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Optimal Control of Heterogeneous Systems with Endogenous Domain of Heterogeneity*

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Abstract

The paper deals with optimal control of heterogeneous systems, that is, families of controlled ODEs parameterized by a parameter running over a domain called *domain of heterogeneity*. The main novelty in the paper is that the domain of heterogeneity is endogenous: it may depend on the control and on the state of the system. This extension is crucial for several economic applications and turns out to rise interesting mathematical problems. A necessary optimality condition is derived, where one of the adjoint variables satisfies a differential inclusion (instead of equation) and the maximization of the Hamiltonian takes the form of "min-max". As a consequence, a Pontryagin-type maximum principle is obtained under certain regularity conditions for the optimal control. A formula for the derivative of the objective function with respect to the control from L_∞ is presented together with a sufficient condition for its existence. A stylized economic example is investigated analytically and numerically.

Keywords: optimal control, distributed control, heterogeneous systems, endogenous domain of heterogeneity, Pontryagin-type maximum principle, set-valued analysis

AMS Subject Classification: 49K20, 49M10, 90A16

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1 Introduction

Generally speaking, in our terminology *heterogeneous control systems* are represented by a family of controlled ODEs parameterized by a parameter q varying in a measurable space $\mathcal{Q}$, called *domain of heterogeneity*. The family is coupled by "aggregated states" (involving integration over $\mathcal{Q}$) that enter in all ODEs and in the respective initial conditions. For each $q \in \mathcal{Q}$ the respective ODE has its own time-interval $[\theta(q), \bar{\theta}(q)]$ in which it matters, at the beginning of which an initial condition is posed. A basic theory of optimal control problems for such heterogeneous systems is presented in [11]. Applications in many areas are indicated in this paper, including problems in population dynamics, epidemiology, air pollution control, and also several problems in economics. In [11] the initial time $\theta(q)$ for the q-th ODE is exogenously given.

In order to explain what are the motivations for the present paper, let us consider a particular economic context originating from [10, 6] in which the parameter q is interpreted as an available technology (or consumption good) and $\theta(q)$ is the time at which the q-th technology emerges. Then, in the spirit of the endogenous economic growth literature ([9, 7, 8]), $\theta(q)$ depends on the rate of technological advancement which depends, in its turn, on the investments in R&D. That is, the time, $\theta(q)$, of appearance of the q-th technology is endogenous. Such a situation, where the domain of heterogeneity at each time is endogenously determined, is not covered by the results in [11] and by the existing literature.[1]

In the present paper we address the issue of endogenous domain of heterogeneity. The class of problems that we consider is relatively simple and does not contain many of the economic applications. Such will be presented in follow-up publications in economics-oriented journals. This is because here we want to stress the mathematical challenges that the endogenous domain of heterogeneity brings in the optimal control context. The main trouble is caused by the fact that the objective value considered as a function of the control is, in general, non-differentiable (in any reasonable space setting). This effect does not arise in standard optimal control problems with smooth data if the set of admissible controls is a subset of L_∞. As a result of this intrinsic non-differentiability, the necessary optimality condition of Pontryagin's type takes a non-standard form in which the adjoint systems is represented by a differential inclusion (rather than equation) although the data are assumed smooth with respect to the state variables. However, this situation may happen only if the optimal control is "irregular" with respect the parameter of heterogeneity q, which is hard to exclude a priori.

The "irregularity" of the optimal control that requires the abovementioned non-standard form of the maximum principle does not happen in the economic application we have in mind, thus it is perhaps mainly of "academic interest". If the optimal control is "regular" enough, the optimality conditions take a form corresponding to the heuristic application of the Lagrange principle. However, the trouble with non-differentiability of the objective function still remains. It creates certain difficulties

[1] In the endogenous growth literature involving a variety of technologies/products it is assumed that the products are indistinguishable, therefore the amount of physical capital allocated to production of each of them is equal ([3]). This reduces the originally distributed optimal control problem to a lumped one.

in the derivation of the optimality condition in the form of global maximum principle, therefore we present it below in detail. The possible non-differentiability also requires a special care about the numerical approaches to the problem based on gradient-type methods.

The paper is organized as follows. In Section 2 we formulate the problem and the assumptions. In Section 3 we derive a necessary optimality condition without any *a priori* assumptions for the optimal control. In this optimality condition the adjoint variable satisfies a differential inclusion and the maximization of the Hamiltonian takes the form of "min-max". From here, under certain regularity of the optimal control we derive also a Pontryagin-type maximum principle. Section 4 presents a formula for the derivative of the objective function with respect to the control from L_∞ and a sufficient condition for its existence. Moreover, a version of the gradient projection method in the control space is briefly described, which at each iteration involves only controls with respect to which the objective function is differentiable. In Section 5 we give two examples of non-differentiability which justify the special treatment in the preceding two sections. Section 6 gives a stylized economic example. In a simple case we obtain the solution analytically and show that the existence issue is complicated: an optimal solution exists for some configurations of the parameters and fails to exist for others. Also, numerical results are presented and interpreted. One longer proof is shifted to Appendix.

2 Formulation of the problem

Let $[0,T]$ be a fixed time-interval and let $[0,\bar{Q}]$ be an interval where the parameter of heterogeneity, q, will take values (here $\bar{Q} > 0$ could be $+\infty$, in which case the interval should be interpreted as $[0,\infty)$). Denote $D = [0,T] \times [0,\bar{Q}]$. The state variables in the model below will be the functions

$$x : D \mapsto \mathbf{R}^n, \quad Q : [0,T] \mapsto [0,\bar{Q}], \quad y : [0,T] \mapsto \mathbf{R}^m,$$

while $u : D \mapsto U \subset \mathbf{R}^r$ will be a control function. The optimal control problem we consider reads as follows:

$$\max_u \int_0^T \int_0^{Q(t)} L(t,q,x(t,q),Q(t),y(t),u(t,q))\,\mathrm{d}q\,\mathrm{d}t, \tag{1}$$

subject to the equations

$$\dot{Q}(t) = g(t,Q(t),y(t)), \quad Q(0) = Q^0 \geq 0, \;\; t \in [0,T], \tag{2}$$

$$y(t) = \int_0^{Q(t)} h(t,q,u(t,q))\,\mathrm{d}q, \tag{3}$$

$$\dot{x}(t,q) = f(t,q,x(t,q),Q(t),y(t),u(t,q)), \tag{4}$$

$$x(0,q) = x^0(q), \;\; q \in [0,Q^0],$$

$$x(t,Q(t)) = x^b(t), \;\; t \in [0,T],$$

$$u(t,q) \in U. \tag{5}$$

Here

$$L : D \times \mathbf{R}^n \times [0,\bar{Q}] \times \mathbf{R}^m \times U \mapsto \mathbf{R},$$

$$f : D \times \mathbf{R}^n \times [0,\bar{Q}] \times \mathbf{R}^m \times U \mapsto \mathbf{R}^n, \quad g : D \times [0,\bar{Q}] \times \mathbf{R}^m \mapsto \mathbf{R}, \quad h : D \times U \mapsto \mathbf{R}^m,$$

$\dot{x}(t,q)$ is the derivative with respect to t.

The informal meaning is as follows. Given a control function u with values in U, equations (2) and (3) define the interval $[0,Q(t)]$ in which the parameter q takes values at time t. The state $y(t)$ represents an aggregated (over the domain of heterogeneity $[0,Q(t)]$) quantity. Equation (4) with the respective side conditions defines the distributed state x. Then the objective functional (1) is to be maximized with respect to the control u.

Before giving the formal definition of the problem we enlist *Standing Assumptions* (i) – (vi) which will hold throughout the paper:
(i) The set $U \subset \mathbf{R}^r$ is compact.
(ii) The functions L, f, g, h are measurable in (t,q) and continuous in the rest of the variables, locally essentially bounded, differentiable in (x,Q,y), with locally Lipschitz partial derivatives, uniformly with respect to $u \in U$ and $(t,q) \in D$. The function h is locally Lipschitz continuous in u uniformly with respect to $(t,q) \in D$.
(iii) $g(t,Q,y) \geq \alpha_0 > 0$ for every $Q \in [Q^0,\bar{Q}]$ and $y \in \int_0^Q h(t,q,U)\,\mathrm{d}q$.
(iv) $x^b : [0,T] \mapsto \mathbf{R}^n$ is continuously differentiable, $x^0 : [0,Q_0] \mapsto \mathbf{R}^n$ is measurable and bounded.

Denote $\mathcal{U} = \{u \in L_\infty(D) : \; u(t,q) \in U\}$. Since for any given $u \in \mathcal{U}$ one can represent

$$g(t,Q,y(t)) = g\left(t, Q, \int_0^Q h(t,q,u(t,q))\,\mathrm{d}q\right)$$

and the function in the right-hand side is Lipschitz in Q, equation (2) has locally a solution $Q = Q[u]$ and it is unique on its maximal interval of existence in $[0,T]$.

Standing Assumption (v): For every $u \in \mathcal{U}$ the solution $Q[u]$ exists in $[0,\bar{Q}]$ on the whole interval $[0,T]$.

Given $u \in \mathcal{U}$, we define for $q \in [0,\bar{Q}]$

$$\theta[u](q) = \begin{cases} 0 & \text{if } q \in [0,Q^0], \\ Q[u]^{-1}(q) & \text{if } q \in (Q^0, Q[u](T)), \\ T & \text{if } q \in [Q[u](T),\bar{Q}]. \end{cases} \tag{6}$$

The definition is correct, since $Q[u]$ is invertible according to Assumption (iii) and its image is $[Q^0, Q[u](T)]$. Notice that $\theta[u]$ is Lipschitz continuous with constant $1/\alpha_0$ due to Assumption (iii). We extend the definition of x^b by setting

$$x^b(t,q) = \begin{cases} x^0(q) & \text{if } q \in [0,Q^0], \\ x^b(t) & \text{if } q \in (Q^0,\bar{Q}]. \end{cases} \tag{7}$$

Then we may view (4) as a family of ODEs (one for each $q \in [0, \bar{Q}]$), where the functions $y = y[u]$ and $Q = Q[u]$ are already defined from (2), (3) as described above. For each such q the solution $x[u]$ of (4) with the additional condition $x(\theta[u](q), q) = x^b(\theta[u](q), q)$ locally exists (around $\theta[u](q)$) and is unique on its maximal interval of existence in $[0, T]$.

Standing Assumption (vi): For every $u \in \mathcal{U}$ and for almost every $q \in [0, \bar{Q}]$ the solution $x[u](\cdot, q)$ exists on $[0, T]$.

Due to the continuous dependence of the solution of the ODE (4) on the parameter q and on the initial data, and due to the measurability of a superposition of a measurable and a continuous function, $x[u]$ is measurable with respect to q. Then the meaning of the optimization problem (1) is clear.

We mention that several of the Standing Assumptions could be relaxed, however at a certain price. In some extensions this price is just a technical complication, as for example considering a function h in (3) depending also on x, or requiring only continuity (instead on Lipschitz continuity) with respect to u in Assumption (ii). Other extensions require a substantial additional analysis. For example, dependence of the side condition $x^b(t)$ on y or on an additional non-distributed control $v(t)$. A third class of extensions require conceptual clarification. This concerns mainly Assumption (iii). How the solution should be defined if $Q(t)$ is not strictly monotone increasing? Apparently the "right" definition depends on the particular application. All these extensions have clear interpretations in several economic contexts and will be investigated for particular economic models in forthcoming papers. In this paper, however, we focus on the mathematical complication that the endogenous heterogeneity brings already in the simplest case which is general enough to cover some applications (see Section 6 for an example).

3 The optimality conditions

In this section we derive necessary optimality conditions of Pontryagin's type for the problem (1)–(5). What makes this derivation not a routine work, is that the objective functional (1) is, in general, non-differentiable with respect to the control function, as it will be demonstrated in Section 5. Moreover, the form of the maximum principle a la Pontryagin is non-standard, in general, although under an additional (non-restrictive for the typical applications) condition it takes a form that could be heuristically derived by an appropriate application of the Lagrange multiplier rule. The problem of existence of an optimal solution is not systematically investigated in this paper, although it is also challenging, as we show in Section 6.

To make the expressions below more compact we interpret x, y and u as columns, in contrast to the adjoint variables λ and ν that will be involved later, which are viewed as row-vectors with corresponding dimensions.

We start with a variational analysis the result of which will be summarized in Proposition 1 below. It will be used in the proofs of the three theorems to follow in this and in the next section.

Let $u \in \mathcal{U}$ be fixed and let $u^\sigma \in \mathcal{U}$ be a sequence of controls parameterized by positive $\sigma \to 0$. We shorten the notations $Q = Q[u]$, $Q^\sigma = Q[u^\sigma]$, etc, and denote $\Delta u = u^\sigma - u$, $\Delta Q = Q^\sigma - Q$, etc.

For the sequence of controls u^σ we require that there exists a constant c such that for all sufficiently small σ

$$\|\Delta u\|_{L_1 \times L_1} + \|\Delta y\|_{L_1} + \|\Delta Q\|_C + \|\Delta x\|_{C \times L_1} \le c\sigma, \tag{8}$$

$$\|\Delta u\|_{L_1 \times L_\infty} + \|\Delta y\|_{L_\infty} + \|\Delta x\|_{C \times L_\infty} \le c\sqrt{\sigma}. \tag{9}$$

There are two essential cases of sequences u^σ for which the above requirements are satisfied:

Case 1: L_1(-simple needle)-variation, where

$$u^\sigma(t,q) = \begin{cases} v & \text{if } (t,q) \in [\tau, \tau + \sqrt{\sigma}] \times [\kappa, \kappa + \sqrt{\sigma}], \\ u(t,q) & \text{elsewhere} \end{cases}$$

and $\tau \in [0,T)$, $\kappa \in [0,\bar{Q})$ and $v \in U$ are arbitrarily fixed.

Case 2: L_∞-variation, where

$$u^\sigma = u + \sigma v$$

and $v \in L_\infty(D)$.

Using assumptions (i), (ii), (iii) and (v) it can be easily verified that in both cases requirements (8) and (9) are fulfilled.

Let us denote by $J(v)$ the value of the objective function (1) corresponding to $v \in \mathcal{U}$. By a similar analysis as in the usual proof of the Pontryagin maximum principle for ODE control systems with unconstrained state we obtain the following result. For every function $\lambda : D \mapsto \mathbf{R}^n$ which is absolutely continuous in t for a.e. $q \in [0,\bar{Q}]$, with $\dot{\lambda} \in L_\infty(D)$ and $\lambda(T,q) = 0$, for every absolutely continuous function $\mu : [0,T] \mapsto \mathbf{R}$ satisfying $\mu(T) = 0$, and for every $\nu \in L_\infty(0,T)$ the following variational representation holds (the proof is not straightforward and will be presented in Appendix):

$$\begin{aligned}
\Delta J \;=\; & \int_0^T \int_0^{Q(t)} [L_x + \dot{\lambda} + \lambda f_x] \Delta x \,\mathrm{d}q\,\mathrm{d}t \qquad (10) \\
& + \int_0^T \left[\dot{\mu} + \mu g_Q + \int_0^{Q(t)} (L_Q + \lambda f_Q)\,\mathrm{d}q + \frac{1}{\Delta Q(t)} \int_{Q(t)}^{Q^\sigma(t)} [L + \lambda(f - \dot{x}^b) + \nu h]\,\mathrm{d}q \right] \Delta Q(t)\,\mathrm{d}t \\
& + \int_0^T \left[-\nu + \mu g_y + \int_0^{Q(t)} (L_y + \lambda f_y)\,\mathrm{d}q \right] \Delta y\,\mathrm{d}t + \int_0^T \int_0^{Q(t)} [\Delta_u L + \lambda \Delta_u f + \nu \Delta_u h]\,\mathrm{d}q\,\mathrm{d}t + o(\sigma).
\end{aligned}$$

In the case $\Delta Q(t) = 0$ the term $\frac{1}{\Delta Q(t)} \int_{Q(t)}^{Q^\sigma(t)}$ may be defined as any real number. Here the arguments $(t, q, x(t, q), Q(t), y(t), u(t, q))$ are skipped, $\Delta_u L = L(u^\sigma(t, q)) - L$ (similarly for the other terms $\Delta_u \ldots$), and as usual $o(\sigma)/\sigma \to 0$. This notational convention will be systematically used below: the arguments of the functions will appear only if they are different from those mentioned above, or in order to avoid confusion, or for more clarity.

Let λ be defined as the solution of the *adjoint equation*

$$-\dot{\lambda}(t, q) = L_x(t, q) + \lambda(t, q) f_x(t, q), \quad \lambda(T, q) = 0. \tag{11}$$

Then the first term in the right-hand side of (10) disappears. This equation has a unique solution on $[0, T]$ in a similar sense as (4): for every $q \in [0, \bar{Q}]$ it is a linear ODE with end condition $\lambda(T, q) = 0$. Clearly, $\dot{\lambda} \in L_\infty(D)$, as required.

Given the control $u \in \mathcal{U}$ and the corresponding solution of (2)–(4) and adjoint function λ, let us define for $t \in [0, T]$ and $\mu \in \mathbf{R}$ the set

$$\Gamma(t, \mu) = \operatorname*{Limsup}_{\alpha \to 0, \alpha \neq 0} \frac{1}{\alpha} \int_{Q(t)}^{Q(t)+\alpha} \Big[L(t, q) + \lambda(t, q)(f(t, q) - \dot{x}^b(t)) + (\mu g_y(t) + \eta(t)) h(t, q) \Big] \, \mathrm{d}q, \tag{12}$$

where

$$\eta(t) = \int_0^{Q(t)} (L_y(t, q) + \lambda(t, q) f_y(t, q)) \, \mathrm{d}q \tag{13}$$

and $\operatorname*{Limsup}_{\alpha \to 0, \alpha \neq 0} G(\alpha)$ is the Kuratowski upper limit of a function G at $\alpha = 0$, consisting of all condensation points of sequences $G(\alpha_k)$ with $\alpha_k \to 0$, $\alpha_k \neq 0$. Thanks to the continuity of the right-hand side in (12) with respect to $\alpha > 0$ it is easy to prove that $\Gamma(t, \mu)$ is a compact interval. For the same reason it is easy to prove that $\Gamma : [0, T] \times \mathbf{R} \Rightarrow \mathbf{R}$ is measurable in t and Lipschitz in μ (theorems 8.2.8 and 8.2.5 in [2] are used in the proof of the measurability.). The Lipschitz continuity holds due to the boundedness of $\|g_y \, h\|_{L_\infty(D)}$, which is a Lipschitz constant of Γ with respect to μ. Then the differential inclusion

$$-\dot{\mu}(t) \in \mu(t) g_Q(t) + \xi(t) + \Gamma(t, \mu(t)), \quad \mu(T) = 0 \tag{14}$$

with

$$\xi(t) := \int_0^{Q(t)} [L_Q(t, q) + \lambda(t, q) f_Q(t, q)] \, \mathrm{d}q \tag{15}$$

has at least one solution (see e.g. [1, Theorem 11.7.1]), therefore its reachable set $R(t)$ is nonempty for every $t \in [0, T]$.

Let us define the measurable functions

$$s^\sigma(t) = \begin{cases} 1 & \text{if } \; Q^\sigma(t) = Q(t), \\ 0 & \text{if } \; Q^\sigma(t) \neq Q(t), \end{cases}$$

and

$$d^\sigma(t) = \Delta Q(t) + \sigma^2 s^\sigma(t).$$

Now consider the equation

$$\begin{aligned} -\dot{\tilde{\mu}}^\sigma(t) &= \tilde{\mu}^\sigma(t) g_Q(t) + \xi(t) \\ &\quad + \frac{1}{d^\sigma(t)} \int_{Q(t)}^{Q(t)+d^\sigma(t)} [L + \lambda(f - \dot{x}^b(t)) + (\tilde{\mu}^\sigma(t) g_y(t) + \eta(t))h] \, \mathrm{d}q, \quad \tilde{\mu}^\sigma(T) = 0. \end{aligned}$$

Notice that $d^\sigma(t) \neq 0$ for any $\sigma > 0$ and t. Since the right-hand side of (16) is linear in $\tilde{\mu}^\sigma$ uniformly in t and measurable in t, it has a solution $\tilde{\mu}^\sigma(t)$. Then denoting

$$\tilde{\nu}^\sigma(t) = \tilde{\mu}^\sigma(t) g_y(t) + \eta(t) \tag{16}$$

and inserting $\lambda, \tilde{\mu}^\sigma$, $\tilde{\nu}^\sigma$ in (10) we obtain

$$\Delta J = \int_0^T \int_0^{Q(t)} [\Delta_u L + \lambda \Delta_u f + \tilde{\nu}^\sigma \Delta_u h] \, \mathrm{d}q \, \mathrm{d}t + o(\sigma). \tag{17}$$

We have

$$\mathrm{dist}\left(\frac{1}{d^\sigma(t)} \int_{Q(t)}^{Q(t)+d^\sigma(t)} [L + \lambda(f - \dot{x}^b(t)) + (\tilde{\mu}^\sigma(t) g_y(t) + \eta(t))h] \, \mathrm{d}q, \, \Gamma(t, \tilde{\mu}^\sigma(t)) \right) := \beta^\sigma(t) \underset{\rho \to 0}{\longrightarrow} 0$$

for every t, where β^σ is bounded in σ and t. This easily follows from the continuity with respect to μ of the mappings involved, the uniform boundedness of $\tilde{\mu}^\sigma$ and the definition of Γ. By the Filippov theorem [2, Theorem 10.4.1] we obtain that there exists a solution $\mu^\sigma(t)$ of (14) such that

$$|\mu^\sigma(t) - \tilde{\mu}^\sigma(t)| \leq C \int_0^T \beta^\sigma(t) \, \mathrm{d}t \underset{\rho \to 0}{\longrightarrow} 0.$$

Then using (16), (8) and (17) and the Lipschitz continuity of h with respect to u (cf. *Standing Assumption* (ii)) we obtain that

$$\Delta J = \int_0^T \int_0^{Q(t)} [\Delta_u L + \lambda \Delta_u f + \nu^\sigma \Delta_u h] \, \mathrm{d}q \, \mathrm{d}t + o(\sigma), \tag{18}$$

where

$$\nu^\sigma(t) = \mu^\sigma(t) g_y(t) + \eta(t). \tag{19}$$

Let us summarize what we have obtained so far.

Proposition 1 *Let $u \in \mathcal{U}$ be arbitrarily fixed and let $u^\sigma \in \mathcal{U}$ be a sequence of controls satisfying (8), (9). Then equation (18) holds true with the adjoint function $\lambda \in L_\infty(D)$ defined in (11) and some solution μ^σ of (14) and the corresponding ν^σ defined by (19). Here λ and μ^σ are absolutely continuous in t with $\dot{\lambda} \in L_\infty(D)$, $\dot{\mu}^\sigma \in L_\infty[0,T]$, η and ξ are defined by (13) and (15), respectively.*

We shall make use of the above finding in two ways. First, in this section, we shall consider the needle variation defined in Case (i) in order to obtain a specific global maximum principle for the problem in consideration, then in Section 4 we shall use variations as in Case (ii) to express the derivative of the objective functional with respect to the control under additional conditions that ensure its existence.

Define $\tilde{H} : D \times \mathbf{R}^{n+1+m+r+n+m} \mapsto \mathbf{R}$ as

$$\tilde{H}(t,q,x,Q,y,u,\lambda,\nu) = L(t,q,x,Q,y,u) + \lambda\, f(t,q,x,Q,y,u) + \nu\, h(t,q,u).$$

Theorem 1 *Let $u \in L_\infty(D)$ be an optimal control in the problem (1)–(5) and let $z := (x,Q,y)$ be the corresponding trajectory. Let λ be the solution of the adjoint equation (11) (it exists and is unique on D) and let η, ξ and Γ be defined by (13), (15) and (12), respectively. Then the reachable set $R(t)$ of the differential inclusion (14) is nonempty, and for almost every $(t,q) \in D(u) := \{(t,q) : t \in [0,T),\ q \in [0,Q(t))\}$*

$$\max_{v\in U} \min_{\nu \in R(t) g_y(t)+\eta(t)} \Big(\tilde{H}(t,q,z(t,q),v,\lambda(t,q),\nu) - \tilde{H}(t,q,z(t,q),u(t,q),\lambda(t,q),\nu)\Big) \le 0. \tag{20}$$

Proof. We shall sketch the proof emphasizing only the points that are unusual. Assume that the claim of the theorem is not true. Denote by $\varphi(t,q,v)$ the function "min…" in (20). This function is measurable in (t,q), according to [2, Theorem 8.2.11]. In a routine way we obtain that there exists a set $\Omega \subset D(u)$ with positive measure, $\varepsilon > 0$ and $v \in U$ such that $\varphi(t,q,v) \ge \varepsilon$ for $(t,q) \in \Omega$. Let (τ,κ) be a Lebesgue point of Ω, that is,

$$\mathrm{meas}(\Omega \cap B^\sigma) = \sigma + o(\sigma), \tag{21}$$

where B^σ is the box $[\theta, \theta+\sqrt{\sigma}] \times [\kappa, \kappa+\sqrt{\sigma}]$. Let u^σ be the simple needle control variation defined in Case (i), and let $(\lambda, \mu^\sigma, \nu^\sigma)$ be the functions from Proposition 1. The according to (18)

$$\Delta J = \int_{B^\sigma} [\tilde{H}(t,q,v,\nu^\sigma(t)) - \tilde{H}(t,q,\nu^\sigma(t))]\,\mathrm{d}q\,\mathrm{d}t + o(\sigma),$$

where, consistently with our notational convention, $\tilde{H}(t,q,v,\nu) = \tilde{H}(t,q,z(t,q),v,\lambda(t,q),\nu)$, $\tilde{H}(t,q,\nu) = \tilde{H}(t,q,z(t,q),u(t,q),\lambda(t,q),\nu)$. Then taking into account that $\nu^\sigma(t) \in R(t)g_y(t) + \eta(t)$ we obtain from the inequality $\varphi(t,q,v) \ge \varepsilon$ and (21) that

$$\Delta J \ge \int_{B^\sigma \cap \Omega} \varepsilon\, \mathrm{d}(q,t) + \int_{B^\sigma \setminus \Omega} \ldots\, \mathrm{d}(q,t) + o(\sigma) = \varepsilon\sigma + o(\sigma).$$

Since for sufficiently small $\sigma > 0$ the right-hand side is strictly positive we come to a contradiction.
Q.E.D.

The above theorem gives information about the optimal control only for $q \in [0, Q(t)]$. Obviously the values of u for $q > Q(t)$ are irrelevant for the objective value.

In the applications we have in mind the optimal control is regular enough to reduce the inclusion in (14) to an equation. Below we elaborate on this case.

Assume, in addition to the Standing Assumptions, that L, f and h are continuous with respect to q (uniformly in the rest of the variables), and consider an optimal control u which is continuous from the left with respect to q at $q = Q(t)$ for a.e. $t \in [0, T]$. Since, as just mentioned, the values of u for $q > Q(t)$ are irrelevant for the objective function (1), we may redefine u as $u(t, q) = u(t, Q(t))$ for $q > Q(t)$. Moreover, the functions $x(t, \cdot)$ and $\lambda(t, \cdot)$ are continuous in q at $q = Q(t)$, thus $x(t, Q(t))$ and $\lambda(t, Q(t))$ are well defined. Then the set $\Gamma(t, \mu)$ is a singleton:

$$\Gamma(t, \mu) = L(t, Q(t)) + \lambda(t, Q(t))(f(t, Q(t)) - \dot{x}^b(t)) + (\mu g_y(t) + \eta(t))h(t, Q(t)).$$

Then the solution $\mu^\sigma = \mu$ of (14) and the corresponding $\nu^\sigma = \nu$ are independent of σ and solve the equations

$$-\dot{\mu}(t) = \mu(t)g_Q(t) + L(t, Q(t)) + \lambda(t, Q(t))(f(t, Q(t)) - \dot{x}^b(t)) + \nu(t)h(t, Q(t)) \quad (22)$$
$$+ \int_0^{Q(t)} [L_Q(t, q) + \lambda(t, q)f_Q(t, q)]\, \mathrm{d}q, \quad \mu(T) = 0,$$

$$\nu(t) = \mu(t)g_y(t) + \int_0^{Q(t)} [L_y(t, q) + \lambda(t, q)f_y(t, q)]\, \mathrm{d}q, \quad (23)$$

(Notice that according to the notational convention and the side condition $x(t, Q(t)) = x^b(t)$ we have $L(t, Q(t)) = L(t, Q(t), x^b(t), Q(t), y(t), u(t, Q(t)))$ and similarly for the other functions above.)

Now we introduce the function (having in many respects the traditional meaning of "hamiltonian") $H : D \times \mathbf{R}^{n+1+m+r+n+1+m} \mapsto \mathbf{R}$ as

$$H(t, q, x, Q, y, u, \lambda, \mu, \nu) = L(t, q, x, Q, y, u) + \lambda\, f(t, q, x, Q, y, u) + \mu\, g(t, Q, y) + \nu\, h(t, q, u)$$

where the arguments in the l.-h. side should be inserted in the r.-h. side wherever appropriate. Then Theorem 1 implies the next one.

Theorem 2 *Let L, f and h are continuous in q (uniformly in the rest of the variables) and let $u \in L_\infty(D)$ be an optimal control in the problem (1)–(5) and $z := (x, Q, y)$ be the corresponding trajectory. Assume that for almost every t the function $u(t, \cdot)$ is (equivalent to) continuous from the left at $q = Q(t)$. Then the adjoint system (11), (22)–(23) has a unique solution $\pi := (\lambda, \mu, \nu)$ and for a.e. $t \in [0, T]$ and a.e. $q \in [0, Q(t)]$*

$$H(t, q, z(t, q), u(t, q), \pi(t, q)) = \max_{u \in U} H(t, q, z(t, q), u, \pi(t, q)).$$

4 A numerical approach

For solving numerically problem (1)–(5) we apply a version of the gradient projection method in the control space $L_\infty(D)$. Denoting as before by $J(u)$ the objective value corresponding to control $u \in \mathcal{U}$, the gradient projection method consists of the following. Given the current "approximation" $u_k \in \mathcal{U}$ of the optimal control we find a next approximation u_{k+1} as

$$u_{k+1} = \mathcal{P}_{\mathcal{U}}(u_k + \alpha_k J'(u_k)).$$

where $J'(u)$ is the derivative of J with respect to u in $L_\infty(D)$ (if it exists) and $\mathcal{P}_X$ means the metric projection on X. Clearly

$$u_{k+1}(t,q) = \mathcal{P}_U\left(u_k(t,q) + \alpha_k J'(u_k)(t,q)\right).$$

The choice of the parameter $\alpha_k \geq 0$ is a subject of many publications and will not be discussed here.

We include this section in the paper for the following reason. It turns out (as shown in the next section by two examples) that the objective functional $J(u)$ is not differentiable in $u \in L_\infty(D)$, in general. Even more intriguing, although the vale of $J(u)$ does not depend on the values of $u(t,q)$ for $q > Q[u](t)$, the differentiability property may depend on how $u(t,q)$ is defined above the graph of Q. This fact requires a special analysis which is the main point in this section.

Denote by $\mathcal{U}_{\rm lc}$ the set of those $u \in \mathcal{U}$ that are continuous from the left at $q = Q[u](t)$ for a.e. $t \in [0,T]$ and by $\mathcal{U}_{\rm c}$ the set of those $u \in \mathcal{U}$ that are continuous at $Q[u](t)$ for a.e. $t \in [0,T]$. First of all, in addition to assumptions (i)–(vii) we introduce the following one.

Assumption (viii). The functions L, f, h are continuously differentiable with respect to u, uniformly with respect to the rest of the variables.

Theorem 3 *On the assumptions (i)–(viii) the objective function $J(u)$ is Gato-differentiable at every $u \in \mathcal{U}_{\rm c}$ and the derivative can be represented as*

$$J'(u)(t,q) = \begin{cases} H_u(t,q) & \text{if } q \leq Q(t), \\ 0 & \text{if } q > Q(t), \end{cases} \tag{24}$$

where (λ, μ, ν) is the solution of the adjoint system (11), (22)–(23).

Proof. Let u^σ be defined as in Case 1: $u^\sigma = u + \sigma v$, where $v \in L_\infty(D)$. Then

$$\begin{aligned} \Delta J &= \int_0^T \int_0^{Q(t)} [H(t,q,u(t,q) + \sigma v(t,q)) - H(t,q,u(t,q))] \, \mathrm{d}q \, \mathrm{d}t + o(\sigma) \\ &= \sigma \int_0^T \int_0^{Q(t)} H_u(t,q) v(t,q) \, \mathrm{d}q \, \mathrm{d}t + o(\sigma), \end{aligned}$$

due to the uniform continuous differentiability of H in u, which means that the functional J is Gato differentiable at u in the space $L_\infty(D)$ and its derivative is (represented by) (24). Q.E.D.

We shall modify the gradient projection procedure in such a way that $u_k \in \mathcal{U}_c$. Indeed, assume that $u_k \in \mathcal{U}_c$. Then it is easy to verify (using the continuous dependence of the solution of an ODE on parameters) that H_u is also continuous in q at $q = Q(t)$ for a.e. t. Then $J'(u)$ is continuous from the left at $q = Q(t)$ due to (24). Since $\mathcal{P}_U$ is a continuous mapping, we obtain that $\tilde{u}_{k+1} := \mathcal{P}_U\left(u_k(t,q) + \alpha_k J'(u_k)(t,q)\right)$ has the same property. Define the operator $\mathcal{I} : \mathcal{U}_{\rm lc} \to \mathcal{U}_{\rm c}$ as $\mathcal{I}(u)(t,q) = u(t,q)$ for $q \le Q[u](t)$ and $\mathcal{I}(u)(t,q) = u(t, Q[u](t))$ for $q > Q[u](t)$. (Notice that $J(\mathcal{I}(u)) = J(u)$, but J is differentiable at $\mathcal{I}(u)$, while it need not be differentiable at u.) Then we define the next iteration as

$$u_{k+1} = \mathcal{I}\left(\mathcal{P}_U\left(u_k(t,q) + \alpha_k J'(u_k)\right)\right).$$

Of course the numerical implementation involves discretization with respect to t and q which contains a delicate points due to the changing domain $[0, Q(t)]$, but this issue is outside the scope of the present paper.

5 Two examples of non-differentiability of the objective function

As mentioned in the previous section the objective value $J(u)$ in (1) considered as a function of the control could be non-differentiable in the space $L_\infty(D)$. We distinguish two different cases of non-differentiability: one is harmless, while the other is not and requires the non-standard form of the maximum principle considered in the previous section.

1. Let $u \in \mathcal{U}$ and let $Q(t) = Q[u](t)$ be the corresponding solution of (2), (3). As argued in Section 2, the respective solution $(x[u], Q[u], y[u])$ is independent of the values of u for $q > Q(t)$, and if u is continuous from the left with respect to q at $q = Q(t)$ for a.e. t, then u can be redefined as continuous in q for $q > Q(t)$ (we used the notation $u^\#$ for the redefined control). Then J is Gato differentiable at $u^\#$. This is the "harmless" case of possible non-differentiability of $J(u)$, which can be avoided by the redefinition of u.

Although the objective value $J(u)$ does not depend on values of u for $q > Q(t)$, its differentiability does, and requires that u is continuous also from the right at $q = Q(t)$ (which means that the redefinition of u as $u^\#$ is essential). The example below shows this.

Example 1. Consider the system

$$\begin{aligned} \dot{Q}(t) &= y(t), \qquad Q(0) = 1. \\ y(t) &= \int_0^{Q(t)} u(t,q)\,\mathrm{d}q \end{aligned}$$

and the objective function

$$J(u) = \int_0^1 \int_0^{Q(t)} 1 \,\mathrm{d}q \,\mathrm{d}t = \int_0^1 Q(t) \,\mathrm{d}t.$$

Fix the control

$$u(t,q) = \begin{cases} 1 & \text{if } q \in [0, e^t], \\ a & \text{if } q > e^t, \end{cases}$$

where a is a real number. Then consider the controls $u_1^h(t,q) = u(t,q) - h$ and $u_2^h(t,q) = u(t,q) + h$ and the corresponding solutions (Q_i^h, y_i^h), and compare with the solution for u, which is obviously $Q(t) = e^t$. It can also be directly checked that

$$Q_1^h(t) = e^{(1-h)t} = Q(t) - hte^t + O(h^2).$$

Since $u_2^h > u$, we have $Q_2^h(t) \geq Q(t)$. Hence

$$\begin{aligned} y_2^h(t) &= \int_0^{Q(t)} u_2^h(t,q) \,\mathrm{d}q + \int_{Q(t)}^{Q_2^h(t)} u_2^h(t,q) \,\mathrm{d}q = (1+h)Q(t) + (a+h)(Q_2^h(t) - Q(t)) \\ &= (a+h)Q_2^h(t) + (1-a)Q(t). \end{aligned}$$

Then the equation for Q becomes

$$\dot{Q}_2^h(t) = (a+h)Q_2^h(t) + (1-a)Q(t), \qquad Q_2^h(0) = 1.$$

Using the Cauchy formula we obtain after routine calculations that

$$Q_2^h(t) = e^t \left(1 + \frac{h}{1-a-h}\right) - \frac{h}{1-a-h}\, e^{(a+h)t} = Q(t) + \frac{h}{1-a-h}\left(e^t - e^{at}\right) - \frac{h}{1-a-h} hte^{at} + O(h^2).$$

Now we consider the case $a \neq 1$ (hence u is discontinuous from the right). Then the second last term in the above equality is also $O(h^2)$, thus

$$Q_2^h(t) = Q(t) + \frac{h}{1-a}\left(e^t - e^{at}\right) + O(h^2).$$

Then

$$J(u-h) - J(u) = h \int_0^1 te^t \,\mathrm{d}t + O(h^2),$$

while

$$J(u+h) - J(u) = \frac{h}{1-a} \int_0^1 \left(e^t - e^{at}\right) \mathrm{d}t + O(h^2).$$

Comparing the two principle terms one can easily verify that they are not opposite numbers, whatever is $a \neq 1$. Thus J is not differentiable if $a \neq 1$.

In contrast, if $a = 1$ the objective j is differentiable in the direction ± 1 (as it was proved in the previous section) since in this case

$$Q_2^h(t) = Q(t) + hte^{at} + O(h^2).$$

The above example shows, in particular, that the special attention that we attribute to the definition of u above $Q[u](t)$ in the previous section is essential.

2. Below we give another example, which shows that the "remedy" of redefinition of u above $Q(t)$ in order to get differentiability of $J(u)$ is not applicable if u is too "bad" below $Q(t)$. This example justifies the non-standard form of the maximum principle in the case of an optimal control that is not continuous from the left at $Q(t)$.

Example 2. Consider the system

$$\dot{Q}(t) = y(t), \qquad Q(0) = 1, \qquad t \in [0, 1] \tag{25}$$

$$y(t) = \int_0^{Q(t)} u(t, q)\, \mathrm{d}q, \tag{26}$$

and the functional

$$J(u) = \int_0^1 \int_0^{Q(t)} u(t, q)\, \mathrm{d}q\, \mathrm{d}t.$$

This is not interpreted as an optimal control problem. We just study the differentiability of the functional J, defined for $u \in L_\infty([0, 1] \times [0, \infty))$. We shall prove that for some u the functional J is not even directionally differentiable at u in the direction of

$$w(t, q) = \begin{cases} 0 & \text{for } (t, q) \in [0, \bar{t}] \times [0, \infty) \\ -1 & \text{for } (t, q) \in (\bar{t}, 1] \times [0, \infty), \end{cases}$$

where $\bar{t} \in (0, 1)$ is to be chosen later sufficiently close to 1. Namely, we define

$$u(t, q) \equiv u(q) = \begin{cases} v(1 - q) & \text{for} \quad q \in [0, 1) \\ 0 & \text{for} \quad q \geq 1 \end{cases}$$

where $v(q)$ is defined on $(0, 1]$ as $v(1) = -1$ and

$$v(q) = \begin{cases} 1 & \text{for } q \in \left[\frac{1}{3^k}, \frac{2}{3^k}\right) \quad k = 1, 2, \ldots, \\ -1 & \text{for } q \in \left[\frac{2}{3^k}, \frac{3}{3^k}\right) \quad k = 1, 2, \ldots. \end{cases}$$

The solution of (25), (26) for u is $Q(t) = 1$, $y(t) = 0$, since

$$y(t) = \int_0^1 v(1 - q)\, \mathrm{d}q = \int_0^1 v(q)\, \mathrm{d}q = 0.$$

Now, let $u^\sigma = u + \sigma w$, where σ is a "small" positive real number. Denote by Q^σ and y^σ the corresponding solution of (25), (26). It is clear that $u^\sigma(t,q) \equiv u(t,q)$ on $[0,\bar{t}] \times [0,\infty)$ and, hence, $Q^\sigma(t) \equiv Q(t) \equiv 1$ on $[0,\bar{t}]$. Is is also clear that $Q^\sigma(t) \leq Q(t)$ for $t \in [\bar{t},1]$.

Denoting $V(q) = \int_0^q v(s)\,\mathrm{d}s$ for $q \in (-\infty,\infty)$, we have $0 \leq V(q) \leq \frac{1}{2}q$ for $q \in [0,1]$ and also $V\left(\frac{3}{3^k}\right) = 0$ and $V\left(\frac{2}{3^k}\right) = \frac{1}{2} \cdot \frac{2}{3^k}$ for $k = 1,2,3,\dots$.

For $t \in [\bar{t},1]$ we have $\dot{Q}^\sigma(t) = \int_0^{Q^\sigma(t)} [u(q) - \sigma]\,\mathrm{d}q = -\sigma + \int_1^{Q^\sigma(t)} [u(q) - \sigma]\,\mathrm{d}q$.
Defining $\varphi_\sigma(t) := Q^\sigma(t) - (1 - \sigma(t-\bar{t}))$ for $t \in [\bar{t},1]$, we have $\varphi_\sigma(\bar{t}) = 0$ and

$$\dot{\varphi}_\sigma(t) = \int_1^{1-\sigma(t-\bar{t})+\varphi_\sigma(t)} [u(q) - \sigma]\,\mathrm{d}q = -\int_0^{\sigma(t-\bar{t})-\varphi_\sigma(t)} [v(s) - \sigma]\,\mathrm{d}s = -V[\sigma(t-\bar{t}) - \varphi_\sigma(t)] + \sigma[\sigma(t-\bar{t}) - \varphi_\sigma(t)].$$

<u>Case 1.</u> Assume $\varphi_\sigma(\tau) > 0$ for $\tau \in [t_0,t] \subset [\bar{t},1]$ where $\varphi_\sigma(t_0) = 0$. Then

$$0 < \varphi_\sigma(\tau) \leq \int_{t_0}^{\tau} \sigma^2(\mu - \bar{t})\,\mathrm{d}\mu \leq \frac{1}{2}\sigma^2(\tau-\bar{t})^2 = O(\sigma^2) \quad \text{for} \quad \tau \in [t_0,t] \quad \text{i.e.} \tag{27}$$

$$Q^\sigma(\tau) = 1 - \sigma(\tau - \bar{t}) + O(\sigma^2) \quad \text{for} \quad \tau \in [t_0,t]. \tag{28}$$

<u>Case 2.</u> Assume $\varphi_\sigma(\tau) < 0$ for $\tau \in [t_0,t] \subset [\bar{t},1]$ where $\varphi_\sigma(t_0) = 0$. Then

$$V(q) \leq \frac{1}{2}q \quad \text{and} \quad \varphi_\sigma(\tau) = \int_{t_0}^{\tau} [-V(\sigma(\mu-\bar{t}) - \varphi_\sigma(\mu)) + \sigma(\sigma(\mu-\bar{t}) - \varphi_\sigma(\mu))]\,\mathrm{d}\mu$$

$$\text{yield} \quad 0 < -\varphi_\sigma(\tau) \leq \left(\frac{1}{2}\sigma - \sigma^2\right)\frac{1}{2}(\tau-\bar{t})^2 + \int_{t_0}^{\tau}(-\varphi_\sigma(\mu))\,\mathrm{d}\mu.$$

Applying the Gronwall inequality (cf., e.g., [5], p. 14) we obtain

$$0 < -\varphi_\sigma(\tau) \leq \left(\frac{1}{2}\sigma - \sigma^2\right)\frac{1}{2}(\tau-\bar{t})^2 e^{t-t_0} \leq \frac{1}{4}\sigma(\tau-\bar{t})^2 e^{1-\bar{t}} \quad \text{for} \quad \tau \in [t_0,t].$$

$$\text{This implies} \qquad Q^\sigma(\tau) \geq 1 - \sigma(\tau-\bar{t}) - \frac{1}{4}\sigma(\tau-\bar{t})^2 e^{1-\bar{t}} \quad \text{for} \quad \tau \in [t_0,t]. \tag{29}$$

We have

$$J(u^\sigma) - J(u) = \int_0^1 \int_0^{Q^\sigma(t)} [u(q) + \sigma w(t,q)]\,\mathrm{d}q\,\mathrm{d}t - \int_0^1 \int_0^1 u(q)\,\mathrm{d}q\,\mathrm{d}t$$

$$= \quad -\sigma(1-\bar{t}) - \int_{\bar{t}}^1 \int_{1-\sigma(t-\bar{t})}^1 u(q)\,\mathrm{d}q\,\mathrm{d}t + \int_{\bar{t}}^1 \int_{1-\sigma(t-\bar{t})}^{Q^\sigma(t)} u(q)\,\mathrm{d}q\,\mathrm{d}t + \int_{\bar{t}}^1 \int_{Q^\sigma(t)}^1 \sigma\,\mathrm{d}q\,\mathrm{d}t.$$

From (28) and (29) we obtain that the last term above is $O(\sigma^2)$. We next estimate the third term in the last row above. From (27) and (29) we obtain that if the positive σ is small

enough, $\quad |Q^\sigma(t) - [1 - \sigma(t - \bar{t})]| \le \dfrac{e^{1-\bar{t}}}{4}\,(t - \bar{t})^2\,\sigma$ holds true for every $t \in [1 - \bar{t}, 1]$.

$$\text{Hence,} \qquad \left| \int_{\bar{t}}^{1} \int_{1-\sigma(t-\bar{t})}^{Q^\sigma(t)} u(q)\,\mathrm{d}q\,\mathrm{d}t \right| \le \sigma \frac{e^{1-\bar{t}}}{4} \int_{\bar{t}}^{1} (t - \bar{t})^2\,\mathrm{d}\tau = \sigma \frac{e^{1-\bar{t}}}{12} (1 - \bar{t})^3. \tag{30}$$

We thus obtained that $\quad J(u^\sigma) - J(u) = -\sigma(1 - \bar{t}) - Z^\sigma + Y^\sigma + O(\sigma^2)$.

where

$$Z^\sigma = \int_{\bar{t}}^{1} \int_{1-\sigma(t-\bar{t})}^{1} u(q)\,\mathrm{d}q\,\mathrm{d}t \quad \text{and} \quad Y^\sigma = \int_{\bar{t}}^{1} \int_{1-\sigma(t-\bar{t})}^{Q^\sigma(t)} u(q)\,\mathrm{d}q\,\mathrm{d}t.$$

In order to prove non-differentiability of J at u in the direction of w it is enough to show that $(Z^\sigma - Y^\sigma)/\sigma$ does not converge with $\sigma \to 0$. We have

$$\begin{aligned} \frac{Z^\sigma}{\sigma} &= \frac{1}{\sigma} \int_{\bar{t}}^{1} \int_{1-\sigma(t-\bar{t})}^{1} v(1 - q)\,\mathrm{d}q\,\mathrm{d}t = \frac{1}{\sigma} \int_{\bar{t}}^{1} \int_{0}^{\sigma(t-\bar{t})} v(q)\,\mathrm{d}q\,\mathrm{d}t \\ &= \frac{(1 - \bar{t})^2}{\sigma(1 - \bar{t})} \int_{0}^{\sigma(1-\bar{t})} v(q)\,\mathrm{d}q - \frac{(1 - \bar{t})^2}{(\sigma(1 - \bar{t}))^2} \int_{0}^{\sigma(1-\bar{t})} q\,v(q)\,\mathrm{d}q. \end{aligned} \tag{31}$$

We shall consider the last two integrals separately, for values $\sigma'_k = \dfrac{2/3^k}{1 - \bar{t}}$ and $\sigma''_k = \dfrac{3/3^k}{1 - \bar{t}}$ of $k = 1, 2, \ldots$. Easy calculations give for the first integral in (31)

$$\frac{1}{\sigma'_k(1 - \bar{t})} \int_{0}^{\sigma'_k(1-\bar{t})} v(q)\,\mathrm{d}q - \frac{1}{\sigma''_k(1 - \bar{t})} \int_{0}^{\sigma''_k(1-\bar{t})} v(q)\,\mathrm{d}q = \frac{1}{2} - 0 = \frac{1}{2}.$$

More cumbersome calculations, which we skip, give for the second integral in (31)

$$\frac{1}{(\sigma'_k(1 - \bar{t}))^2} \int_{0}^{\sigma'_k(1-\bar{t})} q\,v(q)\,\mathrm{d}q - \frac{1}{(\sigma''_k(1 - \bar{t}))^2} \int_{0}^{\sigma''_k(1-\bar{t})} q\,v(q)\,\mathrm{d}q = \frac{11}{32} + \frac{1}{8} = \frac{15}{32}.$$

Comparing the expressions for the two integrals in (31) we conclude that the variation of $\frac{Z^\sigma}{\sigma}$ remains strictly positive (at least $(1-\bar{t})^2/32$) for arbitrarily small σ. On the other hand, from (30) we obtain that if $\bar{t} \in (0, 1)$ is close enough to 1, $\frac{|Y^\sigma|}{\sigma} \le (1 - \bar{t})^2/64$ holds true. Hence $(Z^\sigma - Y^\sigma)/\sigma$ does not converge with $\sigma \to 0$. This completes the proof of the non-differentiability of J at u in the direction w.

The example, discussed above, can be easily modified to include the case of strictly increasing $Q(\cdot)$ by replacing the equation in (25) by $\dot{Q}(t) = 1 + y(t)$. Since the proof of the non-differentiability of J is this case requires longer and more cumbersome calculations, we omit it.

6 An economic example

In this section we present a stylized economic model of endogenous economic growth to which the above results can be applied. Economically more meaningful extensions will be presented elsewhere.

We consider a finite time horizon $[0, T]$ (presumably rather large, so that T is an "approximation" of the infinity) and a large corporation producing at time t diverse goods labeled by the real number $q \in [0, Q(t)]$. Here $Q(t)$ is the newest good (technology) available at time t. Each of the goods q is produced by a separate firm that at time t has physical capital stock $x(t, q)$. The q-th firm ($q \in [0, Q(t)]$) invests at time t an amount $u(t, q)$ that is split in two parts: $\alpha u(t, q)$ is allocated to increase the capital stock, while $(1 - \alpha)u(t, q)$ is the contribution of the q-th firm to the R&D activity of the corporation which results in development of new technologies (goods) and hence in increase of $Q(t)$.

The model reads as follows:

$$
\begin{aligned}
\dot{x}(t, q) &= -\delta x(t, q) + \alpha u(t, q), \quad x(0, q) = x^0(q) \;\text{ for }\; q \in [0, Q^0], \\
&\qquad\qquad\qquad\qquad\qquad\quad x(t, Q(t)) = 0 \;\text{ for }\; t > 0, \\
\dot{Q}(t) &= (1 - \alpha)y(t), \quad Q(0) = Q^0, \\
y(t) &= \int_0^{Q(t)} u(t, q)\,\mathrm{d}q.
\end{aligned}
$$

Here $y(t)$ is the total investment in R&D, $\delta \geq 0$ is the depreciation rate of the physical capital, $x^0(q)$ is the initially available capital stock for producing goods $q \in [0, Q^0]$, $Q^0 > 0$ is the newest technology available at time $t = 0$. The objective function to be maximized is

$$
\int_0^T e^{-rt} \int_0^{Q(t)} \Big[p(q, Q(t))x(t, q) - bu(t, q) - cu(t, q)^2\Big]\,\mathrm{d}q\,\mathrm{d}t, \tag{32}
$$

subject to the control constraint $u(t, q) \geq 0$. Here $r \geq 0$ is the discount rate, $p(q, Q)$ is the market price of the good $q \in [0, Q]$, given that goods up to level Q are available, $bu + cu^2$, $c > 0$, is the cost of investments u. The dependence of the price p on q and Q reflects the fact that the market price of any available good decreases when newer products emerge (that is, when Q increases). For the present illustrative purpose we chose the specification $p(q, Q) = e^{-\gamma(Q-q)}$ with $\gamma \geq 0$. So the price of the newest product is normalized to one (which is supported by the data for personal computers and mobile telephones, where the price of the new products does not substantially change with time, while the quality increases).

Adjoint equations (11), (22)-(23) for this example, written in the terms of the "current value" adjoint variables $\tilde{\lambda} = e^{rt}\lambda$, $\tilde{\mu} = e^{rt}\mu$, $\tilde{\nu} = e^{rt}\nu$ read as follows

$$
-\dot{\tilde{\lambda}}(t, q) = -(\delta + r)\tilde{\lambda}(t, q) + e^{-\gamma(Q(t)-q)}, \quad \tilde{\lambda}(T, q) = 0, \tag{33}
$$

$$-\dot{\tilde{\mu}}(t) = -r\tilde{\mu}(t) - [bu(t,Q(t)) + cu(t,Q(t))^2] + \alpha\tilde{\lambda}(t,Q(t))u(t,Q(t)) + \tilde{\nu}(t)u(t,Q(t))$$

$$- \gamma\int_0^{Q(t)} e^{-\gamma(Q(t)-q)}x(t,q)\,\mathrm{d}q, \quad \tilde{\mu}(T) = 0, \tag{34}$$

$$\tilde{\nu}(t) = \tilde{\mu}(t)(1-\alpha), \tag{35}$$

and Theorem 2 implies that if $u(t,q)$ is an optimal control that is continuous from the left at $q = Q[u](t)$, then

$$u(t,q) = \max\{0, (\alpha\tilde{\lambda}(t,q) + \tilde{\nu}(t) - b)/2c\}. \tag{36}$$

1. Let us consider first the case $\gamma = 0$, where the solution can be studied analytically. It follows from (36) and (33) that the optimal control does not depend on q, so $\tilde{\lambda}(t,q) = \tilde{\lambda}(t)$ and $u(t,q) = u(t)$. We will solve equations (33)–(36) backwards starting from time T. It follows from $\tilde{\mu}(T) = 0$, $\tilde{\lambda}(T) = 0$, and (36) that $u(T) = 0$. Let $[t_0, T]$ be the maximal interval which ends at T and in which $u(t) = 0$. To find t_0 we substitute $u(t) = 0$ in (34) and obtain (using that $\gamma = 0$) the solution $\tilde{\mu}(t) = \tilde{\nu}(t) = 0$ for $t \in [t_0, T]$. Now we get from (36) that $\tilde{\lambda}(t_0) = b/\alpha$ and substitute here the solution $\tilde{\lambda}(t) = (1 - e^{-(r+\delta)(T-t)})/(r+\delta)$ of equation (33). Thus, we obtain $t_0 = T + \frac{1}{\delta+r}\ln\left(1 - b\,\frac{\delta+r}{\alpha}\right)$. It the case $b\,(\delta+r)/\alpha \geq 1$ the optimal control is identically zero.

In a time interval, where the control constraint is not active ($u(t) > 0$) the derivative of the hamiltonian must be equal to zero: $\tilde{H}_u(t,q) = -b - 2cu(t,q) + \alpha\tilde{\lambda}(t,q) + \tilde{\nu}(t) = 0$. Using this condition and equations (33)–(35) we can derive the following Riccati equation for the optimal control for $t \in [0, t_0]$

$$\dot{u}(t) = ru(t) - \frac{1-\alpha}{2}u(t)^2 - \frac{r}{2c}\left(\frac{\alpha}{\delta+r} - b\right) - \frac{\alpha\delta}{2c(\delta+r)}e^{-(\delta+r)(T-t)}, \quad u(t_0) = 0, \tag{37}$$

which turns out to be positive on $[0, t_0)$.

1.1. The solution of equation (37) can be obtained in special functions. Under the additional condition $\delta = 0$ it takes the simpler form

$$u\,(t) = \frac{r}{1-\alpha} - \frac{\sqrt{A}}{c\,(1-\alpha)}\,\tanh\!\left(\frac{\sqrt{A}}{2\,c}(t_0 - t) + \operatorname{arctanh}\!\left(\frac{r\,c}{\sqrt{A}}\right)\right)$$

if

$$A := c\,\left(r^2c - (1-\alpha)(\alpha - b\,r)\right) > 0,$$

If $A < 0$ the solution is

$$u\,(t) = \frac{r}{1-\alpha} + \frac{\sqrt{-A}}{c\,(1-\alpha)}\,\tan\!\left(\frac{\sqrt{-A}}{2\,c}(t_0 - t) - \arctan\!\left(\frac{r\,c}{\sqrt{-A}}\right)\right),$$

which becomes infinite if $t_0 - t$ may take sufficiently large values when $t \in [0, t_0]$. This is always the case if T is sufficiently large, as it follows from the expression for t_0.

Thus, in order to ensure boundedness of the solution on any time horizon, the discount rate should be big enough:

$$r > \frac{1}{2\,c}\left(\sqrt{b^2(1-\alpha)^2 + 4\,c\,\alpha\,(1-\alpha)} - b\,(1-\alpha)\right). \tag{38}$$

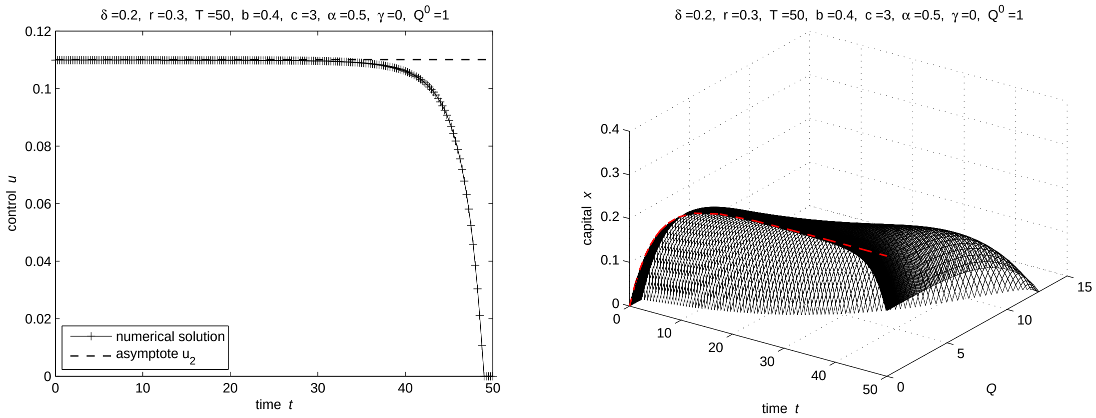

Figure 1: Optimal investments (left) and the corresponding capital stock (right) for $\gamma = 0$. Numerical results are compered with asymptotic solution (dashed lines).

1.2. In the case $\delta > 0$ we apply an asymptotic analysis assuming that T is large. Namely, we can find the asymptotic solution of (37) considering times t such that $t_0 - t$ is big enough for neglecting the last term in (37). Then the two steady state solutions are

$$u_1 = \frac{\sqrt{r}}{1-\alpha}\left[\sqrt{r} + \sqrt{r - \frac{1-\alpha}{c}\left(\frac{\alpha}{\delta + r} - b\right)}\right], \quad u_2 = \frac{\sqrt{r}}{1-\alpha}\left[\sqrt{r} - \sqrt{r - \frac{1-\alpha}{c}\left(\frac{\alpha}{\delta + r} - b\right)}\right] \tag{39}$$

The first of which is an attractor while the second is a repeller. In inverse time $t' = -t$ point u_2 becomes an attractor with the basin of attraction $(u_1, -\infty)$. Thus, u_2 is the horizontal asymptote of the exact solution for $t \to -\infty$ if it is still in the basin $u(t) \in (u_1, -\infty)$ when the last term in (37) is already insignificant, see Figure 1 (left). For a large time horizon $[0, T]$ if the optimal control $u(t)$ exists then most of the time $t \in [0, t_0]$ it is close to u_2 when $u_2 > 0$, or $u(t) \equiv 0$ when $u_2 \le 0$. Nonnegativity of the expression under the square root in (39) gives us condition for existence of asymptote u_2

$$r > \frac{1}{2\,c}\left(\sqrt{(b\,(1-\alpha) - \delta\,c)^2 + 4\,c\,\alpha\,(1-\alpha)} - b\,(1-\alpha) - \delta\,c\right) \tag{40}$$

that generalizes condition (38) to all $\delta \ge 0$. It follows from (39) and (40) that when $\delta > \alpha/b$ even $r = 0$ allows for bounded solution but this solution is trivial $u(t, q) \equiv 0$.

The essence of the above analysis is that even for a very simple economic problem the issue of existence of a solution is not simple. Inequality (40) provides a necessary condition under which a solution exists for any time horizon $[0, T]$.

2. The above considerations concern the "degenerate" case $\gamma = 0$, where all data of the problem are independent of q. In the general case $\gamma \geq 0$ we have obtained only numerical results.[2]

Figure 2 presents the optimal investments (left) and the corresponding capital stock (right) of the firm with $\gamma = 0.7$. The time horizon is $T = 50$ and the initial product variety is $[0, Q^0] = [0, 1]$. Notice that due to the technological development ($Q(t)$ increases from 1 to about 2) the firm completely abandons investments in some older technologies. At time $t = 30$, for example, the firm invests only in technologies $q \in [0.6, 1.63]$. As a result, the physical capital of technologies q much smaller than the technological frontier $Q(t)$ is close to zero.[3] Every section $x(\cdot, q)$ has the meaning of *diffusion curve* of technology q. It starts at the time $\theta(q)$ (see (6)) and reaches its maximum at some later time, after which it begins decreasing.

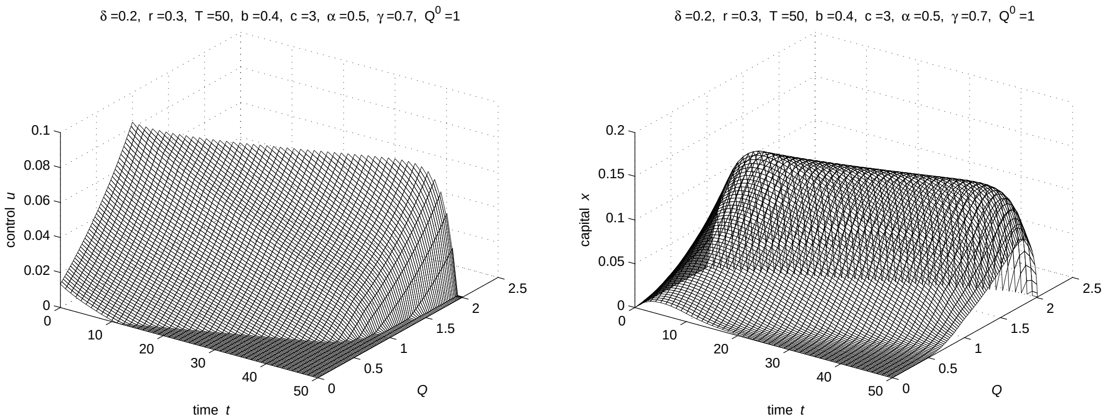

Figure 2: Optimal investments (left) and the corresponding capital stock (right) for $\gamma = 0.7$

3. In the above numerical example the dependence of the optimal control on the technology q results from the dependence of the price function $p(q, Q)$ on q. However, a minor nonlinearity inserted in the problem may lead to technology-dependent investments even if all data of the problem are technology-independent. Let, for example, the revenue function $p(q, Q)\, x$ in (32) be replaced with $p\,(x + x_m)^\sigma$ with $\sigma \in (0, 1)$ and constant $p > 0$ and $x_m > 0$. This corresponds to the revenue of a firm with market power or scarce non-capital production factors. The numerically obtained optimal control is plotted on Figure 3. Clearly, the investments in the newest technologies are larger than those of the older ones due to the higher marginal productivity of the technologies with lower capital stock.

[2] The numerical results are obtained by our own MATLAB solver in which the modification of the gradient projection method described in Section 4 is implemented.

[3] A more profound investigation of the issue of obsolescence in the spirit of [4] will be done elsewhere.

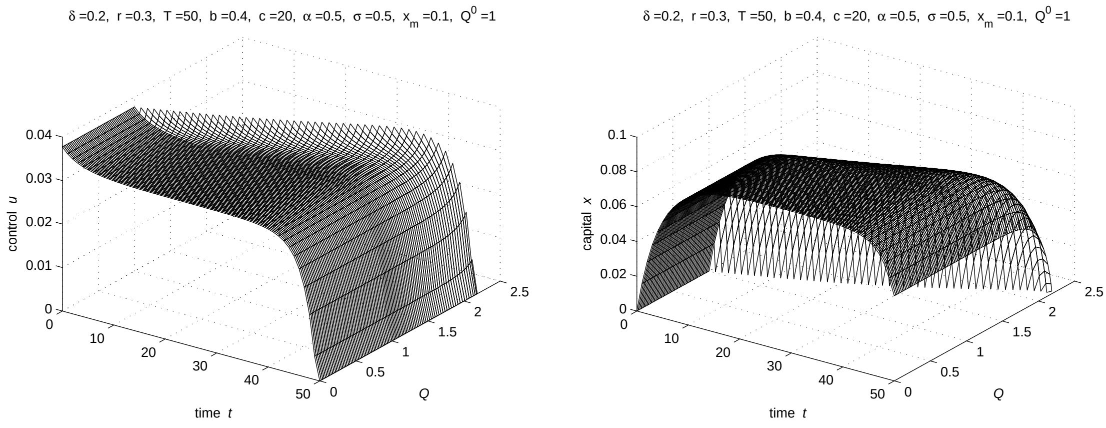

Figure 3: Optimal investments (left) and the corresponding capital stock (right) for nonlinear production function $\sqrt{x+0.1}$

Appendix

Here we prove the variational representation (10) under the conditions of Section 3 and with the notational convention made there. Consider

$$\begin{aligned}\Delta J \quad &:= \quad J(u^\sigma) - J(u) = \int_0^T \int_0^{Q^\sigma(t)} L(t,q,x^\sigma(t,q),Q^\sigma(t),y^\sigma(t),u^\sigma(t,q))\,\mathrm{d}q\,\mathrm{d}t \\ &\quad - \int_0^T \int_0^{Q(t)} L(t,q,x(t,q),Q(t),y(t),u(t,q))\,\mathrm{d}q\,\mathrm{d}t \\ &= \quad \int_0^T \int_0^{Q^\sigma(t)} [L + L_x \Delta x + L_Q \Delta Q + L_y \Delta y + \Delta_u L]\,\mathrm{d}q\,\mathrm{d}t - \int_0^T \int_0^{Q(t)} L\,\mathrm{d}q\,\mathrm{d}t + o(\sigma).\end{aligned}$$

The same convention is systematically used below. The above equality follows in a standard way from Assumption (ii) and (8), (9)[4]. Using again (8), (9) again, we obtain

$$\Delta J = \int_0^T \int_{Q(t)}^{Q^\sigma(t)} L\,\mathrm{d}q\,\mathrm{d}t + \int_0^T \int_0^{Q(t)} [L_x \Delta x + L_Q \Delta Q + L_y \Delta y + \Delta_u L]\,\mathrm{d}q\,\mathrm{d}t + o(\sigma). \tag{41}$$

Let $\lambda : D \mapsto \mathbf{R}^n$ be absolutely continuous in t for a.e. $q \in [0,\bar{Q}]$, $\dot{\lambda} \in L_\infty(D)$ and $\lambda(T,q) = 0$, $q \in [0,\bar{Q}]$. We remind that λ is considered as a row-vector.

Consider the value

$$\int_0^T \int_0^{Q^\sigma(t)} \lambda(t,q)[\dot{x}^\sigma(t,q) - \dot{x}^b(t)]\,\mathrm{d}q\,\mathrm{d}t - \int_0^T \int_0^{Q(t)} \lambda(t,q)[\dot{x}(t,q) - \dot{x}^b(t)]\,\mathrm{d}q\,\mathrm{d}t$$

[4] It is to be mentioned that $o(\sigma)$ is not necessarily of second order with respect to σ. It can be of order 3/2.

$$= \int_0^T \int_0^{Q^\sigma(t)} \lambda(t,q)[\dot{x}^\sigma(t,q) - \dot{x}^b(t,q)]\,\mathrm{d}q\,\mathrm{d}t - \int_0^T \int_0^{Q(t)} \lambda(t,q)[\dot{x}(t,q) - \dot{x}^b(t,q)]\,\mathrm{d}q\,\mathrm{d}t$$

$$= \int_0^{Q^\sigma(T)} \int_{\theta^\sigma(q)}^T \lambda[\dot{x}^\sigma - \dot{x}^b]\,\mathrm{d}t\,\mathrm{d}q - \int_0^{Q(T)} \int_{\theta(q)}^T \lambda[\dot{x} - \dot{x}^b]\,\mathrm{d}t\,\mathrm{d}q$$

$$= -\int_0^{Q^\sigma(T)} \int_{\theta^\sigma(q)}^T \dot{\lambda}[x^\sigma - x^b]\,\mathrm{d}t\,\mathrm{d}q + \int_0^{Q(T)} \int_{\theta(q)}^T \dot{\lambda}[x - x^b]\,\mathrm{d}t\,\mathrm{d}q,$$

where we use the side conditions for x, x^σ and λ (see also (6) and (7)). Changing back the order of integration and denoting $f^\sigma(t,q) = f(t,q,x^\sigma,Q^\sigma,y^\sigma,u^\sigma)$ we obtain from the above equalities that

$$\int_0^T \int_0^{Q^\sigma(t)} \lambda(t,q)[f^\sigma(t,q) - \dot{x}^b(t)]\,\mathrm{d}q\,\mathrm{d}t - \int_0^T \int_0^{Q(t)} \lambda(t,q)[f(t,q) - \dot{x}^b(t)]\,\mathrm{d}q\,\mathrm{d}t$$

$$= -\int_0^T \int_0^{Q^\sigma(t)} \dot{\lambda}(t,q)[x^\sigma(t,q) - x^b(t,q)]\,\mathrm{d}q\,\mathrm{d}t + \int_0^T \int_0^{Q(t)} \dot{\lambda}(t,q)[x(t,q) - x^b(t,q)]\,\mathrm{d}q\,\mathrm{d}t.$$

In a similar way as for L and using the same notational convention we represent the left-hand side as

$$\int_0^T \int_0^{Q(t)} \lambda[f_x\Delta x + f_Q\Delta Q + f_y\Delta y + \Delta_u f]\,\mathrm{d}q\,\mathrm{d}t + \int_0^T \int_{Q(t)}^{Q^\sigma(t)} \lambda[f - \dot{x}^b(t)]\,\mathrm{d}q\,\mathrm{d}t + o(\sigma). \tag{42}$$

The right-hand side can be rewritten as

$$-\int_0^T \int_0^{Q(t)} \dot{\lambda}\Delta x\,\mathrm{d}q\,\mathrm{d}t - \int_0^T \int_{Q(t)}^{Q^\sigma(t)} \dot{\lambda}(t,q)[x^\sigma(t,q) - x^b(t)]\,\mathrm{d}q\,\mathrm{d}t.$$

Now we shall argue that the second term in the last expression is $o(\sigma)$. We estimate this term by

$$\int_0^T \left| \int_{Q(t)}^{Q^\sigma(t)} |x^\sigma(t,q) - x^b(t)|\,\mathrm{d}q \right| \mathrm{d}t\,\|\dot{\lambda}\|_{L_\infty} \le \int_0^T \left| \int_{Q(t)}^{Q^\sigma(t)} |x^\sigma(t,q) - x^\sigma(\theta^\sigma(q),q)|\,\mathrm{d}q \right| \mathrm{d}t\,\|\dot{\lambda}\|_{L_\infty}$$

(43)

$$\int_0^T \left| \int_{Q(t)}^{Q^\sigma(t)} |x^\sigma(\theta^\sigma(q),q) - x^b(\theta^\sigma(q))|\,\mathrm{d}q \right| \mathrm{d}t\,\|\dot{\lambda}\|_{L_\infty} + \int_0^T \left| \int_{Q(t)}^{Q^\sigma(t)} |x^b(\theta^\sigma(q)) - x^b(t)|\,\mathrm{d}q \right| \mathrm{d}t\,\|\dot{\lambda}\|_{L_\infty}.$$

The second term in the right-hand side equals zero due to the definition of x. Let us consider the first term. Since for a.e. q the function $x(\cdot,q)$ is Lipschitz with a constant C (independent of q) we have

$$\int_0^T \left| \int_{Q(t)}^{Q^\sigma(t)} |x^\sigma(t,q) - x(\theta^\sigma(q),q)|\,\mathrm{d}q \right| \mathrm{d}t \le C \int_0^T \left| \int_{Q(t)}^{Q^\sigma(t)} |t - \theta^\sigma(q)|\,\mathrm{d}q \right| \mathrm{d}t$$

$$= C \int_0^T \left| \int_{Q(t)}^{Q^\sigma(t)} |\theta^\sigma(Q^\sigma(t)) - \theta^\sigma(q)|\,\mathrm{d}q \right| \mathrm{d}t \le \frac{C}{\alpha_0} \int_0^T \left| \int_{Q(t)}^{Q^\sigma(t)} |Q^\sigma(t) - q|\,\mathrm{d}q \right| \mathrm{d}t \le \frac{C}{2\alpha_0} Tc^2\sigma^2,$$

where we have used (8) in the last inequality. The same argument applies to the last term in (43) since x^b is Lipschitz.

Thus we obtain from (42) the equality

$$\int_0^T \int_0^{Q(t)} [\dot{\lambda}\Delta x + \lambda(f_x \Delta x + f_Q \Delta Q + f_y \Delta y + \Delta_u f)]\, \mathrm{d}q\, \mathrm{d}t + \int_0^T \int_{Q(t)}^{Q^\sigma(t)} \lambda[f - \dot{x}^b(t)]\, \mathrm{d}q\, \mathrm{d}t = o(\sigma). \quad (44)$$

Now we introduce an absolutely continuous function $\mu : [0,T] \mapsto \mathbf{R}$ satisfying $\mu(T) = 0$. By the same argument that we used in obtaining (44) we obtain the equality

$$\int_0^T [\mu(g_Q \Delta Q + g_y \Delta y) + \dot{\mu}\Delta Q]\, \mathrm{d}t = o(\sigma). \quad (45)$$

As before, $o(\sigma) \approx \sigma^{3/2}$.

Finally we introduce an $(1 \times m)$-dimensional function $\nu \in L_\infty(0,T)$ and obtain from (3) by the same argument as before that

$$-\int_0^T \nu \Delta y\, \mathrm{d}t + \int_0^T \int_0^{Q(t)} \nu \Delta_u h\, \mathrm{d}q\, \mathrm{d}t + \int_0^T \int_{Q(t)}^{Q^\sigma(t)} \nu h\, \mathrm{d}q\, \mathrm{d}t = o(\sigma). \quad (46)$$

Summing up the equalities (41), (44), (45), (46) we obtain (10).

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