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J.P. Caulkins, G. Feichtinger, D. Grass,
R.F. Hartl, P.M. Kort, A. Seidl

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Operations Research and Control Systems
Institute of Mathematical Methods in Economics
Vienna University of Technology

Research Unit ORCOS
Argentinierstraße 8/E105-4,
1040 Vienna, Austria
E-mail: orcos@eos.tuwien.ac.at

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Abstract

Since the end of the seventies Skiba points have been studied in infinite time optimal control problems with multiple steady states. At such a Skiba point the decision maker is indifferent between choosing trajectories that approach different steady states. This paper extends this theory towards free end time optimal control problems, where the decision maker collects a salvage value at the endogenous horizon date. In particular, besides operating forever, the decision maker can choose to stop operations immediately, or to operate during a finite time interval after which it stops and collects the salvage value.

This paper partly develops the new theory by analyzing a capital accumulation problem of the firm where the manager has the option to sell the firm at any time. This problem is relevant especially in the high tech sector.

1 Introduction

Sethi (1977, 1979); Skiba (1978); Dechert and Nishimura (1983) established the existence of indifference points, or Skiba points,[1] in infinite time optimal control models with multiple steady states. At the Skiba point the decision maker is indifferent between choosing trajectories that approach different steady states.

This paper extends the notion of Skiba points to the class of optimal control problems where the decision maker can stop operations at any time and collect a salvage value. In such a framework essentially three qualitatively different solution candidates can be distinguished. Besides operating forever, the decision maker can decide to stop operations immediately and collect the salvage value, or operate for a finite time and stop afterwards. We show that Skiba points exist where the decision maker is indifferent between "operating forever" and "operating for a finite time and then stopping", or between "operating forever" and "stopping immediately".

We develop our results by studying the problem of a company where the manager has the option to sell the firm (be acquired). This problem is relevant especially with regard to high growth, technology oriented

[1] Skiba points are also known in literature as DNSS points referring to the authors of the previously mentioned contributions.

companies. Some technology companies remain independent, and some grow to become very profitable (e.g. Google, whose founders are now billionaires, Apple, Facebook, and Adobe Systems). Yet, for every Google, there are many other startups that make their founders rich in a different way, by allowing the founders to grow the technology for a time, but then be acquired voluntarily. Often that technology is worth more when combined with the acquiring company's other assets (including brand recognition and marketing reach), than it is for the startup's own operations and products. Examples might be Hotmail, Android, YouTube and PayPal (acquired by Microsoft, Google and eBay).

While there are bookshelves of capital accumulation models (e.g., among many others, Jorgenson, 1963; Gould, 1968; Barucci, 1998; Feichtinger et al., 2006), to our knowledge none of them focus on the possibility that it could be optimal to sell the firm after some time. Therefore, our aim is to extend this literature by explicitly taking into account the possibility that a startup may want to be acquired by another (usually larger) firm.

After having analyzed this model, we return to a general formulation, where we focus on Skiba points where the decision maker is indifferent between the policies "operate forever" or "operate for a finite time and collect the salvage value at the endogenous horizon date". One might expect that in a one-dimensional model, if the state starts to increase at the Skiba point, then for the other policy the state must start to decrease. We prove that is true – almost always. There is a hairline case in which both policies can move away from the Skiba point in the same direction, but in that case they apply the same control and all points traversed are Skiba points; the two alternate solutions move from one Skiba point to another Skiba point until reaching a point where one is indifferent between continuing to operate forever and ceasing operation immediately.

The remainder of the paper is organized as follows. The next section presents the capital accumulation model of the firm where the manager has the option to sell it at any time. Section 3 analyzes this model. In Section 4 we consider different scenarios, where in each scenario we show when the firm should operate forever as a separate entity, when it should be sold immediately, or when it should remain active during a finite time period after which it should be sold. Section 5 shows for a general framework that whenever Skiba points exist where the firm is indifferent between a policy of operating forever or being sold at a finite point of time, it happens that when the state variable starts by growing (contracting) from there for the policy of staying alive forever, exactly the opposite happens when the decision maker chooses to stop operations in finite time. Section 6 concludes.

2 The Model of the Firm

We start out from a concave revenue function version of the capital accumulation model by Barucci (1998) and Hartl and Kort (2000, 2004). We extend this model by introducing the option that the start-up firm can be taken over by another, established company. This other company has to pay a takeover price that equals the value of the startup firm, which consists of the intellectual property and the technological capital stock. The Economist (2005) estimated that up to 75% of the value of US public companies is now based on their intellectual property.

The firm starts with some initial assets which we model as a fixed constant, m, as they are not the focus of the dynamics under investigation here. In addition it has a capital stock, K, that it exploits to produce revenue and in which it can invest to expand that capacity. Since our analysis is inspired by high-tech companies, we will think of this as intellectual property and call the state variable a technological capital stock. Denoting the value of one unit of this capital stock, by s, the takeover price $S(K)$ is

$$S(K) = sK + m.$$

The company chooses the point in time (if any) at which it will be taken over. At the takeover time it receives the value $S(K)$, so it must factor this option into its investment decisions. To do so the startup company has to solve a free end time problem where the takeover price $S(K)$ is the salvage value. If T is the moment at which the startup company is taken over, the dynamic model of the firm becomes

$$\max_{I,T} \int_0^T \mathrm{e}^{-rt} \left(aK - \frac{b}{2}K^2 - cI - \frac{I^2}{2} \right) \mathrm{d}t + \mathrm{e}^{-rT} \left(sK(T) + m \right),$$
$$\dot{K} = I - \delta K, \quad K(0) = K_0,$$
$$I \geq 0.$$

The model can be explained as follows. Over time the startup company determines the investment, I, such that the value of the firm is maximized. Besides the discounted (with rate r) takeover value at the free end time T, this value also consists of the discounted cash flow stream over the life time of the firm. The cash inflow is provided by the revenue, $aK - \frac{b}{2}K^2$ with a and b being positive constants. Such a quadratic expression with negative second order term could arise, for instance, when the production process is constant returns to scale and the demand function is linear. The cash outflow consists of the cost of investment, cI where c is the unit acquisition price, and the adjustment costs $\frac{I^2}{2}$. The capital stock decreases with depreciation with δ being the constant depreciation rate. The initial capital stock can have any non-negative value. Small values make sense when the only initial funds available are provided by the founders themselves, and larger values make sense if their ideas have been able to attract venture capital.

Two candidate takeover policies can be detected. First, due to the fact that the initial intellectual property is (partly) independent of the capital stock, takeover may be optimal when the capital stock is small. This may occur, for instance, when initially the owners of the startup firm have to come up with financial funds themselves without a venture capital firm. Then it may be optimal not to startup the firm, but instead the owners should bring their intellectual property directly to the (potentially) acquiring firm.

Hence, instead of starting up their own firm they should immediately sell their technology to the (potentially) acquiring firm. Against this policy speaks the fact that marginal revenue is large for low capital stock values. For this reason it may be beneficial for the owners to begin operating the startup.

Second, the startup might decide to build up a large capital stock before being acquired. Since marginal revenue is decreasing in the capital stock while the takeover price grows proportionally with the capital stock, at some point it may be better to be taken over by the other firm. The reason that the takeover value is linearly increasing in capital stock while revenue is concave, is that the acquiring firm benefits not only from the increased production generated by the acquired technological capital, but also from the fact that the acquired intellectual property adds value to the other operations of the conglomerate.

3 Analysis of the Model of the Firm

To analyze the model we apply Pontryagin's maximum principle (see, e.g., Feichtinger and Hartl (1986); Grass et al. (2008)). The Hamiltonian is

$$H = aK - \frac{b}{2}K^2 - cI - \frac{I^2}{2} + \lambda\left(I - \delta K\right),$$

and the optimality condition for an interior control is

$$\lambda = c + I. \tag{1}$$

The adjoint equation is

$$\dot{\lambda} = \left(r + \delta\right)\lambda - a + bK,$$

with, in the case that T is finite, the transversality condition

$$\lambda\left(T\right) = S'\left(K\left(T\right)\right) = s \Rightarrow I\left(T\right) = s - c. \tag{2}$$

It follows that the canonical system in the state and the control is given by

$$\begin{aligned} \dot{K} &= I - \delta K, \\ \dot{I} &= \left(r + \delta\right)c - a + \left(r + \delta\right)I + bK. \end{aligned}$$

The isoclines are the straight lines

$$\begin{aligned} \dot{K} = 0 \qquad &\Leftrightarrow \qquad I = \delta K, \\ \dot{I} = 0 \qquad &\Leftrightarrow \qquad I = -\frac{\left(r + \delta\right)c - a + bK}{r + \delta}. \end{aligned}$$

We conclude that there is a unique steady state

$$\hat{K} = \frac{a - c\left(r + \delta\right)}{b + \delta\left(r + \delta\right)}, \quad \hat{I} = \delta\frac{a - c\left(r + \delta\right)}{b + \delta\left(r + \delta\right)}. \tag{3}$$

The Jacobian is

$$J=\begin{pmatrix}-\delta & 1\\ b & r+\delta\end{pmatrix},\qquad \det J=-\delta\left(r+\delta\right)-b<0,$$

which is why the steady state is a saddle point.

Because the model is linear-quadratic, the policy functions and the value function can be computed analytically. However, these formulas are quite messy and do not provide much insight. Hence, we just provide the elements needed in the phase plane analysis:

From

$$\begin{pmatrix}K\\ I\end{pmatrix}=\begin{pmatrix}\hat{K}\\ \hat{I}\end{pmatrix}+x\begin{pmatrix}1\\ \xi_1+\delta\end{pmatrix},$$

where x and ξ_1 are the corresponding eigenvector and eigenvalue of the Jacobian (see Appendix), respectively, we derive that also the stable path (saddle point path) is a straight line

$$I=\delta\frac{a-c\left(r+\delta\right)}{b+\delta\left(r+\delta\right)}+\left(K-\frac{a-c\left(r+\delta\right)}{b+\delta\left(r+\delta\right)}\right)\left(\frac{r}{2}-\sqrt{\left(\frac{r}{2}+\delta\right)^2+b}+\delta\right), \tag{4}$$

and so is the unstable path:

$$I=\delta\frac{a-c\left(r+\delta\right)}{b+\delta\left(r+\delta\right)}+\left(K-\frac{a-c\left(r+\delta\right)}{b+\delta\left(r+\delta\right)}\right)\left(\frac{r}{2}+\sqrt{\left(\frac{r}{2}+\delta\right)^2+b}+\delta\right). \tag{5}$$

While the stable path (4) is downward sloping, the latter is upward sloping.

In case the only option of the firm were to stay in business forever (i.e. $T=\infty$) then the optimal solution would consist of converging towards the equilibrium $\hat{K}$ and $\hat{I}$ by choosing the policy function (4). Only for values where this stable path would yield negative values for the investment rate, would $I=0$ be chosen.

In case the firm has the option to sell earlier (i.e. T finite) the optimal solutions will, as usual, be represented by hyperbola like curves in the (K,I) phase plane that have the stable and the unstable path as asymptotes. It remains to determine, which of these hyperbola like curves must be chosen. One condition for determining this is the transversality condition (2), which says that the solution will end at the horizontal line $I=s-c$.

Furthermore, we can exploit the condition for the optimal takeover time T, which is

$$H\left(T\right)=rS\left(K\left(T\right)\right)\quad\text{if}\quad T\text{ is finite and positive,} \tag{6}$$

$$H\left(0\right)<rS\left(K\left(0\right)\right)\quad\text{if}\quad T=0\text{ is optimal,} \tag{7}$$

$$\text{if }H\left(t\right)>rS\left(K\left(t\right)\right)\text{ for all }t\text{ then }T=\infty. \tag{8}$$

The left hand side of (6) can be evaluated by substituting the expressions (2) for I and λ into the Hamiltonian:

$$H\left(T\right)=\frac{1}{2}\left(c-s\right)^2+\frac{K\left(T\right)}{2}\left(2a-2s\delta-bK\left(T\right)\right). \tag{9}$$

The right hand side of (6) is

$$rS\left(K\left(T\right)\right) = r\left(sK\left(T\right) + m\right) \tag{10}$$

Assuming a finite optimal $T > 0$, equation (6) can be solved for $K\left(T\right)$ to yield

$$K_1\left(T\right) = \frac{1}{b}\left(a - s\left(r + \delta\right) + \sqrt{\left(a - s\left(r+\delta\right)\right)^2 + b\left(c - s\right)^2 - 2bmr}\right), \tag{11}$$

$$K_2\left(T\right) = \frac{1}{b}\left(a - s\left(r + \delta\right) - \sqrt{\left(a - s\left(r+\delta\right)\right)^2 + b\left(c - s\right)^2 - 2bmr}\right). \tag{12}$$

If a solution with finite optimal free end time T exists, i.e. at time T the firm will be sold, the corresponding trajectories in the (K, I) phase plane will end at one of the two candidate points given by the capital stocks K_1 and K_2 from (11) and (12) and the terminal investment rate given by (2).

Now that the solution of the finite and infinite horizon problems are - in principle - established, we can ask for which initial capital stock will each one of the two options ("sell" or "stay in business forever") be optimal. This is done by comparing the two value functions. In the next section, we will provide interesting and surprising answers to this question by performing some numerical experiments. But first, we derive some general results.

From a qualitative point of view, three possible solutions can be distinguished. First, it may be optimal for the firm to never being taken over. In that case expression (8) holds. Second, the firm can be taken over immediately, in which case expression (7) holds. Third, it may be optimal for the firm to be active during a time period of finite length after which it will be taken over. This happens at time T with a capital stock satisfying expression (11).

Proposition 1. *If K_0 is greater than $K(T)$ satisfying* (11) , *the firm sells either immediately (i.e. $T = 0$) or never sells.*

Proof. This follows from (7) . Note that expression (10) is linearly increasing in K, while $H\left(T\right)$ from (9) is concave in K and less steep in the intersection point with (10). ∎

Proposition 2. *In the scenario*

$$rm < \frac{1}{2}\left(c - s\right)^2,$$

selling immediately can never be optimal when

$$K < K_2(T). \tag{13}$$

Proof. It follows from (9) and (10) that $H\left(T\right)$ exceeds $rS\left(T\right)$ if the inequality (13) holds. ∎

Proposition 3. *If*

$$\left(a - s\left(r+\delta\right)\right)^2 + b\left(c - s\right)^2 < 2bmr, \tag{14}$$

the firm is taken over immediately.

Proof. In this scenario $H(T)$ and $rS(K(T))$ never intersect. To see this combine (14) with (11). Now it is easy to see that

$$H(T) < rS(K(T))$$

for all values of K. ∎

4 Economic Results

We will now investigate which policy ("sell" or "stay in business forever") is optimal for each initial capital stock and show that the result is quite complex and is non-monotonic in the initial capital endowment. For this we perform numerical analysis using the parameter values

$$a = 100, b = 1, c = 100, r = .1, \delta = .2.$$

For the steady state we obtain that

$$\hat{K} = 66.04, \hat{I} = 13.21, \hat{\lambda} = 113.21.$$

Note that these values (as well as isoclines and stable and unstable paths) are independent of the salvage value function. By varying this salvage value function, we can generate different behavior of the solution of the finite horizon problem and its end point given by (2) and (6).

4.1 Investing in One's Technology to Increase Acquisition Value

Parameter values $s = 150$ and $m = 18000$ yield what might be thought of as the classic choice facing the owner of a high-tech startup. If the startup's new technology is really good, specifically if the initial stock of intellectual property is larger than or equal to $K(0) = 107.11$, then the firm will sell itself immediately. That makes sense because the acquiring firm can exploit the technology throughout its presumably larger product line and geographic footprint; the technology is too valuable to use it only within the scope of a relatively small startup firm's operations.

If the technology is not that good, then the firm will remain independent and exploit the technology itself in ongoing operations, adjusting its investments to make the capital stock approach the saddle point equilibrium level of 66.04.

It is not the case that if it is optimal to be acquired, the acquisition should happen immediately. If the value of the startup's other assets were modestly higher (increasing from $m = 18000$ to $m = 19500$) that understandably increases the range of initial capital stocks for which it is optimal to sell the firm, because the startup's value in acquisition is higher. But if the startup knows it will be acquired and its technology is pretty good but not terrific, then the startup should operate independently for a time so it can invest in

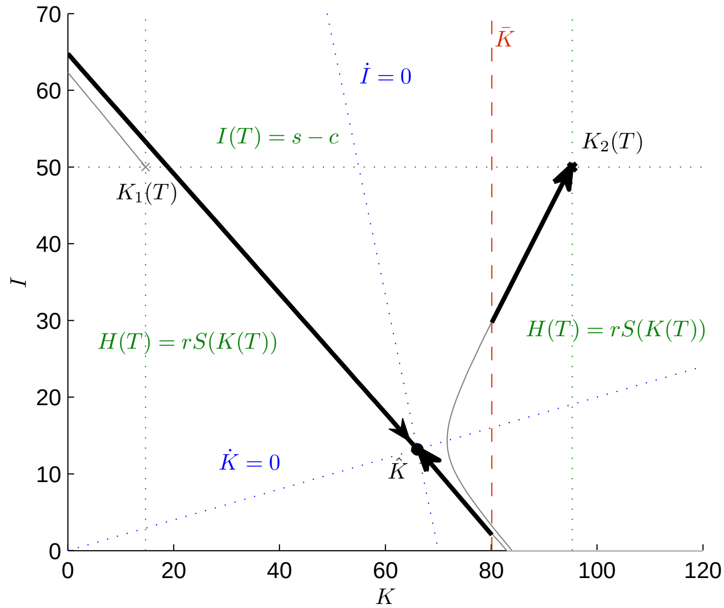

Figure 1: Phase portrait $s = 150$, $m = 19500$. There are two potential end points $K_1(T)$ and $K_2(T)$ for the free end time problem, where the conditions $H(T) = rS(K(T))$ and $I(T) = s - c$ are fulfilled. $\hat{K}$ denotes the steady state, where the two isoclines $\dot{K} = 0$ and $\dot{I}$ intersect. The bold line depicts the optimal solution. $\bar{K}$ denotes the Skiba point.

improving that technology and, hence, its acquisition price. Figure 1 shows the phase portrait for this more interesting case.

Now the minimum level of technology necessary to make immediate sale optimal has decreased to

$$K_2(T) = 95.31.$$

When the initial capital stock is below this but not too far below, specifically such that $K_0 \in \left(\bar{K}, K_2(T)\right)$, then the startup will invest to improve its technology before being acquired.

Figure 2 reveals that for smaller values of the initial capital stock the firm prefers to operate forever as a separate entity. Growing large is too expensive and selling is not a profitable option, because its value independent of the capital stock (m) is too low. Exactly at the Skiba point $K_0 = \bar{K}$, which has a value around eighty, the firm is indifferent between (1) operating forever by staying small and actually letting its capital stock decay a bit or (2) investing to grow its intellectual capital, followed by selling the firm at time T.

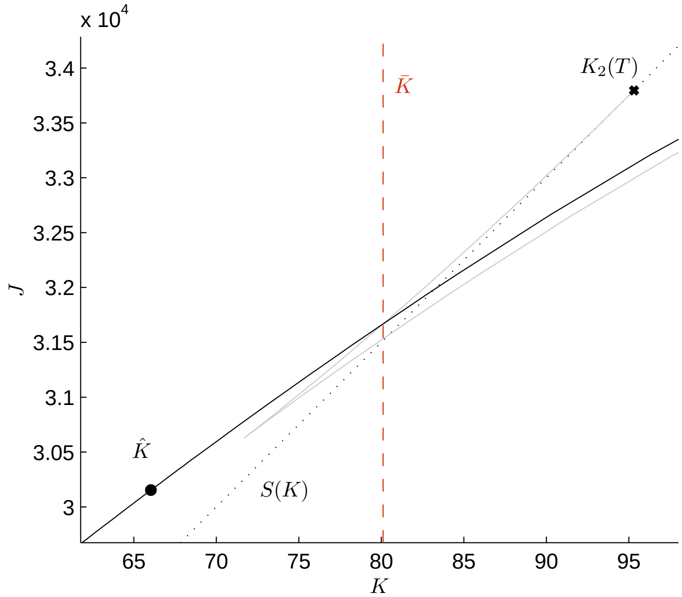

Figure 2: Value of the objective function for $s = 150$, $m = 19500$. The black line shows the objective value in case a strategy over infinite time horizon is applied, while the gray line shows the same for the finite time horizon case. The dotted line depicts the salvage value.

4.2 Milking In-House Technology Before Ceasing Independent Operations

As just discussed, the model can produce what might be thought of the classic result, but there are other possibilities.

If the external world values the startup firm overall, e.g., for its employees, but not so much for its technology per se (i.e., if the salvage value coefficient s is smaller but m is larger), then the mirror image strategy is optimal. (See Figure 3.) If the initial technology stock is large enough ($K(0) > \bar{K} = 63.06$), the startup should exploit that technology internally by staying in business indefinitely. However, if the technology stock is too small for that to be appealing, the best option is to sell. The sale could come immediately (if the initial technology endowment is very poor, i.e., ($K(0) < K_1(T) = 57.2$) or after milking that technology to produce operating profits for a time but not investing aggressively enough to prevent decay in that technology (if $K_1(T) < K_0 < \bar{K}$).

4.3 Optimal Strategy Depends Non-Monotonically on the Technology's Value

One might expect that the optimal strategy is monotonic in the initial capital stock. That is, one might expect the rule to be, if the initial capital stock exceeds some threshold, then the firm should sell and otherwise stay in business or vice versa. But that turns out not to be correct.

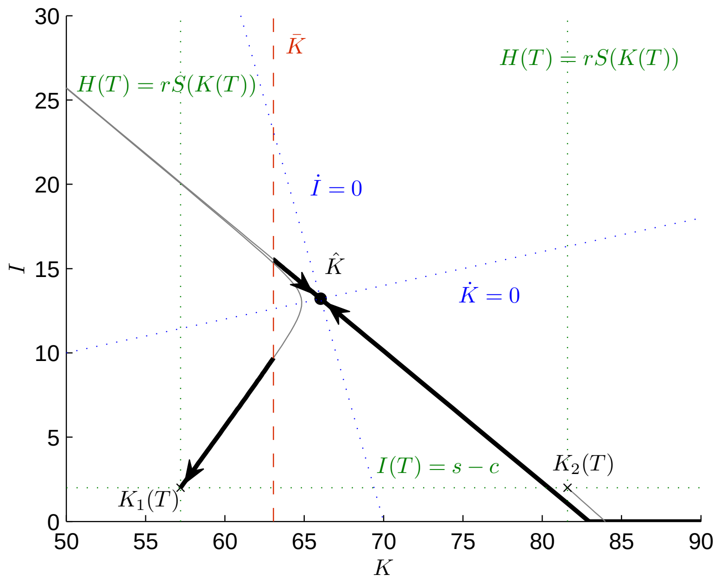

Figure 3: Phase portrait $s = 102$, $m = 23358$

If the valuation of the capital technology (s) and the valuation of the startup's other assets (m) are both intermediate between the two cases just discussed, then it can be optimal to sell if the initial capital stock is either too low or too high, but to operate indefinitely for intermediate values of the initial capital stock. For example, as Figure 4 shows, if $s = 120$ and $m = 21600$ then it is optimal to remain in business indefinitely if $K_0 \in (16.32, 97.72)$, but otherwise to sell immediately.

Tweaking the parameters slightly to $s = 124$ and $m = 21886$ yields an even more interesting result. The solution is qualitatively the same as in the previous case, except that now an interval for the initial capital stock levels exists at which the firm operates during a finite time period and then sells the firm. In particular, for $\bar{K}_2 < K_0 < K_2(T) = 74.74$, the firm increases its capital stock until it reaches $K_2(T)$. Then it is optimal to sell the firm.

5 Skiba points in a Free End Time Problem

This section focuses on a characterization of the optimal paths in one state free end time problems, specifically when a Skiba point exists, where the firm is indifferent between "operating forever" and "operating during a finite time interval after which the firm is sold". In Figures 1, 3, and 5 we see that at the Skiba point it holds that if the capital stock starts growing under one policy, it starts by shrinking for the other policy. This section shows that this result not only holds in our model, but that it can be proved in a general framework. Because of its generality we adopt a more general notation here.

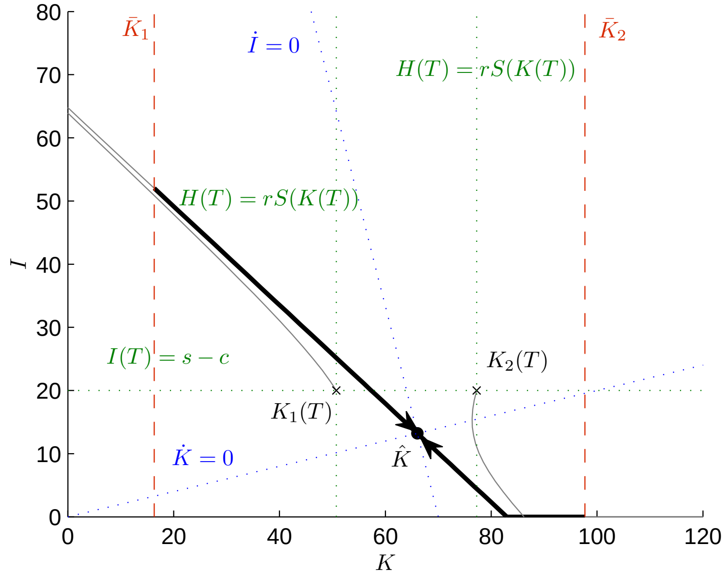

Figure 4: Phase portrait $s = 120$, $m = 21600$

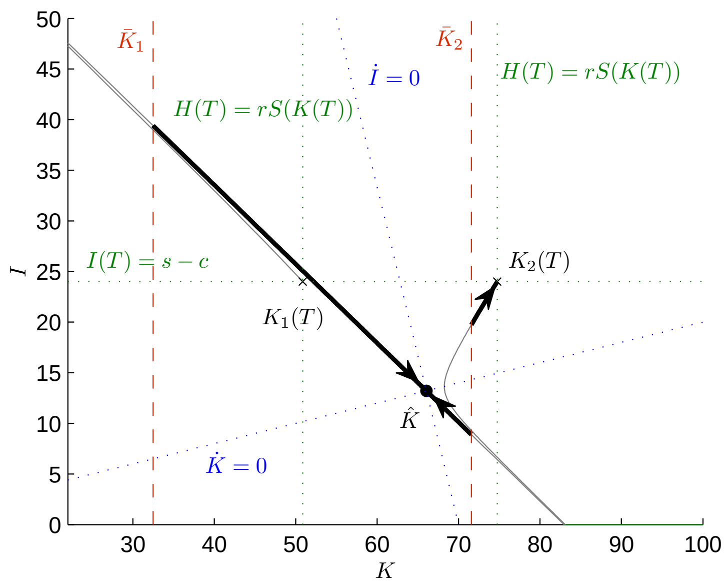

Figure 5: Phase portrait $s = 124$, $m = 21886$

Consider a discounted autonomous optimal control problem with state $x \in \mathbb{R}$ and control $u \in \mathbb{R}^m$

$$V(u(\cdot), x_0, [t_0, T\rangle) := \int_{t_0}^{T} \mathrm{e}^{-rt} F(x(t), u(t))\,\mathrm{d}t + \mathrm{e}^{-rT} S(x(T)) \tag{15a}$$

$$\text{s.t.} \quad \dot{x}(t) = f(x(t), u(t)), \tag{15b}$$

$$u(t) \in \mathcal{U}(x(t)), \tag{15c}$$

$$x(t_0) = x_0 \in \mathbb{R}, \tag{15d}$$

with

$$[t_0, T\rangle := \begin{cases} [t_0, T) & T = \infty, \\ [t_0, T] & T < \infty. \end{cases}$$

A problem satisfying (15) with

$$J_{t_0}(x_0, T) := \max_{u(\cdot)} V(u(\cdot), x_0, [t_0, T\rangle) \tag{16}$$

is called an optimal control problem with fixed end time T. For $T = \infty$ the problem is called an infinite time horizon problem. If the time horizon is also a decision variable, i.e.

$$J_{t_0}(x_0) := \max_{u(\cdot), T} V(u(\cdot), x_0, [t_0, T\rangle), \tag{17}$$

the problem is called a free end time problem. For $t_0 = 0$ the subscript in J_{t_0} is omitted.

Assumption 1. *For every $x_0 \in \mathbb{R}$ there exists an optimal solution and for an admissible path $x(\cdot)$, $|x(t)| \leq C < \infty$, i.e., $x(\cdot)$ is bounded.*

Assumption 2. *The controls $u(\cdot) \in C_p^0([0, \infty))$ are piecewise continuous and at a point τ of discontinuity they are right continuous*

$$u(\tau) := \lim_{t \to \tau^+} u(t).$$

This assumption will only be used in Proposition 4 and is only stated for the ease of presentation. Without this assumption the statement would still hold almost everywhere.

Definition 1. *A solution $(x^*(\cdot), u^*(\cdot))$ of the optimal control problem with fixed end time, or $(x^*(\cdot), u^*(\cdot), T^*)$ for the free end time problem, respectively, is called* time invariant *iff the time shifted solution*

$$(x_\tau(t), u_\tau(t)) := (x^*(t + \tau), u^*(t + \tau)), \quad t_0 \leq \tau \leq T^*$$

with

$$T_\tau := \begin{cases} T^* - \tau & T^* < \infty \\ \infty & T^* = \infty. \end{cases}$$

is an optimal solution with $x_\tau(t_0) = x^*(t_0 + \tau)$.

Definition 2. *Two solutions* $(x_i^*(\cdot), u_i^*(\cdot)),\ i = 1,2$ *are called* different *iff there exists an interval* $I \subset [t_0, T],\ I \neq \emptyset$, *such that* $x_1(t) \neq x_2(t)$ *for all* $t \in I$. *In case of a free end time problem* $(x_i^*(\cdot), u_i^*(\cdot), T_i^*),\ i = 1,2$ *are called* different *iff* $(x_i^*(\cdot), u_i^*(\cdot)),\ i = 1,2$ *are different or if* $T_1^* \neq T_2^*$.

For a time invariant solution the following properties hold.

Lemma 1. *a) Let* $(x^*(\cdot), u^*(\cdot), T^*)$ *be a time invariant solution and* $0 \leq \tau \leq T^*$ *with* $x^*(\tau) = \xi$ *then*

$$J_\tau(\xi) = \mathrm{e}^{-r\tau}\, J(\xi). \tag{18}$$

b) Let $(x_i^*(\cdot), u_i^*(\cdot), T_i^*),\ i = 1,2$, *be two different time invariant optimal solutions, for which the overlap region* M *in the state space is non-empty:*

$$M := \{x_1^*(t) :\ 0 < t \leq T_1^*\} \cap \{x_2^*(t) :\ 0 < t \leq T_2^*\} \neq \emptyset,$$

then for every $\xi \in M$ *with* $x_1^*(t_1) = x_2^*(t_2) = \xi$ *we find*

$$J_{t_1}(\xi) = \mathrm{e}^{-r(t_1 - t_2)}\, J_{t_2}(\xi). \tag{19}$$

And the path

$$(\tilde{x}(t), \tilde{u}(t)) := \begin{cases} (x_1^*(t), u_1^*(t)) & 0 \leq t \leq t_1, \\ (x_2^*(t - t_1 + t_2), u_2^*(t - t_1 + t_2)) & t_1 < t \leq T_2^* + t_1 - t_2 \end{cases}$$

is an optimal solution, which is different from $x_1^*(\cdot)$ *and* $x_2^*(\cdot)$.

Proof. To prove (18) we note that for the time transformation $s = t - \tau$ we find

$$\begin{aligned} J_\tau(\xi) &= \int_\tau^{T^*} \mathrm{e}^{-rt}\, F(x^*(t), u^*(t))\,\mathrm{d}t + \mathrm{e}^{-rT^*}\, S(x^*(T^*)) \\ &= \int_0^{T^*-\tau} \mathrm{e}^{-r(s+\tau)}\, F(x^*(s+\tau), u^*(s+\tau))\,\mathrm{d}s + \mathrm{e}^{-r(T^*-\tau)}\, S(x^*(T^*)) \\ &= \mathrm{e}^{-r\tau} \left(\int_0^{T^*-\tau} \mathrm{e}^{-rs}\, F(x_\tau^*(s), u_\tau^*(s))\,\mathrm{d}s + \mathrm{e}^{-r(T^*-\tau)}\, S(x_\tau^*(T^*-\tau)) \right) \\ &= \mathrm{e}^{-r\tau}\, J(\xi), \end{aligned}$$

where for the last equality we used the optimality of the time shifted path.

The identity (19) follows immediately by applying (18) to the time shifted solutions with $t_i,\ i = 1,2$, yielding

$$J_{t_1}(\xi) = \mathrm{e}^{-rt_1}\, J(\xi) = \mathrm{e}^{-rt_1}\,\mathrm{e}^{rt_2}\, J_{t_2}(\xi) = \mathrm{e}^{-r(t_1-t_2)}\, J_{t_2}(\xi).$$

For the last part we consider the integral

$$
\begin{aligned}
V(\tilde{u}(\cdot), x_0, [0, T_2^*+t_1-t_2\rangle) &= \int_0^{T_2^*+t_1-t_2} \mathrm{e}^{-rt}\, F(\tilde{x}(t), \tilde{u}(t))\,\mathrm{d}t + \mathrm{e}^{-rT_2^*}\, S(\tilde{x}(T_2^*+t_1-t_2)) \\
&= \int_0^{t_1} \mathrm{e}^{-rt}\, F(x_1^*(t), u_1^*(t))\,\mathrm{d}t + \\
&\qquad \int_{t_1}^{T_2^*+t_1-t_2} \mathrm{e}^{-rt}\, F(x_2^*(t-t_1+t_2), u_2^*(t-t_1+t_2))\,\mathrm{d}t + \mathrm{e}^{-rT_2^*}\, S(x_2^*(T_2^*)) \\
&= \int_0^{t_1} \mathrm{e}^{-rt}\, F(x_1^*(t), u_1^*(t))\,\mathrm{d}t + \\
&\qquad \mathrm{e}^{-rt_1} \int_0^{T_2^*-t_2} \mathrm{e}^{-rt}\, F(x_{2,t_2}^*(s), u_{2,t_2}^*(s))\,\mathrm{d}t + \mathrm{e}^{-r(T_2^*-t_1)}\, S(x_{2,t_2}^*(T_2^*-t_2)) \\
&= \int_0^{t_1} \mathrm{e}^{-rt}\, F(x_1^*(t), u_1^*(t))\,\mathrm{d}t + \mathrm{e}^{-rt_1}\, J(\xi) = \mathrm{e}^{-r(t_1-t_2)}\, J_{t_2}(\xi) = J_{t_1}(\xi) \\
&= \int_0^{t_1} \mathrm{e}^{-rt}\, F(x_1^*(t), u_1^*(t))\,\mathrm{d}t + J_{t_1}(\xi) \\
&= \int_0^{T_1^*} \mathrm{e}^{-rt}\, F(x_1^*(t), u_1^*(t))\,\mathrm{d}t + \mathrm{e}^{-rT_1^*}\, S(x_1^*(T_2^*)) = J(x_0).
\end{aligned}
$$

The difference of $\tilde{x}(\cdot)$ follows from the construction and $t_1 > 0$, which ends the proof. ∎

Lemma 2. *An optimal solution $(x^*(\cdot), u^*(\cdot), T^*)$ of the free end time problem is time-invariant.*

Proof. Let $(x^*(\cdot), u^*(\cdot), T^*)$ with $0 < T^* < \infty$ be an optimal solution of (15) then

$$
\int_0^{T^*-\tau} \mathrm{e}^{-rt}\, F(x_\tau(t), u_\tau(t))\,\mathrm{d}t + \mathrm{e}^{-r(T^*-\tau)}\, S(x_\tau(T^*-\tau)) = \mathrm{e}^{r\tau}\left(\int_\tau^{T^*} \mathrm{e}^{-rt}\, F(x(t), u(t))\,\mathrm{d}t + \mathrm{e}^{-rT^*}\, S(x(T^*))\right).
$$

Now assume there exists a solution $(y(\cdot), v(\cdot), \bar{T})$ improving $(x_\tau(t), u_\tau(t), T^*)$ such that

$$
\int_\tau^{T^*} \mathrm{e}^{-rt}\, F(x^*(t), u^*(t))\,\mathrm{d}t + \mathrm{e}^{-rT^*}\, S(x^*(T^*)) < \int_\tau^{\bar{T}} \mathrm{e}^{-rt}\, F(y(t), v(t))\,\mathrm{d}t + \mathrm{e}^{-r\bar{T}}\, S(y(\bar{T})).
$$

Then the combined solution

$$
(\tilde{x}(t), \tilde{u}(t)) := \begin{cases} (x^*(t), u^*(t)) & 0 \le t \le \tau, \\ (y(t), v(t)) & \tau < t \le \bar{T} \end{cases}
$$

yields

$$
\int_0^{\tau} \mathrm{e}^{-rt}\, F(x^*(t), u^*(t))\,\mathrm{d}t + \int_\tau^{\bar{T}} \mathrm{e}^{-rt}\, F(y(t), v(t))\,\mathrm{d}t + \mathrm{e}^{-r\bar{T}}\, S(y(\bar{T})) > J(x^*(\cdot), u^*(\cdot)),
$$

which is a contradiction to the optimality of $(x^*(\cdot), u^*(\cdot), T^*)$. Consequently $(x_\tau(t), u_\tau(t), T^*)$ cannot be improved, proving the time invariance. An analogous argument shows the time invariance for an optimal solution with $T^* = \infty$. ∎

Lemma 3. *Let $(x^*(\cdot), u^*(\cdot), T^*)$ be a unique and time invariant optimal solution, then $x^*(\cdot)$ is monotonic.*

Proof. The case $T = \infty$ has been proved in Hartl (1987) and is an immediate consequence of Lemma 1. ∎

Proposition 4. *Let x_0 be a Skiba point for the free end time problem satisfying (15), Assumption 1 and Assumption 2. Let the two solutions $(x^i(\cdot), u^i(\cdot), T^i),\ i = f, \infty$ satisfying $T^f < \infty$ and $T^\infty = \infty$, and let $(x^\infty(\cdot), u^\infty(\cdot))$ be the only solution for the infinite time horizon problem. Then the following conditions hold:*

a) $x^\infty(\cdot)$ converges to a steady state $\hat{x}$, i.e., $\lim_{t\to\infty} x^\infty(t) = \hat{x} \in \mathbb{R}$.

b) If the solutions start in the same direction, i.e.

$$\dot{x}^f(0)\dot{x}^\infty(0) > 0, \tag{20}$$

the solutions coincide on the time interval $[0, T^f]$, i.e.

$$(x^f(t), u^f(t)) = (x^\infty(t), u^\infty(t)), \quad 0 \le t \le T^f, \tag{21}$$

and therefore every point $x^f(t),\ 0 \le t \le T^f$ is a Skiba point. Moreover,

$$x_0 \le x(T^f) < \hat{x} \quad \text{or} \quad x_0 \ge x(T^f) > \hat{x},$$

and (in case the model functions f and F are sufficiently well behaved that the maximum principle holds)

$$rS(x^\infty(t)) \begin{cases} \le H(x^\infty(t), u^\infty(t), \lambda(t)) & t \in [0, \infty) \setminus T^f, \\ = H(x^\infty(t), u^\infty(t), \lambda(t)) & t = T^f. \end{cases} \tag{22}$$

hold. If $(x^f(\cdot), u^f(\cdot), T^f)$ is the only solution with a finite time horizon, the strict inequality in (22) holds.

c) If the solutions start to move in different directions, i.e.

$$\dot{x}^f(0)\dot{x}^\infty(0) < 0, \tag{23}$$

then

$$\{x^f(t):\ 0 < t \le T^f\} \cap \{x^\infty(t):\ 0 < t < \infty\} = \emptyset. \tag{24}$$

If in a neighborhood of x_0 there exists a unique solution to the infinite time horizon problem converging to $\hat{x}$ then there exists no other Skiba point with solutions starting in different directions.

Proof. Since the path $x^\infty(\cdot)$ is monotonic (Lemma 3) and bounded (Assumption 1), point a) is proved.

For $T^f = 0$ the proposition is trivial. Therefore, we assume $T^f > 0$. W.l.o.g. we assume $x_0 < \hat{x}$. Considering the overlap region

$$M := \{x^f(t):\ 0 < t \le T^f\} \cap \{x^\infty(t):\ 0 < t < \infty\},$$

the condition $\dot{x}^f(0)\dot{x}^\infty(0) > 0$ yields $M \ne \emptyset$. Using Lemma 1 we know that for every point in M there exist at least two different optimal solutions and therefore every point in M is a Skiba point.

Next, we show that $x_0 < x^f(T^f) < \hat{x}$. Assume on the contrary that $x^f(T^f) \geq \hat{x}$, then there exists a time $\bar{t} \leq T^f$ with $x^f(\bar{t}) = \hat{x}$ and using Lemma 1 another solution for the infinite time horizon problem could be constructed, which violates its uniqueness. Thus we find

$$\{x^f(t):\ 0 < t \leq T^f\} \subset \{x^\infty(t):\ 0 < t < \infty\}.$$

If (21) does not hold there exists a time interval (t_1, t_2) with $0 \leq t_1 < t_2 \leq T^f$ where $x^f(t)$ and $x^\infty(t)$ are different. Applying once again Lemma 1 we could construct an infinite time horizon solution violating its uniqueness, proving (21).

Using (21) the necessary optimality conditions for a finite time horizon solution only have to be checked along $(x^\infty(\cdot), u^\infty(\cdot))$. Thus we immediately find $rS(x^\infty(T^f)) = H(x^\infty(T^f), u^\infty(T^f), \lambda(T^f))$. To see the inequality in (22) we note that otherwise a finite time horizon solution would dominate the infinite time horizon solution, which is a contradiction and therefore (22) has to hold.

If (24) does not hold, i.e., the intersection of both sets is not empty, then we could construct another infinite time horizon solution, violating its uniqueness. To show that x_0 is isolated, we assume to the contrary that there exists a Skiba point $\bar{x} \neq x_0$ for every neighborhood of x_0. Then we could construct another infinite time horizon solution contradicting the uniqueness assumption in the second part of point c), which finishes the proof. ∎

Remark 5. *Without the uniqueness assumption in Proposition 4 the statement holds true that for solutions satisfying*

$$\dot{x}^f(0)\dot{x}^\infty(0) > 0,$$

every point in the intersection

$$\{x^f(t):\ 0 < t \leq T^f\} \cap \{x^\infty(t):\ 0 < t < \infty\} \neq \emptyset$$

is a Skiba point.

Figures 6 and 7 show such a scenario with non-isolated indifference points, where the capital stock grows for the "operating forever" as well as the "selling in finite time" policy. Yet, this is only a hairline case created by choosing a concave salvage value function, which is tangential to the value function at one point.

6 Conclusion

The starting point of this paper is the literature on Skiba points, including the early contributions by Sethi (1977, 1979); Skiba (1978); Dechert and Nishimura (1983). This literature considers infinite time optimal control models with multiple steady states. A Skiba point is defined such that right at this point the decision maker is indifferent between choosing trajectories that approach different steady states. This paper

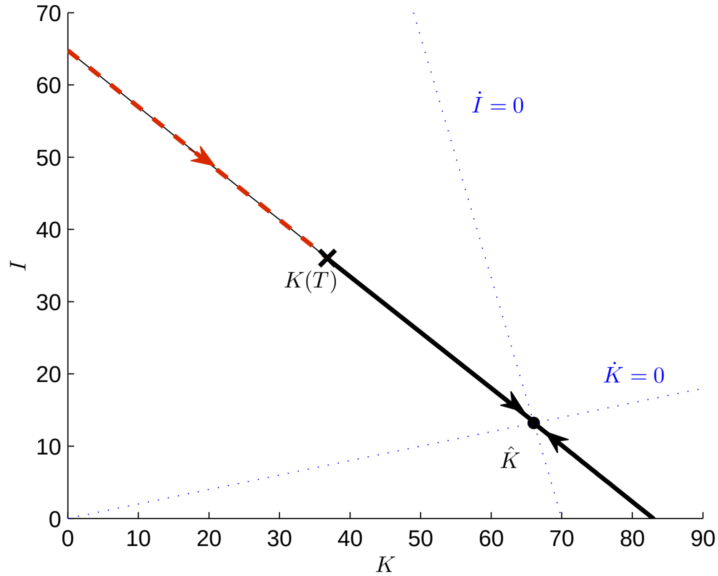

Figure 6: Phase portrait for a hairline case with a concave salvage value function which touches the value function. On the dashed line one is indifferent between operating for a finite time, selling the firm at $K(T)$ or operating forever, approaching the steady state $\hat{K}$. On the solid line the infinite time horizon strategy is optimal.

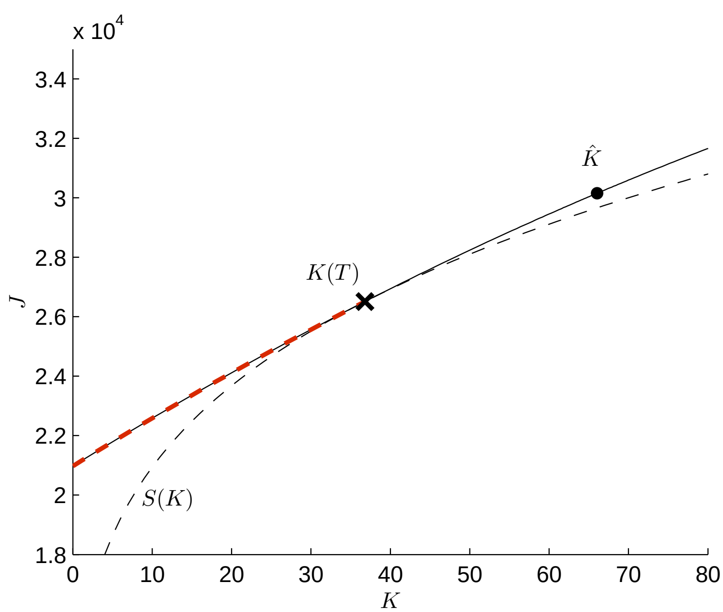

Figure 7: Value of the objective function and salvage value function corresponding to Figure 6.

generalizes the Skiba point literature to models with a free end time, i.e. the decision maker can choose the optimal time to end operations. We find Skiba points at which the decision maker is indifferent between the policies of "stopping immediately" and "operating forever", or between "operating forever" and "operating during a finite time and stopping afterwards". Concerning the latter, we prove that if for one policy the state starts to increase at the Skiba point, then for the other policy it must hold (except in a hairline case) that the state starts to decrease.

We develop our results by considering the application of a capital accumulating firm that has the option to sell (be acquired) at any time. This is a very relevant problem in today's economy, for instance with takeovers of small technology firms by large conglomerates. The analysis yields interesting substantive results, including that whether it is optimal to sell or remain independent need not be a monotonic function of the initial state.

One fruitful avenue for enriching our model of the firm would be to allow the firm to borrow against its other assets in order to generate cash flow that it could invest in its (intellectual) capital stock. The present model was inspired by the image of startup firms for whom bank financing can be difficult to obtain. However, merger and acquisition activity is widespread, not limited only to startups, and the greater range of financing options available to other firms might enhance the range of optimal strategies.

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7 Appendix

In order to compute the solution trajectories, we obtain the eigenvalues

$$\xi_1 = \frac{r}{2} - \sqrt{\left(\frac{r}{2}+\delta\right)^2 + b} < 0, \quad \xi_2 = \frac{r}{2} + \sqrt{\left(\frac{r}{2}+\delta\right)^2 + b} > 0, \tag{25}$$

with the eigenvectors

$$\begin{pmatrix} 1 \\ \xi_1 + \delta \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{r}{2} + \delta - \sqrt{\left(\frac{r}{2}+\delta\right)^2 + b} \end{pmatrix}$$

corresponding to ξ_1, and

$$\begin{pmatrix} 1 \\ \xi_2 + \delta \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{r}{2} + \delta + \sqrt{\left(\frac{r}{2}+\delta\right)^2 + b} \end{pmatrix}$$

corresponding to ξ_2. The general solution of the canonical system (transformed into the state-control space) is

$$\begin{pmatrix} K \\ I \end{pmatrix} = \begin{pmatrix} \hat{K} \\ \hat{I} \end{pmatrix} + \sigma_1 \begin{pmatrix} 1 \\ \xi_1 + \delta \end{pmatrix} \mathrm{e}^{\xi_1 t} + \sigma_2 \begin{pmatrix} 1 \\ \xi_2 + \delta \end{pmatrix} \mathrm{e}^{\xi_2 t}.$$

In order to determine the constants σ_1 and σ_2 we have the boundary conditions

$$K(0) = K_0 = \hat{K} + \sigma_1, \tag{26}$$

$$\lambda(T) = s, \quad \text{i.e.} \quad I(T) = s - c, \tag{27}$$

so that

$$\sigma_1 = -\frac{c - s + \hat{I} + \mathrm{e}^{\xi_2 T}(\delta + \xi_2)\left(K_0 - \hat{K}\right)}{\mathrm{e}^{\xi_1 T}(\delta + \xi_1) - \mathrm{e}^{\xi_2 T}(\delta + \xi_2)},$$

$$\sigma_2 = \frac{c - s + \hat{I} + \mathrm{e}^{\xi_1 T}(\delta + \xi_1)\left(K_0 - \hat{K}\right)}{\mathrm{e}^{\xi_1 T}(\delta + \xi_1) - \mathrm{e}^{\xi_2 T}(\delta + \xi_2)}.$$

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