



The reproductive value as part of the shadow price of population

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Abstract

The reproductive value (see Fisher [16]) arises as part of the shadow price of the population in a large class of age-structured optimal control models.

1 Relationship

The reproductive value at age a , introduced by Fisher [16], is defined as

$$v(a) = \int_a^\beta e^{-r(s-a)} \frac{l(s)}{l(a)} m(s) ds, \quad (1)$$

where $l(s)$ denotes the probability to survive from birth until age s , $m(s)$ the fertility rate of age s , β the oldest age of childbearing and r the discount rate, which is equal to Lotka's r (see e.g. Keyfitz [21]).

Consider an optimal control model in which an intertemporal objective functional is to be optimised over a finite time horizon, where the objective itself depends on the size

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and age-structure of some population. Further assume that within such an age-structured optimal control model, population aged a at time t is used as a state variable $N(a, t)$, described by the McKendrick equation (see Keyfitz [21] and Keyfitz and Keyfitz [22]), then the shadow price attached to population $\xi^N(a, t)$ typically embraces a generalization of the reproductive value. $\xi^N(a, t)$ itself can be interpreted as the marginal value of an additional a -year old individual at time t . The relationships become clearer when considering the following decomposition¹

$$\xi^N(a, t) = \xi^d(a, t) + \int_a^\omega e^{-\rho(s-a)} \frac{l(s, \cdot)}{l(a, \cdot)} m(s, \cdot) \xi^N(0, t - a + s) ds, \quad (2)$$

for $t - a + \omega < T$, where ω denotes the maximal length of life, t time, a age and ρ the time discount rate. The *direct* effect $\xi^d(a, t)$ accounts for the marginal value of the population that is currently alive. The *indirect* effect, as given by the second term on the right hand side, accounts for the marginal value of expected newborns and is a generalized form of the classical reproductive value (1). Put differently the indirect effect could represent the economic value of the lineage associated with an additional individual.

Note that the integral in the above result is a variant of Fisher's formulation. The integrand, the discounted expected number of newborns at time $t - a + s$, is weighted by their value to the decision-maker, as given by the shadow price $\xi^N(0, t - a + s)$. In Fisher's case this value equals one, since he regarded the birth of a child as lending to him of a life. Moreover, while the discount rate is equal to Lotka's r in (1), in our expression (2) the discount rate is equal to the time preference ρ and exogenously given. Note that it makes no difference that the upper bound of the integral is the maximal length of life, since the fertility rate is zero after the maximal age of childbearing.

The generality of the above expression has two important implications. Firstly, in Fisher's original formulation (1) the population is stable. The above expression allows for changes in the demography over time.² Secondly, the sign of the reproductive value is always positive. Due to the weighting with $\xi^N(0, t - a + s)$, the general expression can also have a negative sign.

2 Proof

Consider the following age-structured optimal control model^{3,4}

¹Equation (2) is derived in section 2 (see equations (4)-(5)).

²See also Ediev[10] on the dynamics of the reproductive value.

³For simplification a and t are omitted sometimes.

⁴ $V(N(a, 0))$ denotes the optimal value of the objective functional, which depends on $N(a, 0)$. In dynamic optimisation $V(N(a, t))$ is referred to as value function.

$$\begin{aligned}
V(N(a, 0)) = \max_{u \in U} & \int_0^T \int_0^\omega e^{-\rho t} L(a, t, N, u) \, da \, dt \\
\text{s.t.} & \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial t} \right) N(a, t) = -\mu(a, t, u) N(a, t) \\
& N(0, t) = B(t) = \int_0^\omega m(a, t, u) N(a, t) \, da, N(a, 0) = N_0(a)
\end{aligned} \tag{3}$$

where population dynamics is modeled according to the McKendrick equation with an endogenous number of newborns $B(t)$. The decision maker chooses the control $u(a, t)$ from the set of admissible controls U such that the objective, equal to the discounted instantaneous objective functionals⁵ $L(\cdot)$ aggregated over age and time, is maximized over a finite time horizon $T < \infty$ with a zero salvage value. The control $u(a, t)$ itself influences the objective functionals as well as the mortality and the fertility rates.

For age-structured optimal control models with additional cohort states (45-degree line in the Lexis diagram), period states (90-degree line in the Lexis diagram) and a non-zero salvage value we refer to Wrzaczek et al. [30].

Note that by allowing age and time to vary, we consider a dynamic age structure and are able to study not only the stationary but also the transient behavior of the optimal solutions.

According to the Maximum principle for age-structured optimal control models (see Brokate [4] or Feichtinger et al. [14])⁶ we formulate the current-value Hamiltonian, which consists of the instantaneous objective functional (current contribution) and the dynamics (population as well as the integral constraint⁷) weighted by the adjoint variable (future contribution), i.e.

$$\mathcal{H} = L(a, t, N, u) - \xi^N(a, t) \mu(a, t, u) N(a, t) + \xi^N(0, t) m(a, t, u) N(a, t) \tag{4}$$

and maximize it with respect to the control $u(a, t)$. The Hamiltonian may be seen as a generalization of the Lagrangean method (for an excellent reference see Leonard and van Long [25]).

Note that in equation (4) the adjoint variable $\xi^N(a, t)$ refers to the role of mortality while the adjoint variable $\xi^N(0, t)$ refers to the role of fertility in determining the dynamics of the population over time.

⁵For instance, the aggregate objective could be intertemporal social welfare. In this case, the objective functional $L(\cdot)$ would measure the instantaneous welfare of age-group $N(a, t)$. In different settings, $L(\cdot)$ may describe the (economic) value ascribed to an animal or plant population.

⁶The Maximum Principle presented in Feichtinger et al. [14] is a generalization of that in Brokate [4]. Firstly the formulation of the model allows for more general forms of the objective function and the dynamics. Secondly, an additional type of state accounting for interactions between the cohorts (often important in epidemiological models) is allowed. On the other hand, the Maximum Principle of Brokate [4] allows for an infinite life-time horizon, which is not contained in that of Feichtinger et al. [14]

⁷The integrand of $B(t)$ is weighted by a separate adjoint variable $\eta(t)$. However, since $B(t)$ only acts as boundary constraint of the population $\eta(t) = \xi^N(0, t)$ follows immediately.

Applying the Maximum Principle we obtain⁸

$$\begin{aligned} \xi^N(a, t) &= \int_a^\omega e^{-\rho(s-a)} \frac{l(s, t-a+s)}{l(a, t)} \frac{\partial L(\cdot)}{\partial N} ds + \\ &+ \int_a^\omega e^{-\rho(s-a)} \frac{l(s, t-a+s)}{l(a, t)} m(s, t-a+s, u(\cdot)) \xi^N(0, t-a+s) ds. \end{aligned} \quad (5)$$

The direct effect is represented by the first integral. It is equal to the marginal effect of the population on the instantaneous objective functional $\frac{\partial L(\cdot)}{\partial N}$. Discounted by ρ and weighted by the survival probability, this effect is aggregated over the remaining life of the cohort born at $t-a$. The second integral is the more general form of Fisher's reproductive value as discussed in the previous section.

For a detailed discussion on a general model and mathematical details (assumptions on the functions involved, conditions for the existence of solutions) we refer to Wrzaczek et al. [30] and Feichtinger et al. [14]. For a model with a male and female population the analysis is analogous and the core result does not change.

3 History and related results

While intertemporal optimization is ubiquitous in population economics,⁹ it is not a standard tool in demography. One main purpose of the present note is to illustrate its applicability in population dynamics. The core concept of dynamic optimization is the shadow price of the state variable. It measures the marginal value (in terms of the objective function) of an increment of the state at time t when moving along the optimal path. This imputed (marginal) value can be seen as a *dynamic* extension of the dual variable in (*static*) mathematical (linear and non-linear) programming. Note that it is no market price (which must be always non-negative), but measures the marginal impact of an additional (infinitesimal) unit of the state on the objective. In optimal control theory this dynamic shadow price is referred to as adjoint variable. Clearly, shadow prices can take negative values. Our indirect effect that shows up in the dynamics of the shadow price (equation (2)) may therefore be seen as a further alternative of the valuation of the reproductive value at age zero, by taking into account the marginal value of a newborn as given by the shadow price at age zero.

Remarkably, there is an interesting analogy with a finding by Goodman [17] who considers an optimal control model to deal with life-history optimization (see e.g. Charlesworth [6] for life history research)¹⁰, where a decision-maker (evolution) chooses optimal trajectories of fertility (rates) in order to maximise the total reproductive contribution of cohort at birth subject to certain physiological constraint. He shows that the reproductive value

⁸Note that the conditional survival probability $\frac{l(s, t-a+s)}{l(a, t)}$ is a transformation of the term $e^{-\int_a^s \mu(\cdot) ds'}$.

⁹For an early example we refer to Arthur and McNicoll [1].

¹⁰For an even earlier dynamic programming and optimal control approach to life history optimisation see Leon [24] or Taylor et al. [29].

(at age a) is the current shadow price of survival of an age a individual in the cohort. Thus, for an optimal life-history model any individual is valued exactly at its reproductive value, whereas in our model, in which population may have a more general value than reproduction alone, the weighted reproductive value constitutes one part of an individual's value.

Notably, the shadow price of population bears close resemblance to the economic-demographic potential (see Ediev [7], [8], [9] based on earlier measures as e.g. by Hersch [18] and Burgeois-Pichat [5]). By deriving the shadow price of population within an explicit optimal control problem, our approach lends a foundation to the economic-demographic potential as measure of an optimal 'policy'. Indeed, by its very nature the shadow price embraces a measure of the future economic-demographic prospects.

In a similar vein, economists have thought since long about the value of (human) life. Hofflander [19] provides an excellent survey of such thought, including prominent contributions by William Farr and Alfred Marshall. Schelling [27] was first to (re-)introduce the concept in a modern way, which was subsequently formalized by Shepard and Zeckhauser [28]. Recent work by Birchenall and Soares [3] and by Kuhn et al. [23] accounts for the presence of altruism towards descendants in the spirit of Becker and Barro [2]. In this case the value of an individual's life is amended by the value the individual attaches to its progeny (depending inter alia on their future prospects), corresponding to the indirect effect in (2).

4 Applications

In section 1 we mentioned that the generalized reproductive value can have a negative sign. As an example consider a model minimizing the impact of a population of a pest on a valuable store of a resource. If the pest population (destroying the resource) is modeled in the above way, the corresponding shadow price will be negative. This implies that also the generalized reproductive value (i.e. the impact of yet to be born pest individuals on the future stock of the resource) is negative.

For another application consider an age-specific predator-prey model, embracing cattle and wolves, say, as presented in Wrzaczek et al. [30]. Also in this example the reproductive term can be negative depending on the type of objective function (e.g. the reproductive value of the predator if in contrast to the prey it has no direct economic value in and of itself). A further epidemiological application involving the interaction of infected and susceptible individuals is presented in Wrzaczek et al. [30]. Finally, we want to emphasize that the concept of the reproductive value is not only applicable for humans and animals, but also for self-renewable machines or even capital (for examples see e.g. Sethi and Thompson [26], Feichtinger et al. [12] and references therein).

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