regions is stable and denoted by $N^H(t)$ and $N^F(t)$, $c_n(t)$ and $c_n^*(t)$ refer to per-capita consumption of the agricultural good in region H and region F respectively, $c_m^H(t)$ and $c_m^{*H}(t)$ denote per-capita consumption in region H and F of the manufactured good produced in H. Analogously, $c_m^F(t)$ and $c_m^{*F}(t)$ denote per-capita consumption in region H and F of the manufactured good produced in F. Thus the superscripts H and F describe where the manufactured good has been produced, while an asterisk denotes the place of its consumption (without asterisk: consumption in region H, with asterisk: consumption in F). If a good is consumed in a different region than it has been produced, iceberg transport costs φ have to be taken into account. To get to aggregate expressions, per capita consumption has to be multiplied by the size of the population of the corresponding region. Therefore the terms in brackets refer to total consumption of one specific manufactured product.

The capital stock at the beginning is strictly positive, i.e. $K^H(0) > 0$ and $K^F(0) > 0$. Otherwise our model would not work, as no production capital is available which is necessary for positive output. Finally we add a nonnegativity constraint for both capital stocks, which is obvious from an economic point of view,

$$K^H(t) \geq 0, K^F(t) \geq 0. \tag{7}$$

The central planner chooses optimal consumption for both regions simultaneously. Intertemporal utility is of the Benthamite type, i.e. per-capita instantaneous utility of consumption is weighted by the population size.[4,5] The individual utility $u(\cdot)$ consists of several components. The interaction between the agricultural and the manufactured goods is of the Cobb-Douglas form with parameter $0 < \xi < 1$, i.e. (for region H representatively)

$$u\big(c_n^{1-\xi}(t) c_m^{agg,\xi}(t)\big). \tag{8}$$

The component from the manufactured goods is denoted by $c_m^{agg}(t)$ and $c_m^{*,agg}(t)$ and is defined by a CES subutility function with $\sigma > 1$ being the constant elasticity of substitution, i.e.

$$\begin{aligned} c_m^{agg}(t) &= \Big[K^H(t) c_m^H(t)^{\frac{\sigma-1}{\sigma}} + K^F(t) c_m^F(t)^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{\sigma}{\sigma-1}} \\ c_m^{*,agg}(t) &= \Big[K^H(t) c_m^{*H}(t)^{\frac{\sigma-1}{\sigma}} + K^F(t) c_m^{*F}(t)^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{\sigma}{\sigma-1}}. \end{aligned} \tag{9}$$

In order to obtain a strictly concave shape of the instantaneous utility function, the logarithm is used, i.e. $u(\cdot) = \ln(\cdot)$. Putting all things together, the objective function of the central planner reads

$$\int_0^\infty e^{-\rho t}\Big[N^H(t) \ln\big(c_n^{1-\xi}(t) c_m^{agg,\xi}(t)\big) + N^F(t) \ln\big(c_n^{*,1-\xi}(t) c_m^{*,agg,\xi}(t)\big)\Big]\, dt, \tag{10}$$

[4] Note that this implies that the social planner treats each individual in the same way. Therefore a region with a higher population has a higher weight in the utility function.

[5] In the numerical calculations a stationary population ($\dot{N}^k(t) = 0$, $k = \{H, F\}$) is assumed. In that case there is no difference between the Benthamite and the Millian approach.

where $\rho > 0$ is the time discount rate. Equation (10) has to be maximized with respect to equation (5) and equation (7) over nonnegative consumption, i.e.

$$c_n(t) \geq 0, c_n^*(t) \geq 0, c_m^H(t) \geq 0, c_m^{*H}(t) \geq 0, c_m^F(t) \geq 0, c_m^{*F}(t) \geq 0. \tag{11}$$

2.1 Optimality Conditions

For the solution of the model we use Pontryagin's Maximum Principle. In order to deal with the nonnegativity constraints of the capital stocks of both regions, we define the following two functions according to Feichtinger and Hartl [9] (from now on we skip the time argument if it is not of particular importance):

$$\begin{aligned}
\mathcal{H} &= N^H \ln\left[c_n^{1-\xi}\left(K^H c_m^{H\frac{\sigma-1}{\sigma}} + K^F c_m^{F\frac{\sigma-1}{\sigma}}\right)^{\xi\frac{\sigma}{\sigma-1}}\right] + \\
&\quad + N^F \ln\left[c_n^{*,1-\xi}\left(K^H c_m^{*H\frac{\sigma-1}{\sigma}} + K^F c_m^{*F\frac{\sigma-1}{\sigma}}\right)^{\xi\frac{\sigma}{\sigma-1}}\right] + \\
&\quad + \lambda^H\left[\frac{1}{F}\Big(L^H - \alpha c_n N^H - \beta K^H\big(c_m^H N^H + \varphi c_m^{*H} N^F\big)\Big) - \delta K^H\right] + \\
&\quad + \lambda^F\left[\frac{1}{F}\Big(L^F - \alpha c_n^* N^F - \beta K^F\big(c_m^{*F} N^F + \varphi c_m^F N^H\big)\Big) - \delta K^F\right] \qquad (12) \\
\mathcal{L} &= \mathcal{H} + \eta^H K^H + \eta^F K^F, \qquad (13)
\end{aligned}$$

where λ^H and λ^F denote the adjoint variables (or dynamic shadow prices) of the capital stocks K^H and K^F, respectively. The expressions η^H and η^F refer to Lagrange multipliers for the nonnegativity constraint of the corresponding capital stock. Consequently, we can apply Theorem 6.2 of Feichtinger and Hartl [9] to obtain the following necessary first order conditions (for positive consumption values)

$$
\begin{aligned}
\mathcal{L}_{c_n} &= N^H(1-\xi)\frac{1}{c_n} - \alpha\lambda^H\frac{1}{F}N^H \stackrel{!}{=} 0 \\
\mathcal{L}_{c_n^*} &= N^F(1-\xi)\frac{1}{c_n^*} - \alpha\lambda^F\frac{1}{F}N^F \stackrel{!}{=} 0 \\
\mathcal{L}_{c_m^H} &= N^H\xi\Big(K^H c_m^{H\frac{\sigma-1}{\sigma}} + K^F c_m^{F\frac{\sigma-1}{\sigma}}\Big)^{-1} K^H c_m^{H-\frac{1}{\sigma}} - \lambda^H\frac{\beta}{F}K^H N^H \stackrel{!}{=} 0 \\
\mathcal{L}_{c_m^F} &= N^H\xi\Big(K^H c_m^{H\frac{\sigma-1}{\sigma}} + K^F c_m^{F\frac{\sigma-1}{\sigma}}\Big)^{-1} K^F c_m^{F-\frac{1}{\sigma}} - \lambda^F\frac{\beta}{F}K^F N^H\varphi \stackrel{!}{=} 0 \\
\mathcal{L}_{c_m^{*H}} &= N^F\xi\Big(K^H c_m^{*H\frac{\sigma-1}{\sigma}} + K^F c_m^{*F\frac{\sigma-1}{\sigma}}\Big)^{-1} K^H c_m^{*H-\frac{1}{\sigma}} - \lambda^H\frac{\beta}{F}K^H N^F\varphi \stackrel{!}{=} 0 \\
\mathcal{L}_{c_m^{*F}} &= N^F\xi\Big(K^H c_m^{*H\frac{\sigma-1}{\sigma}} + K^F c_m^{*F\frac{\sigma-1}{\sigma}}\Big)^{-1} K^F c_m^{*F-\frac{1}{\sigma}} - \lambda^F\frac{\beta}{F}K^F N^F \stackrel{!}{=} 0
\end{aligned} \tag{14}
$$

and the following dynamics of the adjoint variables

$$
\begin{aligned}
\dot{\lambda}^H(t) &= \Big(\rho+\delta+\frac{\beta}{F}(c_m^H N^H + \varphi c_m^{*H}N^F)\Big)\lambda^H - \\
&\quad -N^H\xi\frac{\sigma}{\sigma-1}\Big(K^H c_m^{H\frac{\sigma-1}{\sigma}} + K^F c_m^{F\frac{\sigma-1}{\sigma}}\Big)^{-1} c_m^{H\frac{\sigma-1}{\sigma}} - \\
&\quad -N^F\xi\frac{\sigma}{\sigma-1}\Big(K^H c_m^{*H\frac{\sigma-1}{\sigma}} + K^F c_m^{*F\frac{\sigma-1}{\sigma}}\Big)^{-1} c_m^{*H\frac{\sigma-1}{\sigma}} - \eta^H \\
\dot{\lambda}^F(t) &= \Big(\rho+\delta+\frac{\beta}{F}(c_m^{*F} N^F + \varphi c_m^{F}N^H)\Big)\lambda^F - \\
&\quad -N^H\xi\frac{\sigma}{\sigma-1}\Big(K^H c_m^{H\frac{\sigma-1}{\sigma}} + K^F c_m^{F\frac{\sigma-1}{\sigma}}\Big)^{-1} c_m^{F\frac{\sigma-1}{\sigma}} - \\
&\quad -N^F\xi\frac{\sigma}{\sigma-1}\Big(K^H c_m^{*H\frac{\sigma-1}{\sigma}} + K^F c_m^{*F\frac{\sigma-1}{\sigma}}\Big)^{-1} c_m^{*F\frac{\sigma-1}{\sigma}} - \eta^F.
\end{aligned} \tag{15}
$$

Due to the state constraints, we also have the following slackness conditions

$$
\begin{aligned}
&\eta^H \geq 0, \eta^H K^H = 0 \\
&\eta^F \geq 0, \eta^F K^F = 0.
\end{aligned} \tag{16}
$$

As the nonnegativity constraints for both capital stocks are pure state constraints, it is possible that the adjoint variables as well as the Hamiltonian jump at junction times τ.[6] Thus for the junction times we additionally have[7]

$$\begin{aligned}
\lambda^H(\tau^-) &= \lambda^H(\tau^+) + \bar{\eta}^H(\tau) \\
\lambda^F(\tau^-) &= \lambda^F(\tau^+) + \bar{\eta}^F(\tau) \\
\mathcal{H}[\tau^-] &= \mathcal{H}[\tau^+] - \bar{\eta}^H(\tau)\frac{\partial K^H}{\partial t}(\tau) - \bar{\eta}^F(\tau)\frac{\partial K^F}{\partial t}(\tau)
\end{aligned} \tag{17}$$

with

$$\bar{\eta}^H \geq 0, \bar{\eta}^H K^H = 0 \qquad \text{and} \qquad \bar{\eta}^F \geq 0, \bar{\eta}^F K^F = 0, \tag{18}$$

where $\bar{\eta}^H$ and $\bar{\eta}^F$ are jump values.

2.2 Inner Equilibrium

At the inner equilibrium, both capital stocks are positive. This implies that the corresponding multipliers are zero, i.e. $\eta^H = \eta^F = 0$. By manipulation of (5) and (15) and using the necessary first order conditions we obtain the following four dimensional dynamical system for state and adjoint variables

$$\begin{aligned}
\dot{K}^H(t) &= \frac{L^H}{F} - N^H\frac{1-\xi}{\lambda^H} - \frac{\xi}{\lambda^H}K^H\Big[\frac{N^H}{K^H + K^F\phi\big(\frac{\lambda^H}{\lambda^F}\big)^{\sigma-1}} + \frac{N^F\phi}{K^H\phi + K^F\big(\frac{\lambda^H}{\lambda^F}\big)^{\sigma-1}}\Big] - \delta K^H \\
\dot{K}^F(t) &= \frac{L^F}{F} - N^F\frac{1-\xi}{\lambda^F} - \frac{\xi}{\lambda^F}K^F\Big[\frac{N^F}{K^F + K^H\phi\big(\frac{\lambda^F}{\lambda^H}\big)^{\sigma-1}} + \frac{N^H\phi}{K^F\phi + K^H\big(\frac{\lambda^F}{\lambda^H}\big)^{\sigma-1}}\Big] - \delta K^F \\
\dot{\lambda}^H(t) &= (\rho+\delta)\lambda^H - \frac{\xi}{\sigma-1}\Big[\frac{N^H}{K^H + K^F\phi\big(\frac{\lambda^H}{\lambda^F}\big)^{\sigma-1}} + \frac{N^F\phi}{K^H\phi + K^F\big(\frac{\lambda^H}{\lambda^F}\big)^{\sigma-1}}\Big] \\
\dot{\lambda}^F(t) &= (\rho+\delta)\lambda^F - \frac{\xi}{\sigma-1}\Big[\frac{N^F}{K^F + K^H\phi\big(\frac{\lambda^F}{\lambda^H}\big)^{\sigma-1}} + \frac{N^H\phi}{K^F\phi + K^H\big(\frac{\lambda^F}{\lambda^H}\big)^{\sigma-1}}\Big],
\end{aligned} \tag{19}$$

where $\phi = \varphi^{1-\sigma}$ $(\sigma > 1)$, similarly to φ, is a measure of trade barriers. If $\phi = 1$ (corresponding to $\varphi = 1$) there are no trade barriers and if $\phi = 0$ (corresponding to $\varphi = +\infty$), trade barriers are prohibitive.

Using the necessary first order conditions (14), this can be transformed

[6] The time points when a state touches, enters or exits the constraint.

[7] τ^+ and τ^- denote the limit from the right and the left respectively.

to the more intuitive state-control (K^H, K^F, c_n, c_n^*)-space

$$
\begin{aligned}
\dot{K}^H &= \frac{L^H}{F} - c_n \frac{\alpha}{F}\Big[N^H + \frac{\xi K^H}{1-\xi}\Big(\frac{N^H}{K^H + K^F\phi(\frac{c_n^*}{c_n})^{\sigma-1}} + \frac{N^F\phi}{K^H\phi + K^F(\frac{c_n^*}{c_n})^{\sigma-1}}\Big)\Big] - \delta K^H \\
\dot{K}^F &= \frac{L^F}{F} - c_n^* \frac{\alpha}{F}\Big[N^F + \frac{\xi K^F}{1-\xi}\Big(\frac{N^F}{K^F + K^H\phi(\frac{c_n}{c_n^*})^{\sigma-1}} + \frac{N^H\phi}{K^F\phi + K^H(\frac{c_n}{c_n^*})^{\sigma-1}}\Big)\Big] - \delta K^F \\
\dot{c}_n &= \frac{\xi}{1-\xi}\frac{c_n^2}{\sigma-1}\frac{\alpha}{F}\Big[\frac{N^H}{K^H + K^F\phi(\frac{c_n^*}{c_n})^{\sigma-1}} + \frac{N^F\phi}{K^H\phi + K^F(\frac{c_n^*}{c_n})^{\sigma-1}}\Big] - (\rho+\delta)c_n \\
\dot{c}_n^* &= \frac{\xi}{1-\xi}\frac{c_n^{*2}}{\sigma-1}\frac{\alpha}{F}\Big[\frac{N^F}{K^F + K^H\phi(\frac{c_n}{c_n^*})^{\sigma-1}} + \frac{N^H\phi}{K^F\phi + K^H(\frac{c_n}{c_n^*})^{\sigma-1}}\Big] - (\rho+\delta)c_n^*. \qquad (20)
\end{aligned}
$$

Using the necessary first order conditions for the remaining controls, we obtain

$$
\begin{aligned}
c_m^H &= c_n \frac{\xi}{1-\xi}\frac{\alpha}{\beta}\Big(K^H + K^F\Big(\frac{1}{\varphi}\frac{c_n^*}{c_n}\Big)^{\sigma-1}\Big)^{-1} \\
c_m^{*H} &= \frac{c_n}{\varphi}\frac{\xi}{1-\xi}\frac{\alpha}{\beta}\Big(K^H + K^F\Big(\varphi\frac{c_n^*}{c_n}\Big)^{\sigma-1}\Big)^{-1} \\
c_m^F &= \frac{c_n^*}{\varphi}\frac{\xi}{1-\xi}\frac{\alpha}{\beta}\Big(K^F + K^H\Big(\varphi\frac{c_n}{c_n^*}\Big)^{\sigma-1}\Big)^{-1} \\
c_m^{*F} &= c_n^* \frac{\xi}{1-\xi}\frac{\alpha}{\beta}\Big(K^F + K^H\Big(\frac{1}{\varphi}\frac{c_n}{c_n^*}\Big)^{\sigma-1}\Big)^{-1}. \qquad (21)
\end{aligned}
$$

In the following two subsections we distinguish between two cases. Firstly, we assume that both regions are identical with respect to population and the available workforce. This allows to obtain an analytic solution of the equilibrium and its stability analysis. Secondly, we assume that the population or the workforce (or both) are different. In this case the equilibrium can only be numerically determined. In both cases we assume that the population N^k $(k = \{H, F\})$ as well as the available workforce L^k $(k = \{H, F\})$ are independent of time. Consequently, the presented model is autonomous.

2.2.1 Identical regions

If both regions are identical, i.e. $L^H = L^F$ and $N^H = N^F$, it is possible to find one equilibrium (which is the optimal steady state) analytically, where both regions behave in the same way and have equal capital stocks. The equilibrium values, which are denoted by an additional hat (e.g. $\hat{K}^H$ denotes the equilibrium capital of region H), are (below we always use the H superscript for the population and the available workforce, as they are equal in both regions)

$$\begin{aligned}
\hat{K}^H &= \hat{K}^F = \frac{\xi}{F}\frac{L^H}{\delta\xi + (\rho+\delta)(\sigma-1)} \\
\hat{c}_n &= \hat{c}_n^* = \frac{L^H}{N^H}\frac{\sigma-1}{\alpha}\frac{(1-\xi)(\rho+\delta)}{\delta\xi + (\delta+\rho)(\sigma-1)} \\
\hat{c}_m^H &= \hat{c}_m^{*F} = \frac{F}{\beta}\frac{\sigma-1}{N^H}\frac{1}{1+\phi}(\rho+\delta) \\
\hat{c}_m^F &= \hat{c}_m^{*H} = \frac{1}{\varphi}\frac{F}{\beta}\frac{\sigma-1}{N^H}\frac{\phi}{1+\phi}(\rho+\delta).
\end{aligned} \tag{22}$$

From the last two expressions it is obvious that $c_m^{*H} = \frac{c_m^H}{\varphi^\sigma}$ and $c_m^F = \frac{c_m^{*F}}{\varphi^\sigma}$. As $\varphi \geq 1$ and $\sigma > 1$[8], this implies that consumption in region k ($k \in \{H, F\}$) of the manufactured good produced in k is never smaller than consumption of that good produced in the other region. If trade barriers are prohibitive, consumption of the manufactured goods produced in the other region is zero. If there is free trade on the other hand, consumption of both manufactured goods is equal.

Using these values and (1)-(3) we are further able to calculate the labour shares devoted to the investment, manufacturing and agricultural sector explicitely, i.e.

$$\begin{aligned}
\hat{L}_I^H = \hat{L}_I^F &= L\frac{\delta\xi}{\delta\xi + (\delta+\rho)(\sigma-1)} \\
\hat{L}_m^H = \hat{L}_m^F &= L\frac{\xi(\rho+\delta)(\sigma-1)}{\delta\xi + (\delta+\rho)(\sigma-1)} \\
\hat{L}_n^H = \hat{L}_n^F &= L\frac{(1-\xi)(\rho+\delta)(\sigma-1)}{\delta\xi + (\delta+\rho)(\sigma-1)}.
\end{aligned} \tag{23}$$

Interestingly, the shares do not depend on φ or ϕ. A similar result also holds for the equilibrium capital. Therefore the central planner chooses the optimal capital stock and production, and then devides it optimally between the two regions. Trade barriers do not influence the amount of production, but the proportion of the home and foreign consumption only.

Furthermore it is possible to study the dependencies within the possible parameter ranges of the labor shares, which is summarized in Table 3. For the complete expressions we refer to appendix B.

In the following, we discuss the dependencies on the four parameters σ, ξ, ρ and δ seperately. Firstly, a higher constant elasticity of subtitution σ in

[8] A complete list of all possible parameter ranges is provided in appendix A.

	$\hat{L}_I$	$\hat{L}_m$	$\hat{L}_n$
$\frac{\partial}{\partial\sigma}$	< 0	> 0	> 0
$\frac{\partial}{\partial\xi}$	> 0	> 0	< 0
$\frac{\partial}{\partial\rho}$	< 0	> 0	> 0
$\frac{\partial}{\partial\delta}$	> 0	< 0	< 0

Table 1: Dependence of the labour shares on the parameters

the CES subutility function (accounting for manufactured products) implies higher production of manufactured goods, as c_m^{agg} increases. This is resonable, as $\frac{\sigma-1}{\sigma}$ and its reciprocal tends to 1 for increasing σ. As a result, less labour is devoted to the investment sector implying a lower equilibrium capital stock and a lower labour share for the investment sector. A bit surprising is that also the labour share for the agricultural sector and the corresponding consumption increases. However, this is induced by the decrease of $\hat{L}_I$ and to the fact that the remaining labor force is devided between the other two sectors. Secondly, an increase of the exponent of the Cobb-Douglas function ξ implies that the subutility of manufactured products is weighted more heavily compared to that of agricultural goods. Thus the manufacturing labour share is increasing, while that of the agricultural production is decreasing. Moreover a higher weight on the manufactured products also induces a higher importance of the product variety. Consequently, also the labour share devoted to investments is increased. Thirdly, if the time discount rate ρ increases the valuation of the future is decreasing. To put it differently the central planner is not interested that much in the amount of the product varieties, but more on the instantaneous utility. I.e. the consumption (of both types) increases implying a higher labour in these sectors. Thus the more farsighted the central planner is, the higher the product diversity will be. Finally, if the depreciation rate δ increases a higher labour share is necessary to sustain the level of production capital. Therefore the labour shares for agricultural and manufactured products decrease.

In order to study the stability of the equilibrium we have to evaluate the Jacobian of the dynamical system (20) evaluated at the equilibrium

$$J = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}, \tag{24}$$

where J_1, J_2, J_3 and J_4 are symmetric 2×2-matrices reading

$$
\begin{aligned}
J_1 &= \begin{pmatrix} -\delta-(\rho+\delta)A & (\rho+\delta)A \\ (\rho+\delta)A & -\delta-(\rho+\delta)A \end{pmatrix} \\
J_2 &= \frac{\alpha}{F}\frac{1}{1-\xi}N^H \begin{pmatrix} -1-\xi A & \xi A \\ \xi A & -1-\xi A \end{pmatrix} \\
J_3 &= \frac{1}{N^H}\frac{F}{\alpha}\frac{\xi-1}{\xi}\frac{(\sigma-1)(\rho+\delta)^2}{(1+\phi)^2} \begin{pmatrix} 1+\phi^2 & 2\phi \\ 2\phi & 1+\phi^2 \end{pmatrix} \\
J_4 &= (\rho+\delta)\begin{pmatrix} 1+A & -A \\ -A & 1+A \end{pmatrix}
\end{aligned} \tag{25}
$$

with $A = 2(\sigma-1)\phi\left(\frac{1}{1+\phi}\right)^2$. After solving the characteristic equation we obtain the following general expression for the eigenvalues

$$
\begin{aligned}
eig_1 &= \frac{1}{2\xi}\left[\xi\rho + \sqrt{\xi}\sqrt{4(\sigma-1)(\rho+\delta)^2 + 4\xi\delta(\rho+\delta) + \xi\rho^2}\right] \\
eig_2 &= \frac{1}{2\xi}\left[\xi\rho - \sqrt{\xi}\sqrt{4(\sigma-1)(\rho+\delta)^2 + 4\xi\delta(\rho+\delta) + \xi\rho^2}\right] \\
eig_3 &= \frac{1}{2\xi(1+\phi)}\Big[\xi\rho(1+\phi) + \sqrt{\xi}\Big[8\rho\delta(\sigma-1)(\phi-1)^2 + \\
&\quad +4(\sigma-1)(\delta^2+\rho^2)(1+\phi^2) + \xi(1+\phi^2)(\rho^2+4\delta(\rho+\delta)) + \\
&\quad +8\phi(\delta^2+\rho^2)(2\sigma^2\xi-\sigma+1) + 2\xi\phi(16\sigma^2\delta\rho+\rho^2- \\
&\quad -4(\rho+\delta)(\delta+2\rho\sigma))\Big]^{\frac{1}{2}}\Big] \\
eig_4 &= \frac{1}{2\xi(1+\phi)}\Big[\xi\rho(1+\phi) - \sqrt{\xi}\Big[8\rho\delta(\sigma-1)(\phi-1)^2 + \\
&\quad +4(\sigma-1)(\delta^2+\rho^2)(1+\phi^2) + \xi(1+\phi^2)(\rho^2+4\delta(\rho+\delta)) + \\
&\quad +8\phi(\delta^2+\rho^2)(2\sigma^2\xi-\sigma+1) + 2\xi\phi(16\sigma^2\delta\rho+\rho^2- \\
&\quad -4(\rho+\delta)(\delta+2\rho\sigma))\Big]^{\frac{1}{2}}\Big].
\end{aligned} \tag{26}
$$

As $\sigma \geq 1$ always holds, it can straightforwardly be shown that $eig_1 > 0 > eig_2$ (both real). If furthermore $\phi < \frac{1}{2}$ holds, we obtain analogously $eig_3 > 0 > eig_4$ (both real). For $\phi \geq \frac{1}{2}$, the relation cannot be shown analytically. Therefore we numerically evaluated the eigenvalues of the Jacobian for all parameter constellations[9] within the possible parameter ranges. The parameters are chosen quite differently in the literature. For the labor efficiency

[9]For each parameter we have chosen very small steps within the possible parameter interval.

units and the population (both are exogenous and constant over time) we chose 1 except in cases where we focus on the difference between the regions due to one of that parameters. The capital depreciation rate δ will be varied in the interval $[0, 0.2]$, for the representative case we chose $\delta = 0.05$ implying that capital depreciates in 20 years. The time preference rate ρ will be taken out of the interval $[0, 0.2]$ with $\rho = 0.015$ as representative case. For the unit input coefficient we chose $F = 2$ as in most economic geography papers. Great differences in the literature exist for the parameter σ, which varies from 2 in Baldwin [6] to more than 4. To cover all cases we consider the interval $\sigma \in [2, 8]$. The parameter ξ differs from 0.1 e.g. in Puga [5] to 0.8 in Martin and Ottaviano [10]. Thus we use $\xi \in [0.1, 0.9]$. The possible parameter ranges are once again summarized in appendix A. Applying them to the eigenvalues, it turned out that $eig_3 > 0 > eig_4$ holds for each of the tested parameter constellations even for the case $\phi \geq \frac{1}{2}$. Therefore the symmetric equilibrium seems to be saddle path stable in the interesting region.

In the case of two identical regions it was not possible to find an asymmetric inner equilibrium neither analytically nor numerically for any parameter constellations. As an example we evaluated the symmetric equilibrium for an explicit choice of parameter values. For the representative calculation we chose $L^H = L^F = 1$, $N^H = N^F = 1$, $\rho = 0.015$, $\delta = 0.05$, $\xi = 0.3$, $\sigma = 2$, $F = 2$, $\alpha = 1$, $\beta = 1$, $\varphi = 2$. The resulting equilibrium values are

- $\hat{K}^H = \hat{K}^F = 1.8750$
- $\hat{\lambda}^H = \hat{\lambda}^F = 2.4618$
- $\hat{c}_n = \hat{c}_n^* = 0.5687$
- $\hat{c}_m^H = \hat{c}_m^{*F} = 0.0867$
- $\hat{c}_m^{*H} = \hat{c}_m^F = 0.0217$
- $L_I^H = L_I^F = 0.1875$
- $L_m^H = L_m^F = 0.24375$
- $L_n^H = L_n^F = 0.56875$

with the following eigenvalues

$$eig_1 = 0.1394, \quad eig_2 = -0.1244, \quad eig_3 = 0.1311, \quad eig_4 = -0.1161. \quad (27)$$

Assuming the above parameter constellation, more than half of the available labour is used for the agricultural sector (where labour is the only input factor). Nearly one fifth is used for the investment sector in order to sustain the capital stock. Compared to that the labour share for the production sector is not so high (approximately one fourth), which points towards the importance of the product variety in the utility function.

2.2.2 Different regions

Now let us assume that the regions are different, i.e. $L^H \neq L^F$ and/or $N^H \neq N^F$. The reason for the differences is not modeled in our framework and is exogenous. For this case Baldwin [6] does not provide a solution, neither analytically nor numerically. For the central planner model an analytical solution is also not possible. Therefore we numerically calculated several scenarios (using different values for N^H, N^F, L^H and/or L^F) in order to obtain insight into the solutions. Matlab always found one economically reasonable equilibrium[10] (which is not symmetric), none of them on the boundary (this was done without considering the constraints) and several unreasonable ones.[11] Moreover, the economically reasonable equilibrium is always saddle path stable. Table 2 shows the values of some asymmetric equilibria with different values for L^H, L^F, N^H and N^F together with the corresponding eigenvalues. For the other parameters we used $\rho = \delta = 0.1$, $\xi = 0.3$, $\sigma = 2$, $F = 2$, $\alpha = 1$, $\beta = 1$, $\varphi = 2$, $\phi = \frac{1}{2}$.

L^H	L^F	N^H	N^F	K^H	K^F	eig_1	eig_2	eig_3	eig_4
1	1	1.1	1	1.7860	1.9660	−0.1245	−0.1160	0.1395	0.1310
1	1	2	1	1.2803	2.5659	−0.1281	−0.1127	0.1432	0.1277
1	1.1	1	1	1.7676	2.1712	−0.1245	−0.1160	0.1395	0.1309
1	2	1	1	0.8313	4.8819	−0.1302	−0.1129	0.1452	0.1279
1.5	1	1.5	1	2.8768	1.8078	−0.1244	−0.1161	0.1394	0.1311
1.5	1	2	1	2.5691	2.1462	−0.1250	−0.1151	0.1400	0.1301
1.5	1	1	1.5	3.9062	0.9192	−0.1121	−0.1303	0.1453	0.1271
1.5	1	1	2	4.2946	0.6419	−0.1318	−0.1109	0.1468	0.1258

Table 2: Asymmetric equilibria (calculated with Matlab)

The asymmetric equilibria cannot be compared with the corresponding results of the decentralized model with mortality. In that case the analysis is more involved and still open.

[10] By the term “reasonable equilibrium” we mean that the consumption variables as well as the capital stocks of both regions are non-negative.

[11] By the term “unreasonable equilibrium” we mean that at least one variable (consumption or capital) is negative.

However, we can observe an important implication from Table 2: A higher labor efficiency leads to a higher equilibrium capital stock. This can be seen by considering the first row of the table. The home region has labor efficiency units $L^H = 1$ and population $N^H = 1$. The foreign region has the same available labor efficiency units $L^F = 1$, but with a smaller population, which means that workers are more efficient.[12]

The intuition for this finding is the following: If the labor efficiency is higher in one region, less people can produce the same output as compared to the other region. Thus output per worker is higher. The concave shape of the utility function then implies that only a part of the "extra output" is consumed and the rest is saved as capital. Note that not only the region with higher labor efficiency benefits but also inhabitants of the other region can consume more manufactured goods. However, the extra benefit is not that large because of the transport costs that have to be incurred.

2.3 Boundary Equilibrium

In Baldwin [6] critical values for the trade cost function ϕ are derived for several scenarios (identical and different regions). Beyond these values so called catastrophic agglomeration occurs. Nevertheless, no analytical expressions are provided. However, it would be important to have a look on boundary equilibria as well. In the following, we therefore concentrate on the central planner case and derive an equilibrium that is a boundary solution. We restrict the analysis to the $K^H = 0$-case as the $K^F = 0$ case is similar.

$K^H = 0$ implies $K^F > 0$ due to the logarithmic utility function and thus $\eta^F = 0$. Therefore consumption of the foreign produced good is positive in both regions, i.e. $c_m^F > 0$ and $c_m^{*F} > 0$. From the necessary first order condition for c_m^F and c_m^{*F} we then obtain

$$\begin{aligned} \hat{c}_m^F &= \frac{1}{\varphi}\frac{F}{\beta}\frac{\xi}{K^F}\frac{1}{\lambda_{K^F}} \\ \hat{c}_m^{*F} &= \frac{F}{\beta}\frac{\xi}{K^F}\frac{1}{\lambda_{K^F}} = \varphi c_m^F. \end{aligned} \tag{28}$$

Consumption of goods produced at home is not unique and it can only be stated that $c_m^H \geq 0$ and $c_m^{*H} \geq 0$.[13] However these values do not influence any

[12] Note that the higher labor efficiency is not due to a more developed production technology. This would be covered in the parameter F which is assumed to be equal for both regions.

[13] As in the necessary first order conditions and the dynamics of the states and the adjoint equations the consumption and the corresponding capital stock always appears together, the consumption values on the boundary $K^H = 0$ cannot be calculated uniquely.

dynamics and can be chosen arbitrarily. Using the equilibrium conditions $\dot{K}^H = 0$ and $\dot{\lambda}^H = 0$, we obtain explicit expressions for consumption of agricultural goods and for the shadow price of the corresponding capital

$$\begin{aligned}\hat{c}_n &= \frac{1}{\alpha}\frac{L^H}{N^H} \\ \hat{\lambda}^H &= F(1-\xi)\frac{N^H}{L^H}.\end{aligned} \tag{29}$$

If we use the equilibrium conditions $\dot{K}^F = 0$ and $\dot{\lambda}^F = 0$ on the other hand, we obtain explicit expressions for the foreign capital stock and the corresponding shadow price

$$\begin{aligned}\hat{K}^F &= \frac{L^F\xi}{F}\frac{N^H + N^F}{(\rho(\sigma-1)+\delta\sigma)(N^F+\xi N^H) - \delta(1-\xi)N^F} \\ \hat{\lambda}^F &= \frac{F}{L^F(\rho+\delta)(\sigma-1)}\Big[(\rho(\sigma-1)+\delta\sigma)(N^F+\xi N^H) - \delta(1-\xi)N^F\Big]\end{aligned} \tag{30}$$

and

$$\begin{aligned}\hat{c}_n^* &= \frac{1-\xi}{\alpha}\frac{L^F(\rho+\delta)(\sigma-1)}{(\rho(\sigma-1)+\delta\sigma)(N^F+\xi N^H) - \delta(1-\xi)N^F} \\ \hat{c}_m^F &= \frac{1}{\varphi}\frac{F}{\beta}\frac{(\rho+\delta)(\sigma-1)}{N^H+N^F} \\ \hat{c}_m^{*F} &= \varphi\hat{c}_m^F = \frac{F}{\beta}\frac{(\rho+\delta)(\sigma-1)}{N^H+N^F}.\end{aligned} \tag{31}$$

Similarly to the symmetric equlibrium in the case of identical regions we are able to derive the labour shares for thr three different sectors. The expression reads

$$L_I^H = 0, \qquad L_m^H = 0, \qquad L_n^H = 1 \tag{32}$$

for the home region and

$$\begin{aligned}L_I^F &= L^F(N^H+N^F)\frac{\delta\xi}{(\rho(\sigma-1)+\delta\sigma)(N^F+\xi N^H) - \delta(1-\xi)N^F} \\ L_m^F &= L^F(N^H+N^F)\frac{\xi(\sigma-1)(\delta+\sigma)}{(\rho(\sigma-1)+\delta\sigma)(N^F+\xi N^H) - \delta(1-\xi)N^F} \\ L_n^F &= L^F N^F\frac{(1-\xi)(\sigma-1)(\rho+\delta)}{(\rho(\sigma-1)+\delta\sigma)(N^F+\xi N^H) - \delta(1-\xi)N^F}\end{aligned} \tag{33}$$

for the forgein one. For the home region the result is trivial as no capital implies that nothing can be manufactured and that nothing has to be invested.

For the foreign region it is remarkable that the labour shares as well as the capital stock $\hat{K}^F$ do not depend on the trade barriers. Thus the trade barriers are only responsible for the distribution of the manufactured products not for the amount of manufactured goods. The higher the trade barriers are the lower is the consumption of manufactured goods of the home region and the higher is that of the foreign one (which produces the manufactured goods). Moreover the labour shares of the foreign region now depend on the populations of both regions N^H and N^F. Thus the relative size of the foreign (which is the only region which is manufacturing and investing in capital) is a key point of the labour shares. This dependence and the dependence on other exogenous parameters are summarized in the following table (the full expressions can be looked up in Appendix C).

The labour shares trivially do not depend on the parameters. For the labour shares of the foreign region we discuss the dependencies step by step. Firstly an increase of the population of the home region N^H increases the need for manufactured goods. As a result the labour share of the manufactured products and also that of the investment sector increases. However, an increase of N^H implies that N^F decreases in relation to N^H, thus the labour share for the agricultural sector decreases, although it is in general independent of the home population. Secondly, if the foreign population N^F changes the signs, the reasons are the other way around: the labour share for the investment and the manufacturing sector decreases, while that of the agricultural sector increases. The signs and the corresponding interpretation of the dependencies of the labor shares in equilibrium on σ, ξ, ρ and δ are analogously to the symmetric equilibrium for identical regions.

	$\hat{L}_I^H$	$\hat{L}_m^H$	$\hat{L}_n^H$	$\hat{L}_I^F$	$\hat{L}_m^F$	$\hat{L}_n^F$
$\frac{\partial}{\partial N^H}$	$= 0$	$= 0$	$= 0$	> 0	> 0	< 0
$\frac{\partial}{\partial N^F}$	$= 0$	$= 0$	$= 0$	< 0	< 0	> 0
$\frac{\partial}{\partial \sigma}$	$= 0$	$= 0$	$= 0$	< 0	> 0	> 0
$\frac{\partial}{\partial \xi}$	$= 0$	$= 0$	$= 0$	> 0	> 0	< 0
$\frac{\partial}{\partial \rho}$	$= 0$	$= 0$	$= 0$	< 0	> 0	> 0
$\frac{\partial}{\partial \delta}$	$= 0$	$= 0$	$= 0$	> 0	< 0	< 0

Table 3: Dependence of the labour shares on the parameters

To study the stability of the system on the boundary we have to consider the (K^F, λ^F)-dynamics[14]. The Jacobian J^B of this system equals

$$
\begin{aligned}
J^B(1,1) &= -\delta \\
J^B(1,2) &= \frac{(N^F+\xi N^H)(L^F)^2(\rho+\delta)^2(\sigma-1)^2}{F^2\big((N^F+\xi N^H)(\rho(\sigma-1)+\sigma\delta)-\delta N^F(1-\xi)\big)^2} \\
J^B(2,1) &= \frac{F^2\big((N^F+\xi N^H)(\rho(\sigma-1)+\sigma\delta)-\delta N^F(1-\xi)\big)^2}{\xi(N^H+N^F)(L^F)^2(\sigma-1)} \\
J^B(2,2) &= \rho+\delta
\end{aligned}
\tag{34}
$$

and has the following eigenvalues

$$
\begin{aligned}
eig_1 &= \frac{\rho}{2}+\frac{1}{2\sqrt{\xi}(N^H+N^F)}\Big[(N^H)^2\xi\Big(4(\delta+\rho)^2\sigma-\rho(3\rho+4\delta)\Big)+ \\
&\quad N^HN^F\Big(4(\rho+\delta)^2(\sigma-1+\sigma\xi)+2\xi(2\delta^2-\rho^2)\Big)+ \\
&\quad (N^F)^2\Big(4(\sigma-1)(\delta+\rho)^2+\xi(2\delta+\rho)^2\Big)\Big]^{\frac{1}{2}} \\
eig_2 &= \frac{\rho}{2}-\frac{1}{2\sqrt{\xi}(N^H+N^F)}\Big[(N^H)^2\xi\Big(4(\delta+\rho)^2\sigma-\rho(3\rho+4\delta)\Big)+ \\
&\quad N^HN^F\Big(4(\rho+\delta)^2(\sigma-1+\sigma\xi)+2\xi(2\delta^2-\rho^2)\Big)+ \\
&\quad (N^F)^2\Big(4(\sigma-1)(\delta+\rho)^2+\xi(2\delta+\rho)^2\Big)\Big]^{\frac{1}{2}}.
\end{aligned}
\tag{35}
$$

It is easy to show that both eigenvalues are real and that $eig_1 > 0 > eig_2$ if $\sigma \geq 1$. Consequently, the boundary equilibrium is saddle path-stable along the boundary. If we assume the same parameter values as for the symmetric equilibrium[15], we obtain the following values:

$$
\begin{aligned}
\hat{c}_n &= 1 & \hat{c}_n^* &= \frac{7}{16} = 0.4375 \\
\hat{c}_m^H &\geq 0 & \hat{c}_m^{*H} &\geq 0 \\
\hat{c}_m^F = \frac{1}{10} &= 0.1 & \hat{c}_m^{*F} &= 0.2 \\
\hat{K}^H &= 0 & \hat{K}^F &= \frac{15}{16} = 0.9375 \\
\hat{\lambda}^H = \frac{7}{5} &= 1.4 & \hat{\lambda}^F &= \frac{16}{5} = 3.2
\end{aligned}
\tag{36}
$$

[14]Note that $\dot{\lambda}^H = 0$ if $\dot{K}^H = 0$, which is fulfilled.

[15]$L^H = L^F = 1$, $N^H = N^F = 1$, $\rho = 0.015$, $\delta = 0.05$, $\xi = 0.3$, $\sigma = 2$, $F = 2$, $\alpha = 1$, $\beta = 1$, $\varphi = 2$

with the following associated Jacobian

$$J^B = \begin{pmatrix} -\frac{1}{10} & \frac{65}{512} \\ \frac{256}{375} & \frac{1}{5} \end{pmatrix}. \tag{37}$$

The eigenvalues are: $eig_1 = 0.3804$, $eig_2 = -0.2804$.

In Table 4 we evaluate the capital stock of the foreign region at the boundary equilibrium for several parameter constellations.

δ	ρ	L^F	N^H	N^F	F	σ	ξ	K^F
0.05	0.015	1	1	1	2	2	0.3	2.6201
0.05	0.015	1	1	1	2	4	0.3	1.0582
0.05	0.015	1	1	1	2	6	0.3	0.663
0.05	0.015	1	1	1	2	8	0.3	0.4827
0.05	0.01	1	1	1	2	2	0.3	2.7778
0.05	0.02	1	1	1	2	2	0.3	2.4793
0.1	0.015	1	1	1	2	2	0.3	1.4320
0.15	0.015	1	1	1	2	2	0.3	0.9852
0.05	0.015	1	1	1	2	2	0.1	1.227
0.05	0.015	1	1	1	2	2	0.5	3.3898
0.05	0.015	1	1	1	2	2	0.7	3.8781
0.05	0.015	1	1	1	2	2	0.9	4.2155
0.05	0.015	1.5	1	1	2	2	0.3	3.9301
0.05	0.015	2	1	1	2	2	0.3	5.2402
0.05	0.015	2.5	1	1	2	2	0.3	6.5502
0.05	0.015	3	1	1	2	2	0.3	7.8603
0.05	0.015	1	1.5	1	2	2	0.3	2.8463
0.05	0.015	1	2	1	2	2	0.3	3.0201
0.05	0.015	1	2.5	1	2	2	0.3	3.1579
0.05	0.015	1	3	1	2	2	0.3	3.2698
0.05	0.015	1	1	1.5	2	2	0.3	2.4272
0.05	0.015	1	1	2	2	2	0.3	2.3136
0.05	0.015	1	1	2.5	2	2	0.3	2.2388
0.05	0.015	1	1	3	2	2	0.3	2.1858

Table 4: capital of the foreign region in the boundary equilibrium

For all parameter constellations which have been used in the numerical analysis, the capital stock of the foreign region is greater in the boundary as compared to the inner equilibrium. If $\xi < 1$ and $\sigma > 1$ (which is fulfilled by definition), this can also be shown analytically for the case of identical regions. For the case of different regions, again no analytic expression is available.

However, the equilibrium values of the consumption controls can be compared for both equilibria analytically. Apart from the manufactured goods produced in region H (corresponding consumption can be chosen to equal zero, i.e. $c_m^H \geq 0$ and $c_m^{*H} \geq 0$), consumption of region F is smaller and consumption of region H is larger. Counterintuitive at the first glance this can be explained as follows. In region H no manufactured goods are produced. Consequently, the whole labor force can be used to produce a larger amount of agricultural goods. Furthermore, in region H consumption of manufactured goods consists entirely of goods produced in region F, which is therefore higher. Thus the manufactured goods of region F have to compensate the loss of consumption of own products. Region F on the other hand is responsible for the production of the manufactured goods of both countries alone. This induces firstly an increase in the equilibrium amount of capital and secondly a decrease in consumption of the manufactured goods in the own region. In general, these two effects go into the opposite direction (higher capital induces higher consumption and vice versa), but the effect of region H (which covers the whole consumption of manufactured goods of region F) is even stronger.

Finally, we can give an interpretation for the boundary equilibrium. In economic geography it is common to assume that the model starts in the symmetric equilibrium. Since that equilibrium is aways saddle path stable (at least for all possible parameter constellations), the economy will never collapse into catastrophic agglomeration like in the decentralized Baldwin case. The boundary equilibrium can only be reached when the economy starts there. It remains for further research to find out whether there exist other forces such that also the first-best solution collapses into the boundary equilibrium.

3 Comparison to the decentralized model

This subsection is devoted to the comparison with the model presented in Baldwin [6]. The decentralized model of that article consists also of two regions with immobile labor and capital. The consumption goods can be traded. Depending on the trade barriers the inner equilibrium can be unstable such that a small deviation from the optimal strategy leads to catastrophic agglomeration.

Using our notation, the equilibrium capital of the decentralized model $\hat{K}^d$

equals[16]

$$\hat{K}^d = \frac{\xi}{F}\frac{L^H}{\delta\sigma + \rho(\sigma - \xi)}. \tag{38}$$

It can be shown that $\hat{K}^H$ (equal to $\hat{K}^F$ in the symmetric equilibrium) is strictly greater than $\hat{K}^d$. The reason of the difference is due to two effects: anticipation and monopolistic competition.

Firstly, in contrast to individuals, the central planner anticipates the effect of the capital stock on the product variety. Individuals assume that their decision has no effect on the whole economy and on the decisions of the remaining population (i.e. Nash assumption). As the decentralized model consists of representative individuals, everyone behaves in the same way. However, the central planner considers aggregate assets of both regions and no prices. Thus consumption is maximized not only with respect to utility but also with respect to the intertemporal aggregate asset dynamics, which is equivalent to product diversity. Capital increases the consumption possibilities not only for the life of the currently living generation, but also of that in the future.[17]

Secondly, the decentralized model has the market structure of monopolistic competition. One characteristic of this structure is that prices are higher than socially optimal ones (see e.g. Sala-i-Martin [11]). To put it differently, prices equal the average cost instead of the marginal cost (occurring in perfect competition). Consequently, we have to deal with two effects: a higher price implies higher expenditures directly but it also lowers demand. Thus the sign of the total effect is ambiguous and depends on the slope of the demand curve. However, if the effect of monopolistic competition on the equilibrium capital stock is positive, it is always smaller than the anticipation effect. This is obvious from the fact that the equilibrium capital stock of the decentralized solution is always smaller than that of the central planner solution.

In Table 5 we evaluated the equilibrium capital stock for both models and several parameter constellations. The labor endowment of the countries $L^H = 1$ are equal due to the identical region scenario and $F = 2$ is used for

[16]For clarification once again: K^d denotes the equilibrium capital of the decentralized Baldwin model in the symmetric equilibrium, $\hat{K}^H$ and $\hat{K}^F$ denote the equilibrium capital of the centralized model of the home and the foreign country respectively.

[17]Note that this argument is not valid in Baldwin [6], as it assumes an arbitrary number (e.g. one) of individuals who live forever. For a decentralized model with finitely living individuals we refer to the more sophisticated model proposed in Grafeneder-Weissteiner and Prettner [7].

δ	ρ	σ	ξ	$\hat{K}^H$	$\hat{K}^d$
0.05	0.015	2	0.3	1.875	1.1952
0.075	0.015	2	0.3	1.3333	0.8547
0.1	0.015	2	0.3	1.0345	0.6652
0.125	0.015	2	0.3	0.8451	0.5445
0.15	0.015	2	0.3	0.7143	0.4608
0.05	0.01	2	0.3	2	1.2821
0.05	0.02	2	0.3	1.7647	1.1194
0.05	0.025	2	0.3	1.6667	1.0526
0.05	0.015	4	0.3	0.7143	0.5871
0.05	0.015	6	0.3	0.4412	0.3891
0.05	0.015	8	0.3	0.3191	0.291
0.05	0.015	2	0.1	0.7143	0.3891
0.05	0.015	2	0.5	2.7778	2.0408
0.05	0.015	2	0.7	3.5	2.9289

Table 5: Comparison of the equilibrium capital in the identical region case

all parameter constellations. For the parameters we have used the ranges all presented in appendix A.

Having discussed the equilibrium values of the capital stocks, we can compare the stability properties. In the decentralized model the symmetric inner equilibrium is unstable whenever trade costs ϕ are higher than a critical value ϕ^{cat} which equals

$$\phi^{\text{cat}} = \frac{1-b}{1+b} \qquad \text{with} \qquad b = \frac{\xi\rho}{\sigma(\rho+\delta)}. \tag{39}$$

As already discussed, the symmetric equilibrium of the central planner model is saddle path stable for every parameter constellation within the possible parameter ranges. Thus the central planner model is even more stable than the comparable decentralized model of Baldwin. In a decentralized model the optimization is made for each individual who does not consider externalities. In the central planner model externalities are internalized as the central planner takes into account the individual savings effect on the number of varieties and it takes into account the inefficiencies arising from monopolistic competition.

4 Conclusions and Extensions

In this paper we have calculated the socially optimal solution of a constructed capital model which is comparable to Baldwin [6]. In contrast to the decentralized outcome, the equilibrium never becomes unstable irrespective of trade barriers. Furthermore, the interior equilibrium is unique. In general, the economic geography literature is concerned with the event of agglomeration. However, the concentration of all mobile production factors in one region was shown to be socially not optimal in case of the constructed capital model.

As the force of mortality does not influence the socially optimal decisions, the results of this paper are also comparable to the extension of the Baldwin model presented in Grafeneder-Weissteiner and Prettner [7], which allows for a finite life-span of the individuals. But also in that paper catastrophic agglomeration occurs in certain parameter constellations.

Future research should derive incentives for individuals such that the socially optimal result can be obtained also in a decentralized model. This will be an interesting task but also quite involved because the model includes more than one externality.

Moreover, the model should be extended to a different meaning of the social planner. In the current approach it is assumed that the social planner derives the social optimum of both regions simultaneously. However, it is also possible that there exist two social planners, each of them considered with the social welfare of one region. This case is especially interesting in view of tax competition between two regions.

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A Parameter ranges for the numerical calculations

- $\xi \in [0.1, 0.9]$
- $\sigma \in [2, 8]$
- $\mu \in [0, 1]$
- $\delta \in [0, 0.2]$
- $\rho \in [0, 0.2]$
- $\phi \in [0, 1]$

B Dependence of the labour shares in the inner symmetric equilibrium on the parameters

Here the exogenous values $\hat{L}_n$, $\hat{L}_m$, $\hat{L}_I$ and $\hat{L}$ are valid for both regions, since we assume here identical regions.
Derrivative with respect to σ:

$$\begin{aligned}
\frac{\partial \hat{L}_n}{\partial \sigma} &= \hat{L}\frac{\delta\xi(1-\xi)(\rho+\delta)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2} \\
\frac{\partial \hat{L}_m}{\partial \sigma} &= \hat{L}\frac{\delta\xi^2(\rho+\delta)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2} \\
\frac{\partial \hat{L}_I}{\partial \sigma} &= -\hat{L}\frac{\delta\xi(\rho+\delta)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2}
\end{aligned} \tag{40}$$

Derrivative with respect to ξ:

$$\begin{aligned}
\frac{\partial \hat{L}_n}{\partial \xi} &= -\hat{L}\frac{(\rho+\delta)(\sigma-1)\big(\delta+(\rho+\delta)(\sigma-1)\big)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2} \\
\frac{\partial \hat{L}_m}{\partial \xi} &= \hat{L}\frac{(\rho+\delta)^2(\sigma-1)^2}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2} \\
\frac{\partial \hat{L}_I}{\partial \xi} &= \hat{L}\frac{\delta(\rho+\delta)(\sigma-1)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2}
\end{aligned} \tag{41}$$

Derrivative with respect to ρ:

$$\begin{aligned}
\frac{\partial \hat{L}_n}{\partial \rho} &= \hat{L}\frac{\delta\xi(\sigma-1)(1-\xi)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2} \\
\frac{\partial \hat{L}_m}{\partial \rho} &= \hat{L}\frac{\delta\xi^2(\sigma-1)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2} \\
\frac{\partial \hat{L}_I}{\partial \rho} &= -\hat{L}\frac{\delta\xi(\sigma-1)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2}
\end{aligned} \tag{42}$$

Derrivative with respect to δ:

$$\begin{aligned}
\frac{\partial \hat{L}_n}{\partial \delta} &= -\hat{L}\frac{\rho\xi(1-\xi)(\sigma-1)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2} \\
\frac{\partial \hat{L}_m}{\partial \delta} &= -\hat{L}\frac{\rho\xi^2(\sigma-1)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2} \\
\frac{\partial \hat{L}_I}{\partial \delta} &= \hat{L}\frac{\rho\xi(\sigma-1)}{\big(\delta\xi+(\delta+\rho)(\sigma-1)\big)^2}
\end{aligned} \tag{43}$$

C Dependence of the labour shares in the boundary equilibrium on the parameters

Derrivative with respect to N^H:

$$\begin{aligned}
\frac{\partial \hat{L}_n^F}{\partial N^H} &= -\hat{L}_n^F\frac{\xi(\rho(\sigma-1)+\delta\sigma)}{((\sigma-1)\rho+\delta\sigma)(N^F+\xi N^H)-\delta(1-\xi)N^F} \\
\frac{\partial \hat{L}_m^F}{\partial N^H} &= \frac{\hat{L}_m^F}{N^H+N^F}\frac{(1-\xi)(\rho+\delta)(\sigma-1)N^F}{((\sigma-1)\rho+\delta\sigma)(N^F+\xi N^H)-\delta(1-\xi)N^F} \\
\frac{\partial \hat{L}_I^F}{\partial N^H} &= \frac{\hat{L}_I^F}{N^H+N^F}\frac{(1-\xi)\big((\rho+\delta)(\sigma-1)N^F+((\sigma-1)\rho+\delta\sigma)N^H\big)}{((\sigma-1)\rho+\delta\sigma)(N^F+\xi N^H)-\delta(1-\xi)N^F}
\end{aligned} \tag{44}$$

Derrivative with respect to N^F:

$$\begin{aligned}
\frac{\partial \hat{L}_n^F}{\partial N^F} &= \frac{\hat{L}_n^F}{N^F}\frac{\xi(\rho(\sigma-1)+\delta\sigma)N^H}{((\sigma-1)\rho+\delta\sigma)(N^F+\xi N^H)-\delta(1-\xi)N^F} \\
\frac{\partial \hat{L}_m^F}{\partial N^F} &= -\frac{\hat{L}_m^F}{N^H+N^F}\frac{(1-\xi)(\rho+\delta)(\sigma-1)N^H}{((\sigma-1)\rho+\delta\sigma)(N^F+\xi N^H)-\delta(1-\xi)N^F} \\
\frac{\partial \hat{L}_I^F}{\partial N^F} &= -\frac{\hat{L}_I^F}{N^H+N^F}\frac{(1-\xi)(\rho+\delta)(\sigma-1)N^H}{((\sigma-1)\rho+\delta\sigma)(N^F+\xi N^H)-\delta(1-\xi)N^F}
\end{aligned} \tag{45}$$

Derrivative with respect to σ:

$$\frac{\partial \hat{L}_n^F}{\partial \sigma} = \frac{\hat{L}_n^F}{\sigma - 1} \frac{\delta\xi(N^H + N^F)}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F}$$
$$\frac{\partial \hat{L}_m^F}{\partial \sigma} = -\frac{\hat{L}_m^F}{\sigma - 1} \frac{\delta\xi(N^F + \xi N^H)}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F}$$
$$\frac{\partial \hat{L}_I^F}{\partial \sigma} = -\frac{\hat{L}_I^F}{\delta\xi} \frac{(\rho+\delta)(N^F + \xi N^H)}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F} \tag{46}$$

Derrivative with respect to ξ:

$$\frac{\partial \hat{L}_n^F}{\partial \xi} = \frac{\hat{L}_n^F}{1-\xi} \frac{\delta(1-\xi)N^F - (\rho(\sigma-1) + \delta\sigma)(N^H + N^F)}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F}$$
$$\frac{\partial \hat{L}_m^F}{\partial \xi} = \frac{\hat{L}_m^F}{\xi} \frac{(\rho+\delta)(\sigma-1)N^F}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F}$$
$$\frac{\partial \hat{L}_I^F}{\partial \xi} = \frac{\hat{L}_I^F}{\xi} \frac{(\rho+\delta)(\sigma-1)N^F}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F} \tag{47}$$

Derrivative with respect to ρ:

$$\frac{\partial \hat{L}_n^F}{\partial \rho} = \frac{\hat{L}_n^F}{\rho+\delta} \frac{\delta\xi(N^H + N^F)}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F}$$
$$\frac{\partial \hat{L}_m^F}{\partial \rho} = \frac{\hat{L}_m^F}{\rho+\delta} \frac{\delta\xi(N^H + N^F)}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F}$$
$$\frac{\partial \hat{L}_I^F}{\partial \rho} = -\frac{\hat{L}_I^F}{\delta\xi} \frac{N^F + \xi N^H}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F} \tag{48}$$

Derrivative with respect to δ:

$$\frac{\partial \hat{L}_n^F}{\partial \delta} = -\frac{\hat{L}_n^F}{\rho+\delta} \frac{\rho\xi(N^H + N^F)}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F}$$
$$\frac{\partial \hat{L}_m^F}{\partial \delta} = -\frac{\hat{L}_m^F}{\rho+\delta} \frac{\rho\xi(N^H + N^F)}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F}$$
$$\frac{\partial \hat{L}_I^F}{\partial \delta} = \frac{\hat{L}_I^F}{\delta} \frac{\rho(\sigma-1)(N^F + \xi N^H)}{((\sigma-1)\rho + \delta\sigma)(N^F + \xi N^H) - \delta(1-\xi)N^F} \tag{49}$$