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High-order approximations to nonholonomic affine control systems*

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Abstract. This paper contributes to the theory of approximations of continuous-time control/uncertain systems by discrete-time ones. Discrete approximations of higher than first order accuracy are known for affine control systems only in the case of commutative controlled vector fields. The novelty in this paper is that constructive second order discrete approximations are obtained in the case of two non-commutative vector fields. An explicit parameterization of the reachable set of the Brockett non-holonomic integrator is a key auxiliary tool. The approach is not limited to the present deterministic framework and may be extended to stochastic differential equations, where similar difficulties appear in the non-commutative case.

1 Introduction

In this paper we present a new result about approximation of an affine control system by appropriately constructed discrete-time systems. The novelty is that the approximation is of second order accuracy with respect to the discretization step, while the vector fields defining the dynamics are not assumed commutative. In the commutative case higher than first order approximations are presented e.g. in [13, 4, 9]. However, it is well known that the non-commutativity is a substantial problem for second order approximations of control/uncertain systems and of stochastic differential equations (cf. [8]), due to the appearance of mixed multiple integrals in the approximation by Volterra (Chen-Fliess) series. We present an approach for obtaining higher-order approximations that combines the truncated Chen-Fliess series with the constructive description of the reachable set of an auxiliary control system which depends on the bounds on the controls, but is independent of the particular vector fields in the original system. In the particular case of a two-inputs system, on which we focus in this paper,

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the auxiliary system is the well known Brockett non-holonomic integrator, for the reachable set of which we present an explicit parameterization making use of [12]. The approach is not limited to the present deterministic framework and may be extended to stochastic differential equations, where a similar difficulty appears.

Let us consider an affine control system

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x), \quad x(0) = x^0 \in \mathbf{R}^n, \quad u = (u_1, \ldots, u_m) \in U, \quad t \in [0, T], \quad (1)$$

where $T > 0$ is a fixed time horizon and U is a compact subset of $\mathbf{R}^m$. Admissible inputs are all measurable functions $u : [0, T] \mapsto U$, and the set of all such functions is denoted by $\mathcal{U} = \mathcal{U}(0, T)$. Under the assumptions formulated in the next section every $u \in \mathcal{U}$ defines an unique trajectory of (1) on $[0, T]$, denoted further by $x[u](t)$.

In parallel, for every natural number N (presumably large) we consider a discrete-time system

$$y^{k+1} = g_N(y^k, v^k), \quad y^0 = x^0, \quad v^k = (v_1^k, \ldots, v_p^k) \in V, \quad k = 0, \ldots, N-1, \quad (2)$$

where V is a compact subset of $\mathbf{R}^p$. An admissible control of this system is any element $v = (v^0, \ldots, v^{N-1}) \in (V)^N$, and the set of all such vectors is denoted by $\mathcal{V}_N$. For $v \in \mathcal{V}_N$ denote by $y[v] = (y^0[v], \ldots, y^N[v])$ the corresponding solution of (2).

System (2) will be viewed as an approximation of (1). To make this precise we denote $h = T/N$, $t_k = hk$, and link y^k with $x(t_k)$. In addition, in order to relate a discrete control $v \in \mathcal{V}_N$ with a continuous-time control $u \in \mathcal{U}$, we require that a mapping $E : V \mapsto \mathcal{U}(0, h)$ is defined associating to any $v \in V$ an admissible control $u = E(v)$ on the interval $[0, h]$. Then any $v \in \mathcal{V}_N$ defines an admissible control from $\mathcal{U}$ (denoted by $E_N(v)$) as $E_N(v)(s) = E(v^k)(s - t_k)$ for $s \in [t_k, t_{k+1})$.

Definition 1 *The sequence of systems (2) and mappings E_N provides an approximation of order r to (1) if there is a constant c such that for every sufficiently large N the following holds true:*
(i) for each $u \in \mathcal{U}$ there exists $v \in \mathcal{V}_N$ for which $\left| x[u](t_k) - y^k[v] \right| \le ch^r$, $k = 0, \ldots, N$;
(ii) for each $v \in \mathcal{V}_N$ it holds that $\left| x[E_N(v)](t_k) - y^k[v] \right| \le ch^r$, $k = 0, \ldots, N$.

The above definition requires that every trajectory of (1) is approximated with accuracy $O(h^r)$ by a trajectory of (2), and that from every control of (2) one can recover a control of (1) which provides the same order of approximation for the respective trajectories.

It is well known (see [3]) that the Euler discretization scheme provides even in a much more general formulation a first order approximation. In our case the respective specifications are $V = U$, $g_N(y, v) = y + h \sum_{i=1}^{m} v_i f_i(y)$, $E(v)(t) \equiv v$.

In Section 2 a system of the type of (2) that provides a second order approximation to (1) is implicitly defined. Thanks to the investigation of an auxiliary problem in Section 3, the second order approximation is made explicit in the case of two controls ($m = 2$) in Section 4.

2 A second order approximation: basic idea

Below we assume that the control u is two-dimensional, that the set $U \subset \mathbf{R}^2$ is compact and that the functions $f_i : X \mapsto \mathbf{R}^n$ are differentiable with Lipschitz continuous derivatives. Here $X \subset \mathbf{R}^n$ is a set that contains in its interior the point x^0 and all trajectories of (1) on $[0, T]$.

At a point $x \in \operatorname{int} X$ and for a "short" horizon $[0, h]$ one can approximate in a standard way the solution of (1) starting from x and corresponding to a measurable control u by the truncated Volterra (Chen-Fliess) series (see e.g. [7]):

$$\begin{aligned} x(h) = x &+ \sum_{i=1}^{m} f_i(x) \int_0^h u_i(t)\,\mathrm{d}t + \frac{1}{2} \sum_{i=1}^{m} f_i' f_i(x) \left(\int_0^h u_i(t)\,\mathrm{d}t \right)^2 \\ &+ \sum_{1 \le j < i \le m} f_i' f_j(x) \int_0^h u_i(t)\,\mathrm{d}t \int_0^h u_j(t)\,\mathrm{d}t \\ &+ \sum_{1 \le i < j \le m} [f_i, f_j](x) \int_0^h u_i(t) \int_0^t u_j(s)\,\mathrm{d}s\,\mathrm{d}t + O(h^3), \end{aligned}$$

with $[f_i, f_j] = f_i' f_j - f_j' f_i$, and $|O(h^3)| \le c h^3$, where c depends only on an upper bound of $|f_i|$, the Lipschitz constant of f_i' around x and $|U| = \max_{u \in U} |u|$. Changing the variables of integration we obtain the representation

$$\begin{aligned} x(h) = x &+ h \sum_{i=1}^{m} f_i(x) \int_0^1 u_i(ht)\,\mathrm{d}t + \frac{h^2}{2} \sum_{i=1}^{m} f_i' f_i(x) \left(\int_0^1 u_i(ht)\,\mathrm{d}t \right)^2 \\ &+ h^2 \sum_{1 \le j < i \le m} f_i' f_j(x) \int_0^1 u_i(ht)\,\mathrm{d}t \int_0^1 u_j(ht)\,\mathrm{d}t \\ &+ h^2 \sum_{1 \le i < j \le m} [f_i, f_j](x) \int_0^1 u_i(ht) \int_0^t u_j(hs)\,\mathrm{d}s\,\mathrm{d}t + O(h^3). \end{aligned} \tag{3}$$

This representation is not symmetric, but can easily be put into a symmetric form by summing the versions of the above representation corresponding to all permutations of the control variables. In the case $m = 2$ considered in this paper the symmetric representation gives the alternative formula (to obtain this representation one can use the explicit formulae obtained in [1], [6] and [10] for product expansion of the Chen-Fliess series)

$$x(h) = g_N(x, v) + O(h^3), \tag{4}$$

where

$$g_N(y,v) = y + h(f_1(y)v_1 + f_2(y)v_2) + \frac{h^2}{2} \sum_{i,j=1,2} f_i' f_j(y) v_i v_j + h^2 [f_1, f_2](y) v_3$$

and

$$\begin{aligned} v_1 &= \int_0^1 u_1(ht)\,\mathrm{d}t, \quad v_2 = \int_0^1 u_2(ht)\,\mathrm{d}t, \\ v_3 &= \int_0^1 u_1(ht) \int_0^t u_2(hs)\,\mathrm{d}s\,\mathrm{d}t - \int_0^1 u_2(ht) \int_0^t u_1(hs)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

The above relations define a mapping $L_1([0,h] \mapsto U) \ni u \longrightarrow F(u) = v$ that maps the admissible controls on $[0,h]$ onto a set $V_3 \subset \mathbf{R}^3$. The set V_3 is exactly the reachable set at time $t = 1$ of the auxiliary control system

$$\begin{aligned} \dot{v}_1 &= u_1, & v_1(0) &= 0, \\ \dot{v}_2 &= u_2, & v_2(0) &= 0, \qquad (u_1, u_2) \in U. \\ \dot{v}_3 &= u_1 v_2 - u_2 v_1, & v_3(0) &= 0. \end{aligned} \tag{5}$$

Let E be any selection mapping of the inverse F^{-1}, that is

$$V \ni v \longrightarrow E(v) \in F^{-1}(v),$$

(thus E "recovers" a control function $u = E(v)$ from a vector $v \in V_3$).

By a standard propagation/accumulation error analysis one can obtain the following result.

Theorem 1 *The discrete-time system (2) with the specifications (2),* V_3*,* E*, provides a second order approximation to the control system (1) for the case* $m = 2$*, in the sense of Definition 1.*

If the original control system (1) contains a drift term f, then one can obtain a respective $O(h^3)$ local approximation applying (3) with the substitution $U := 1 \times U$. In this case the mapping g_N and the discrete control set $V = V_5$ are defined as

$$\begin{aligned} g_N(y,v) = y &+ hf(y) + \frac{h^2}{2} f' f(y) + h\left(f_1' f(y) v_1 + f_2' f(y) v_2\right) \\ &+ h\left([f, f_1](y) v_3 + [f, f_2](y) v_4\right) \\ &+ h(f_1(y)v_1 + f_2(y)v_2) + \frac{h^2}{2} \sum_{i,j=1,2} f_i' f_j(y) v_i v_j + h^2 [f_1, f_2](y) v_5, \end{aligned} \tag{6}$$

where $(v_1, \ldots, v_5) \in V_5$, and V_5 is the reachable set of the 5-dimensional control system

$$
\begin{array}{lll}
\dot{v}_1 = \quad\quad u_1, & v_1(0) = 0, & \\
\dot{v}_2 = \quad\quad u_2, & v_2(0) = 0, & \\
\dot{v}_3 = \quad\quad v_1, & v_2(0) = 0, & (u_1, u_2) \in U. \\
\dot{v}_4 = \quad\quad v_2, & v_4(0) = 0, & \\
\dot{v}_5 = u_1 v_2 - u_2 v_1, & v_5(0) = 0. &
\end{array}
$$

A similar theorem holds true with the specifications of g_N in (6), V_5, and E, where $u = E(v)$ is a control with values in U that reaches the point $v \in V_5$.

We stress that the sets V_3 and V_5 are independent of the vector fields f_i, hence they can be "pre-calculated" by a representative set of points for typical control constraining sets U, such as boxes, balls, or ellipsoids. Such a "pre-calculation" is possible for V_3 but is hard for V_5. Moreover, the utilization of the approximation from Theorem 1 with a set V represented by a huge number of points is limited to simulation purposes. An explicit parameterization of the discrete control set V, that may make it efficiently usable also in the optimal control context, is discussed in the next sections.

3 An auxiliary problem

In this section we investigate the reachable set V_3 of the auxiliary control system (5) in the most typical case of $U = [-1, 1] \times [-1, 1]$. This is a particular be-linear system well known in the literature as "Brocket non-holonomic integrator" [2]. The pair of vector fields $B_1(v) = (1, 0, v_2)$ and $B_2(v) = (0, 1, -v_1)$ that defines the system is nilpotent of order two and generates an Lie algebra of rank three at the origin. Thus, the system is small-time locally controllable at the origin. Hence, V_3 contains the origin in its interior (cf. for example [5] or [11]).

The description of the reachable set V_3 is complicated by the fact that it is non-convex. More precisely, the following lemma holds for the reachable set of (5) on $[0, \theta]$, denoted further by $V_3(\theta)$.

Lemma 1. *The reachable set $V_3(\theta)$ is not-convex for each $\theta > 0$.*

The next lemma is obvious.

Lemma 2. *The projection of $V_3(\theta)$ on the (v_1, v_2)-plane is the square $V_2(\theta)$ centered at the origin, with sides parallel to the axes and of length 2θ.*

The following result plays the key role in the characterization of V_3.

Lemma 3. *If a point (v_1, v_2, v_3) belongs to $V_3(\theta)$, then also $(v_1, v_2, \alpha v_3)$ belongs to $V_3(\theta)$ for each $\alpha \in [-1, 1]$.*

This lemma is proved, essentially, in [12], although our proof (to be presented elsewhere) is technically different. It employs the chronological calculus (see [1]) and is applicable for multi-input systems (with $m > 2$) and for systems with a drift term (see the end of the previous section), in contrast to the geometric approach in [12].

The last two lemmas together imply that the set $V_3(\theta)$ can be recovered from its upper boundary, that is, from the surface $\Gamma(\theta) = \{v \in R^3 : (v_1, v_2) \in V_2(\theta),\ v_3 = \max\{w : (v_1, v_2, w) \in V_3(\theta)\}\}$. Notice that according to Lemma 1 this surface is not a graph of a concave function.

Next, we shall define a parameterized family of controls that generate the upper boundary $\Gamma(\theta)$ of the reachable set $V_3(\theta)$. In doing that we rely on the results in [12].

For any $\theta > 0$ define the sets

$$P^1(\theta) = \{(\mu, \tau) : \frac{\theta}{3} \le \tau \le \theta,\ \ 0 \le \mu \le \tau,\ \ \theta - 2\tau \le \mu \le \theta - \tau\},$$

$$P^2(\theta) = \{(\mu, \tau) : \frac{\theta}{4} \le \tau \le \frac{\theta}{2},\ \ 0 \le \mu \le \tau,\ \ \theta - 3\tau \le \mu \le \theta - 2\tau\}.$$

For each parameter $p = (\mu, \tau) \in P^1(\theta)$ we define the control $u^{1,p}$ on $[0, \theta]$ as follows

$$u^{1,p}(t) = \begin{cases} (-1, 1) & \text{for } t \in [0, \mu) \\ (1, 1) & \text{for } t \in [\mu, \mu + \tau) \\ (1, -1) & \text{for } t \in [\mu + \tau, \theta]. \end{cases}$$

Moreover, for each $p = (\mu, \tau) \in P^2(\theta))$ we define $u^{2,p}$ as

$$u^{2,p}(t) = \begin{cases} (-1, 1) & \text{for } t \in [0, \mu) \\ (1, 1) & \text{for } t \in [\mu, \mu + \tau) \\ (1, -1) & \text{for } t \in [\mu + \tau, \mu + 2\tau) \\ (-1, -1) & \text{for } t \in [\mu + 2\tau, \theta]. \end{cases}$$

Let us define the following symmetries in $\mathbf{R}^3$: for $v = (v_1, v_2, v_3)$

$$S_0(v) = (v_2, v_1, v_3), \quad S_1(v) = (-v_1, v_2, v_3),$$
$$S_2(v) = (v_1, -v_2, v_3), \quad S_{12}(v) = (-v_1, -v_2, v_3), \quad S_{012}(v) = (-v_2, -v_1, v_3).$$

Proposition 1. *For $\theta > 0$ define the sets*

$$\Gamma^i(\theta) = \{v[u](\theta) : u \in P^i(\theta)\}, \qquad i = 1, 2,$$

where $v[u](\cdot)$ is the trajectory of (5) corresponding to the control $u : [0, \theta] \mapsto U$. Then the following representation holds true:

$$\Gamma(\theta) = \left[\Gamma^1 \cup S_0(\Gamma^1) \cup S_{12}(\Gamma^1) \cup S_{012}(\Gamma^1)\right] \cup \left[\Gamma^2 \cup S_1(\Gamma^2) \cup S_2(\Gamma^2) \cup S_{12}(\Gamma^2)\right],$$

where the argument θ is suppressed in the right-hand side.

The essence of the above proposition is two-fold. First, it implies that every point of the upper boundary of the reachable set V_3 (in fact also of the whole boundary) can be reached by a bang-bang control with at most three switches. However, this information is not enough to obtain an one-to-one parameterization, since the three would give a redundant parameterization of the boundary, which is two-dimensional. The proposition claims more, namely that in the case of three switches the distance between the first and the second switch equals the distance between the second and the third switch (see formula for $u^{2,p}(t)$ above). Hence the needed switching points are always determined by the two parameters μ and τ. Our proof of this fact (in contrast to the geometric proof in [12]) makes use of the chronological calculus [1] and is applicable to higher-order auxiliary systems, in particular for the 5-dimensional system in the end of Section 2.

4 A constructive second order scheme for two non-commutative fields

Thanks to the results of the previous section we can formulate explicitly a discrete-time system of the type of (2) in the case $m = 2$ and for $U = [-1, 1] \times [-1, 1]$. Namely,

$$y^{k+1} = g_N(y^k, v^k), \quad y^0 = x^0, \quad v^k = (v_1^k, v_2^k, v_3^k) \in V_3 \subset \mathbf{R}^3,$$

$k = 0, \ldots, N-1$ where

$$g_N(y, v) = y + h(f_1(y)v_1 + f_2(y)v_2) + \frac{h^2}{2} \sum_{i,j=1,2} f_i' f_j(y) v_i v_j + h[f_1, f_2] v_3,$$

and

$$\begin{aligned} V_3 = & \left[V^1 \cup S_0(V^1) \cup S_{12}(V^1) \cup S_{012}(V^1)\right] \\ & \cup \left[V^2 \cup S_1(V^2) \cup S_2(V^2) \cup (S_{12})(V^2)\right], \end{aligned}$$

$$\begin{aligned} V_3^1 = \{v = (v_1, v_2, v_3) : \; & v_1 = 2\mu - 1, \; v_2 = 2(\mu + \tau) - 1, \; v_3 = 2\alpha(1 - \tau)\tau, \\ & (\mu, \tau) \in P^1(1), \; \alpha \in [-1, 1]\}, \end{aligned}$$

$$\begin{aligned} V_3^2 = \{v : \; & v_1 = 1 - 4\tau, \; v_2 = 2(\mu + \tau) - 1, \\ & v_3 = 2\alpha[(1 - \mu - 2\tau)(\tau - \mu) + \tau(\mu + \tau)], (\mu, \tau) \in P^2(1), \; \alpha \in [-1, 1]\}. \end{aligned}$$

Notice that the discrete system is not affine although the original one is such. Moreover, the constraint V_3 for the discrete-time control v^k is non-convex. All this makes somewhat questionable whether the obtained second-order approximation would be useful in an optimal control context. This issue deserves additional investigation and perhaps development of specific optimization methods.

The case where the control constraint U is the unit ball in $\mathbf{R}^2$ requires also additional investigation, since to our knowledge the reachable set of the auxiliary system (5) has not been constructively parameterized.

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