TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Operations Research and Control Systems

# Multiple Equilibria and Skiba Points in a Rational Addiction Model

*J.P. Caulkins, G. Feichtinger, R.F. Hartl, P.M. Kort, A.J. Novak, A. Seidl,*

**Research Report 2011-11**

November, 2011

**Operations Research and Control Systems**
Institute of Mathematical Methods in Economics
Vienna University of Technology

Research Unit ORCOS
Argentinierstraße 8/E105-4,
1040 Vienna, Austria
E-mail: orcos@eos.tuwien.ac.at

# Multiple Equilibria and Skiba Points in a Rational Addiction Model

Jonathan P. Caulkins   Gustav Feichtinger   Richard F. Hartl
Peter M. Kort   Andreas J. Novak   Andrea Seidl

November 29, 2011

**Abstract**

Becker and Murphy (1988) have established the existence of unstable steady states leading to threshold behavior for optimal consumption rates in intertemporal rational addiction models. In the present paper a simple linear-quadratic optimal control model is used to illustrate how their approach fits into the framework of multiple equilibria and Skiba points. By changing the degree of addiction and the level of harmfulness we obtain a variety of behavioral patterns.

In particular we show that when the good is harmful as well as very addictive, a Skiba point separates patterns of converging to either zero or maximal consumption, where the latter occurs in case of a high level of past consumption. This implicitly shows that an individual needs to be aware in time of these characteristics of the good. Otherwise, he/she may start consuming so much that in the end he/she is totally addicted.

## 1 Introduction

In the seventies and eighties a theory of rational addiction has been developed in a series of papers. 'Rational' means that agents maximize utility continuously over time. While addiction might be seen just opposite of rational behavior, these works delivered valuable insights into addictive behavior.

An early forward-looking maximization of an intertemporal utility stream with stable preferences was presented by Ryder and Heal (1973). Their utility function does not depend only on the consumption rate, but also on the past (accumulated) consumption. They identified a special property of such a state-dependent utility denoted as 'adjacent complementarity'.

Another important attempt to model rational addiction is by Stigler and Becker (1977); their analysis considers the concepts of beneficial and harmful addiction. If greater present consumption lowers future utility, habits or addictions are called harmful. Extending this approach, Iannaccone (1986) delivers a further clarification of this concept; see also Léonard

(1989); Orphanides and Zervos (1994, 1995, 1998), and Becker (1992). More recent work is by Braun and Vanini (2003) and Gavrila et al. (2005).

Becker and Murphy (1988) pursued the idea that consumers anticipate the future consequence of their choices. Denoting a good as potentially addictive if increasing past consumption raises the utility of current consumption, they showed that steady-state consumption is *unstable* when the degree of addiction is strong. Becker and Murphy were the first authors stressing the importance of unstable equilibria to explain addictive behavior. They state this as follows: '... *powerful complementaries* [i.e. a substantial effect of past consumption on current consumption] *cause some steady states to be unstable... even small deviations from consumption of an unstable steady state can lead to large cumulative rises over time in addictive consumption or to rapid falls in consumption to abstention.*'

This analysis of Becker and Murphy is related to Skiba points[1] without referring to them explicitly. Readers might have guessed from Becker and Murphy (1988) that Skiba is most likely to occur, but still worried that perhaps there is always convergence to one of the steady states. Here we determine the scenarios under which Skiba occurs, and also determine in which cases we always have convergence to one of the steady states. The purpose of the paper is to draw attention to this important but neglected aspect. For this aim the solution structure of a simple linear-quadratic model is analyzed for varying model parameters. The paper by Gavrila et al. (2005) might be seen in this sense as a forerunner of the present paper, who consider the implications of a budget constraint of a consumer of an addictive substance.

The contribution of our paper is that within our framework we can exactly identify the scenarios under which in the long run individuals end up consuming nothing, consuming the maximal amount, or consuming at an intermediate level. Interestingly, the above described Skiba behavior arises when the good is very addictive and harmful. An addictive good has the characteristic that the marginal utility of consumption goes up with past consumption. This implies that consumers have to be aware in time whether a good is strongly addictive and harmful. This is because the danger arises that initially they consume so much that they get addicted and end up consuming large amounts, which gives a lot of harm. On the other hand, when they know in time how addictive/harmful consuming the good is, in the end they will be safe by consuming nothing. Here the level of past consumption is crucial, and we show that a Skiba point separates the regions of past consumption levels leading to these two different behaviors.

In our framework a Skiba point can occur in two different ways. The first way arises when the unstable steady state is an unstable node. Then this unstable node can be a (weak) Skiba point itself. In this scenario the long-run solution always depends only on the initial state value. In the second case the unstable steady state is either a focus or a node and there is an overlap of solution paths in the state-space. Then it is more difficult to determine the location of the Skiba point and has to be done numerically.

Our paper is organized as follows. In Section 2 a simple linear-quadratic model is presented. In Section 3 the first-order optimality conditions are stated and used for a phase

[1] For a discussion of this topic see Grass et al. (2008, Ch. 5).

portrait analysis. Further, different solution structures for varying parameters are calculated. Particular interest is laid on the discussion of Skiba thresholds. Section 4 concludes the model. A brief introduction into multiple equilibria, Skiba points and path dependency can be found in the appendix.

# 2 The Model

The model we analyze is a special case of the one studied by Becker and Murphy (1988). Utility $u$ of an individual depends both on the current consumption rate $c$ as well as on a stock $S$ denoted as 'consumption capital'. The state variable $S$ measures past (accumulated) consumption, sometimes also called habit. Thus,

$$u = u(c, S),$$

where $c = c(t)$, $S = S(t)$ are time-dependent variables connected by

$$\dot{S} = c - \delta S, \tag{1}$$

and the initial value

$$S(0) = S_0 \tag{2}$$

is given.

The instantaneous depreciation rate $\delta$ measures the exogenous rate of depreciation of the consumption stock and is assumed to be constant.

We assume that the maximum possible consumption is 1, thus we have the control constraint $0 \leq c \leq 1$.

**Remark:** The constraint $0 \leq c \leq 1$ implies that the state $S$ is restricted to the interval $[0, 1/\delta]$, provided that $S_0 \in [0, 1/\delta]$.

Similar to Becker and Murphy (1988, p. 679, eq. (7)), we restrict attention to a quadratic utility function

$$u(c, S) = \alpha_c c + \frac{\alpha_{cc}}{2} c^2 + \alpha_S S + \frac{\alpha_{SS}}{2} S^2 + \alpha_{cS} cS. \tag{3}$$

The coefficients $\alpha$ have the following signs

$$\alpha_c > 0, \quad \alpha_{cc} < 0, \tag{4}$$

$$\alpha_S < 0, \quad \alpha_{SS} < 0, \tag{5}$$

$$\alpha_{cS} > 0. \tag{6}$$

The signs (4) are economically clear and reflect the concavity of $u$ w.r.t. $c$ and $S$, respectively. The price of the addictive good is contained in $\alpha_c$.

The negativity of $\alpha_S$ and $\alpha_{SS}$, i.e. assumption (5), is known as harmful addiction in the literature (see, e.g., Iannaccone, 1986; Dockner and Feichtinger, 1993). Consequently, present consumption leads to a lower future utility.

The positive interaction (6) is essential for addictive behavior. It says that

$$\alpha_{cS} = \frac{\partial}{\partial S}\frac{\partial u(c,S)}{\partial c} > 0,$$

i.e. the marginal utility of current consumption *increases* with past consumption.

We do not model budget constraint, the concavity of $u$ with respect to $c$ implicitly bounds consumption rates.

Then the optimization problem reads as follows:

$$\max_c \int_0^\infty e^{-rt} u(c,S)dt, \tag{7}$$

subject to (1) and (2) and $0 \le c \le 1$, where $r > 0$ denotes the time preference rate and $u(c,S)$ is given by (3).[2]

# 3 Classifying Solutions According to Degree of Addiction and Harmfulness

We apply Pontryagin's maximum principle and show that it is possible to produce explicit analytical expressions for the steady states and also to gain insights concerning their stability properties. The current value Hamiltonian is

$$H = \alpha_c c + \alpha_{cS} cS + \frac{\alpha_{cc} c^2}{2} + \alpha_S S + \frac{\alpha_{SS} S^2}{2} + \lambda\left(c - \delta S\right),$$

which gives the necessary optimality conditions

$$\begin{aligned} H_c &= \alpha_c + \alpha_{cS} S + \alpha_{cc} c + \lambda = 0, \\ \dot{\lambda} &= (r+\delta)\lambda - \alpha_S - \alpha_{SS} S - \alpha_{cS} c. \end{aligned}$$

We derive that the optimal consumption $c^*$ is given by

$$c^* = \begin{cases} 0 & \text{if} \quad \lambda < -\alpha_{cS} S - \alpha_c, \\ \frac{\alpha_c + \alpha_{cS} S + \lambda}{-\alpha_{cc}} & \text{if} \quad -\alpha_{cS} S - \alpha_c < \lambda < -\alpha_{cc} - \alpha_{cS} S - \alpha_c, \\ 1 & \text{if} \quad -\alpha_{cc} - \alpha_{cS} S - \alpha_c < \lambda. \end{cases} \tag{8}$$

Since

$$\begin{pmatrix} H_{SS} & H_{cS} \\ H_{cS} & H_{cc} \end{pmatrix} = \begin{pmatrix} \alpha_{SS} & \alpha_{cS} \\ \alpha_{cS} & \alpha_{cc} \end{pmatrix},$$

the Hamiltonian is jointly concave in state and control iff $\alpha_{cc} < 0$, $\alpha_{SS} < 0$, and $\alpha_{cS}^2 < \alpha_{SS}\alpha_{cc}$.

First we analyze the canonical system along the boundary arcs with $c = 0$ and $c = 1$, respectively. The results are summarized in the following proposition.

**Proposition 1:**

[2]Note that here and in the following the time argument $t$ is mostly omitted.

1. There exists a boundary steady state along the arc with $c \equiv 0$ iff $\alpha_S < -\alpha_c(r+\delta)$ and it is given by

$$\left(S_0^\infty = 0, \lambda_0^\infty = \frac{\alpha_S}{r+\delta}, c_0^\infty = 0\right) \tag{9}$$

2. The second boundary steady state is given by

$$\left(S_1^\infty = \frac{1}{\delta}, \lambda_1^\infty = \frac{\delta(\alpha_{cS} + \alpha_S) + \alpha_{SS}}{\delta(r+\delta)}, c_1^\infty = 1\right) \tag{10}$$

and it is feasible iff

$$\frac{-\delta(r+\delta)(\alpha_{cc} + \alpha_c) - \delta\alpha_S - \alpha_{SS}}{r+2\delta} < \alpha_{cS} \tag{11}$$

3. Both boundary steady states (provided they exist) are saddle path stable with real eigenvalues, one positive and one negative.

**Proof:**

1. Along the boundary arc with $c \equiv 0$ the canonical system reduces to :

$$\begin{aligned} \dot{S} &= -\delta S \\ \dot{\lambda} &= (r+\delta)\lambda - \alpha_S - \alpha_{SS} S \end{aligned}$$

The boundary steady state is therefore given by (9). It is feasible, i.e. matches the optimality condition (8) iff $\alpha_S < -\alpha_c(r+\delta)$.

2. Setting $c \equiv 1$ the canonical system along the second boundary arc is given by

$$\begin{aligned} \dot{S} &= 1 - \delta S \\ \dot{\lambda} &= (r+\delta)\lambda - \alpha_S - \alpha_{SS} S - \alpha_{cS} \end{aligned}$$

leading to the steady state (10).

It matches the optimality condition (8) iff

$$-(r+\delta)[\delta(\alpha_{cc} + \alpha_c) + \alpha_{cS}] < \delta(\alpha_{cS} + \alpha_S) + \alpha_{SS}$$

which is equivalent to (11).

3. Both boundary steady states are stable saddle-points as the eigenvalues of the Jacobian

$$J = \begin{pmatrix} -\delta & 0 \\ -\alpha_{SS} & r+\delta \end{pmatrix} \quad \text{are } -\delta, r+\delta.$$

$\boxtimes$

Next we analyze the canonical system in the interior, i.e. for $c$ given by

$$c^* = \frac{\alpha_c + \alpha_{cS} S + \lambda}{-\alpha_{cc}},$$

this yields

$$\begin{aligned} \dot{S} &= -\left(\frac{\alpha_{cS}}{\alpha_{cc}} + \delta\right) S - \frac{\lambda}{\alpha_{cc}} - \frac{\alpha_c}{\alpha_{cc}} \\ \dot{\lambda} &= \left(\frac{\alpha_{cS}^2}{\alpha_{cc}} - \alpha_{SS}\right) S + \left(r + \delta + \frac{\alpha_{cS}}{\alpha_{cc}}\right) \lambda - \alpha_S + \frac{\alpha_{cS}\alpha_c}{\alpha_{cc}}. \end{aligned} \tag{12}$$

The dynamic system gives rise to the following proposition.

**Proposition 2:**

The canonical system (12) possesses an interior steady state at

$$S^\infty = \frac{-\alpha_S - \alpha_c(r+\delta)}{(r+2\delta)\alpha_{cS} + \delta\alpha_{cc}(r+\delta) + \alpha_{SS}} \tag{13}$$

iff either

$$0 < \delta[-\alpha_S - \alpha_c(r+\delta)] < (r+2\delta)\alpha_{cS} + \delta\alpha_{cc}(r+\delta) + \alpha_{SS} \tag{14}$$

or

$$(r+2\delta)\alpha_{cS} + \delta\alpha_{cc}(r+\delta) + \alpha_{SS} < \delta[-\alpha_S - \alpha_c(r+\delta)] < 0. \tag{15}$$

**Proof:**

1. From the canonical system (12) we compute the $\dot{S} = 0-$isocline

   $$\lambda = -(\delta\alpha_{cc} + \alpha_{cS})S - \alpha_c,$$

   as well as the $\dot{\lambda} = 0-$ isocline

   $$\lambda = \frac{\alpha_S\alpha_{cc} - \alpha_c\alpha_{cS} - (\alpha_{cS}^2 - \alpha_{cc}\alpha_{SS})S}{\alpha_{cc}(r+\delta) + \alpha_{cS}},$$

   To calculate the interior steady state we obtain

   $$-(\delta\alpha_{cc} + \alpha_{cS})S - \alpha_c = \frac{\alpha_S\alpha_{cc} - \alpha_c\alpha_{cS} - (\alpha_{cS}^2 - \alpha_{cc}\alpha_{SS})S}{\alpha_{cc}(r+\delta) + \alpha_{cS}}$$

   which implies

   $$S\{\underbrace{\alpha_{cS}^2 - \alpha_{cc}\alpha_{SS} - (\delta\alpha_{cc} + \alpha_{cS})[\alpha_{cc}(r+\delta) + \alpha_{cS}]}_{=-\alpha_{cc}[\alpha_{SS} + 2\delta\alpha_{cS} + \delta^2\alpha_{cc} + r\delta\alpha_{cc} + r\alpha_{cS}]}\} = \alpha_{cc}[\alpha_c(r+\delta) + \alpha_S],$$

   leading to (13).

2. An interior feasible steady state requires $0 < S^\infty < 1/\delta$, i.e.

$$0 < \frac{\delta[-\alpha_S - \alpha_c(r+\delta)]}{(r+2\delta)\alpha_{cS} + \delta\alpha_{cc}(r+\delta) + \alpha_{SS}} < 1.$$

If the denominator is positive, this implies (14); if the denominator is negative it leads to (15).

⊠

**Proposition 3:**
The feasibility of the interior steady state as well as its stability is related to the existence of the boundary steady states. The following four cases can be distinguished based on the level and harmfulness of the addiction.

| | less harmful<br>$-\alpha_c(r+\delta) < \alpha_S$ | more harmful<br>$\alpha_S < -\alpha_c(r+\delta)$ |
|---|---|---|
| low addiction<br>$\alpha_{cS} < \frac{1}{r+2\delta}[\delta(-\alpha_{cc} - \alpha_c)(r+\delta)$<br>$-\delta\alpha_S - \alpha_{SS}]$ | Region 1<br>only (13) is feasible<br>it is stable | Region 3<br>(9) is the only<br>feasible steady state |
| high addiction<br>$\alpha_{cS} > \frac{1}{r+2\delta}[\delta(-\alpha_{cc} - \alpha_c)(r+\delta)$<br>$-\delta\alpha_S - \alpha_{SS}]$ | Region 2<br>(10) is the only<br>feasible steady state | Region 4<br>both (9) & (10) are feasible,<br>(13) is unstable |

**Proof:**
To determine the stability of this interior steady state we compute the Jacobian, which is given by

$$J = \begin{pmatrix} -\frac{\alpha_{cS}}{\alpha_{cc}} - \delta & -\frac{1}{\alpha_{cc}} \\ -\alpha_{SS} + \frac{\alpha_{cS}^2}{\alpha_{cc}} & r + \delta + \frac{\alpha_{cS}}{\alpha_{cc}} \end{pmatrix}$$

with $\text{tr}J = r$ and $\det J = -\frac{\alpha_{cS}}{\alpha_{cc}}[r+2\delta] - \delta(r+\delta) - \frac{\alpha_{SS}}{\alpha_{cc}}$.

As the eigenvalues are given by $\lambda_{1,2} = \frac{\text{tr}J \pm \sqrt{(\text{tr}J)^2 - 4\det J}}{2}$ the interior steady state (13) is therefore a/an

$$\begin{aligned} \text{saddle path stable node} &\Leftrightarrow \det J < 0 \\ \text{unstable node} &\Leftrightarrow 0 < \det J < \left(\frac{\text{tr}J}{2}\right)^2 \\ \text{unstable focus} &\Leftrightarrow \left(\frac{\text{tr}J}{2}\right)^2 < \det J \end{aligned}$$

⊠

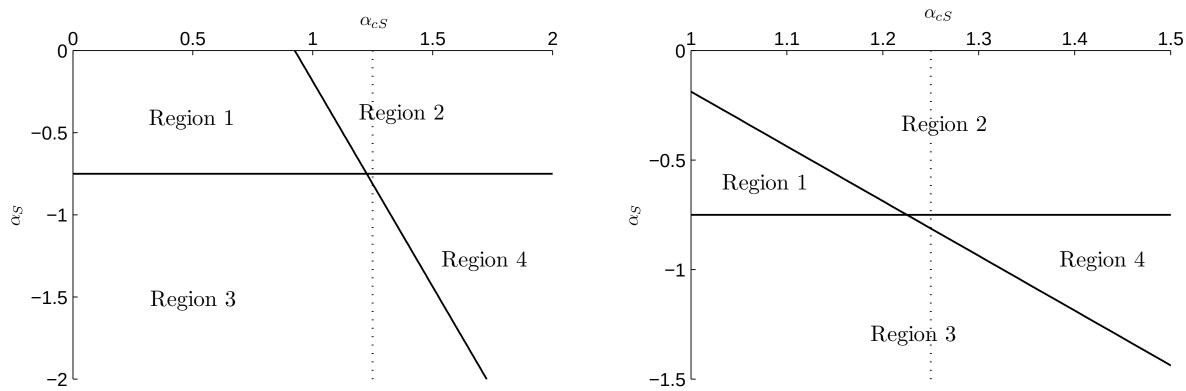


Figure 1: Four different parameter regions can be distinguished, the right panel shows a zooming of the left panel

The case of an unstable interior steady state (south-east case of the table above) can be further divided into two sub-cases. The interior steady state is an unstable node, iff

$$0 < \delta[-\alpha_S - \alpha_c(r+\delta)] < \alpha_{cS}(r+2\delta) + \alpha_{cc}\delta(r+\delta) + \alpha_{SS} < \frac{-\alpha_{cc}r^2}{4}.$$

It is an unstable focus, iff

$$0 < \delta[-\alpha_S-\alpha_c(r+\delta)] < \alpha_{cS}(r+2\delta)+\alpha_{cc}\delta(r+\delta)+\alpha_{SS}, \text{and } \frac{-\alpha_{cc}r^2}{4} < \alpha_{cS}(r+2\delta)+\alpha_{cc}\delta(r+\delta)+\alpha_{SS}$$

In the model section we noted that $\alpha_{cS}$ stands for the effect of past consumption on the marginal utility of current consumption. It is clear that the larger $\alpha_{cS}$ is, the more addictive the good is. Therefore, in Proposition 3 we denote $\alpha_{cS} < \frac{1}{r+2\delta}[\delta\left((\alpha_{cc}-\alpha_c)(r+\delta)-\alpha_S\right)-\alpha_{SS}]$ as a "low addiction" scenario, whereas $\alpha_{cS} > \frac{1}{r+2\delta}[\delta\left((\alpha_{cc}-\alpha_c)(r+\delta)-\alpha_S\right)-\alpha_{SS}]$ is a "high addiction scenario". We know from the literature that the (negative) value of $\alpha_S$ is a measure for how harmful the addiction is. This explains that in Proposition 3 we have to deal with a less harmful addiction when $\alpha_S > -\alpha_c(r+\delta)$ and a more harmful one when $\alpha_S < -\alpha_c(r+\delta)$.

# 4 Numerical Example

Our aim is to investigate the effects of addictiveness and harmfulness of the good on the individual's consumption behavior. For this reason we let the values of $\alpha_{cS}$ (addictiveness) and $\alpha_S$ (harmfulness) vary, while we keep the other parameter values fixed. Concerning the latter we choose the following values: $\delta = 0.1, r = 0.05$, $\alpha_c = 5.0, \alpha_{cc} = -10, \alpha_{SS} = -0.15625$.

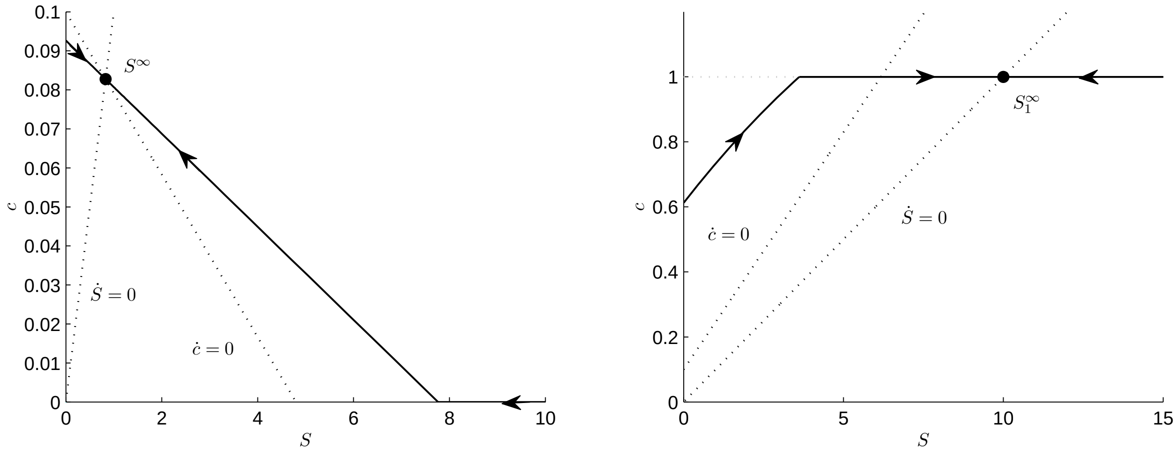


Figure 2: Less harmful habits that are not (left panel; Region 1) or are (right panel; Region 2) very addictive.

| | less harmful<br>$-0.75 < \alpha_S$ | more harmful<br>$\alpha_S < -0.75$ |
|---|---|---|
| weak addiction<br>$\alpha_{cS} < 0.925 - 0.4\alpha_S$ | only interior SS is feasible stable | $(S = 0, c = 0)$ is the only feasible steady state |
| strong addiction<br>$0.925 - 0.4\alpha_S < \alpha_{cS}$ | $(S = 10, c = 1)$ is the only feasible SS | both boundary SS feasible, interior SS unstable |

The Fig. 1 shows the 4 Regions in the $\alpha_S - \alpha_{cS}$ parameter space. The utility function is jointly concave left to the dashed line. In the upper left corner (tiny triangle) of Region 4 the interior steady state is an unstable node, in the remaining part of Region 4 it is an unstable focus.

Figs. 2-5 show optimal solutions in the state-control space. Dashed lines depict the isoclines, dotted lines show where control constraints become active.

- **Region 1:** (less harmful, low addiction) Only interior steady state, which is a saddle, is feasible.

  The parameters chosen for the numerical calculations are $\alpha_S = -0.6, \alpha_{cS} = 0.5$. We can find a steady state at $(0.827586, -4.58621, 0.0827586)$.

  Here we have a kind of normal situation, where, after a possible adjustment phase, the individual consumes a fixed amount, corresponding to a unique steady state. This situation is depicted in the left panel of Figure 2.

- **Region 2:** (less harmful, strong addiction) Only the steady state with $S = 1/\delta, c = 1$ is feasible.

  We choose $\alpha_S = -0.6, \alpha_{cS} = 1.5$. The steady state is located at $(10., -4.41667, 1.)$.

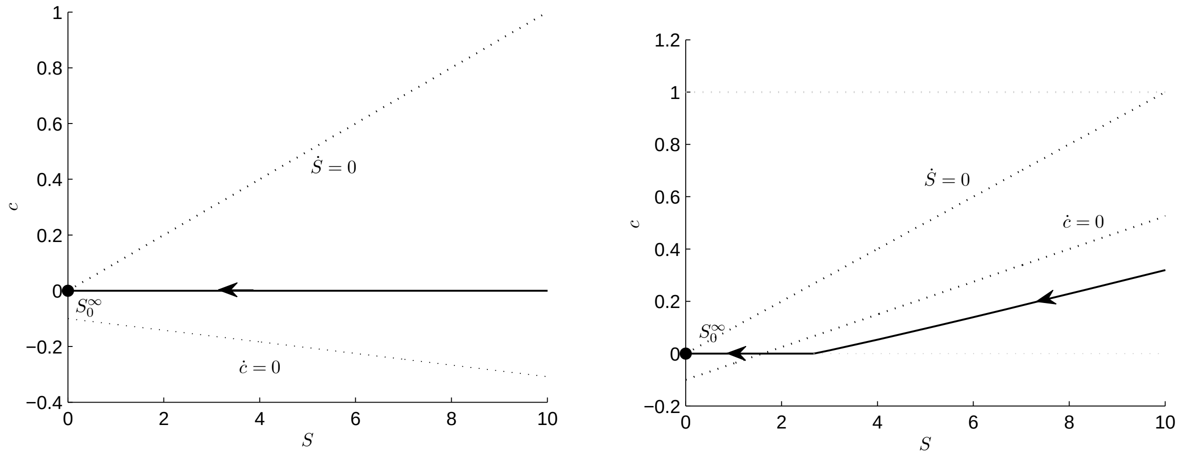


Figure 3: More harmful habits that are not very addictive; Region 3 (Parameters for right panel $\alpha_S = -0.9$ and $\alpha_{cS} = 1$)

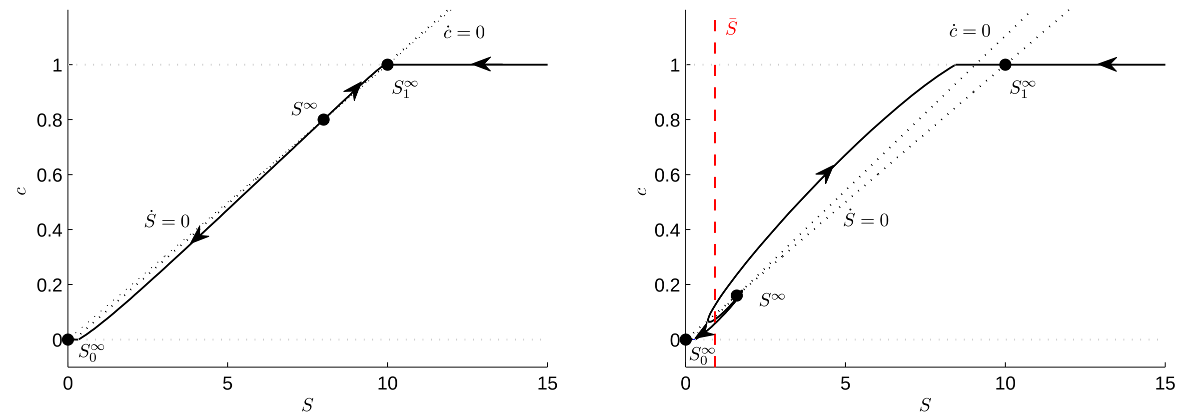


Figure 4: Harmful, addictive habits: Region 4

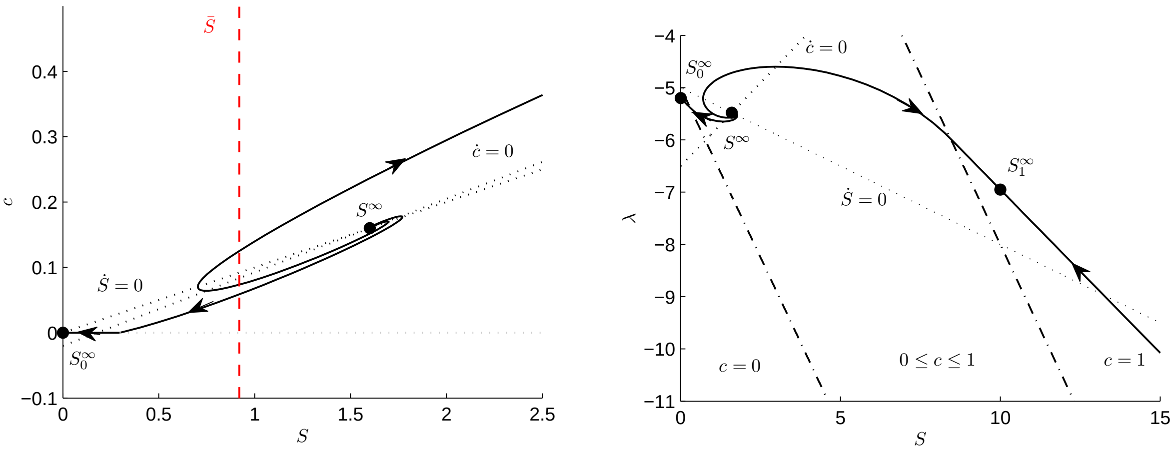


Figure 5: Region 4b: Zooming of the Skiba point and unstable focus; Phase portrait in the state-costate space

The good is strongly addictive, but this is relatively less harmful. Therefore, it makes sense that in the end we are in a situation with maximal consumption; see right panel of Figure 2.

- **Region 3:** (more harmful, low addiction) The only feasible steady state can be found at $S = 0, \lambda = \alpha_S/(r + \delta), c = 0$.

  Here we use $\alpha_S = -0.9, \alpha_{cS} = 0.5.$ and can calculate the steady state at $(0., -6., 0.)$.

  Here we have the opposite situation to Region 2. Now the good is less addictive but more harmful. It makes sense that a rational individual does not consume at all in this case, at least in the long term, which is confirmed in Figure 3.

- **Region 4:** (harmful, strong addiction) All steady states are feasible.

  - **Region 4a:** The interior steady state is an unstable node.

    The used parameters are $\alpha_S = -0.78, \alpha_{cS} = 1.24$ and the steady states $(0., -5.2, 0.)$, $(10., -7.3, 1.)$, and $(8., -6.92, 0.8)$.

  - **Region 4b:** The interior steady state is an unstable focus.

    We use $\alpha_S = -0.78, \alpha_{cS} = 1.3$ and find steady states at $(0., -5.2, 0.)$, $(10., -6.95, 1.)$, and $(1.6, -5.48, 0.16)$.

This is a difficult situation for the consumer. Consumption is harmful, but at the same time very addictive. The latter implies that marginal utility of consumption is strongly increasing in past consumption ($S$). This in turn implies that, although the good is harmful, the individual still increases consumption to the maximal level, or keeps it at the maximal level, when $S$ is large. In such a case the individual is addicted to the good. However, once the individual did not consume a lot of it in the past, it is optimal to reduce consumption and refrain from consuming it at all in the end. Here we observe history dependent behavior: having consumed a lot of the good in the past leads to large consumption levels in the future, whereas otherwise we converge to a situation of no consumption. A Skiba point, $\bar{S}$, separates the levels of past consumption that gives this different behavior, where of course the addicted person starts with an $S_0 > \bar{S}$.

There are two different types of Skiba points. In the left panel of Figure 4 the Skiba point is located at the unstable steady state, i.e. $\bar{S} = S^{\infty}$, which is an unstable node. Hence, by determining the unstable steady state we have also determined the location of the Skiba point. In this case it holds that consumption, $c$, is a continuous function of past consumption, $S$. A sufficient (but not necessary) condition for such history dependent behavior to occur is that the Hamiltonian is concave in the unstable steady state. This was detected for the first time in Wirl and Feichtinger (2005) and extensively analyzed in Hartl et al. (2004) by employing a capital accumulation model.

In the right panel of Figure 4 and in Figure 5, respectively, the unstable steady state is an unstable focus. Here the location of the Skiba point is in principle unknown: it lies somewhere "near" to the unstable focus. Numerical investigations have to be undertaken in order to find the exact location of the Skiba point. Another difference with the case of Figure 4 is that here consumption is not a continuous function of $S$: right at the Skiba point there is a discontinuity.

# 5 Conclusions

According to Becker (1992) habits exist if current consumption is positively related to former consumption. Addiction is defined as strong habit. Depending on their welfare effects, habits may be harmful or beneficial. Examples for harmful addiction are regular consumption of legal or illicit drugs, overeating, gambling, etc.

In a nutshell, Becker and Murphy's main result was that unstable steady states are crucial to understand rational addictive behavior. Our analysis confirms that an increase of the interaction term $\alpha_{cS}$ measuring the degree of (potential) addiction, combined with a significant negative value of $\alpha_S$, the likelihood that the interior steady state is unstable.

It is well-known that the linear-quadratic ansatz admits (at most) one interior steady state. To obtain *multiple equilibria* we have to add further non-linearities, e.g. a cubic term in the utility function. This was already remarked by Becker and Murphy (1988, p. 683). They illustrated this situation by showing that the left interior equilibrium is unstable focus (or node), while the right-one has saddle point quality (see their Fig. 2 at p. 686).

We like to make a short remark on binges. As Hartl (1987) has shown, the state trajectories in *one-state* optimal control models are monotonous (see also Kamien and Schwartz, 1991). This results implies that the model dealt with above admits no cycles as optimal solutions. Accordingly, Becker and Murphy (1988, Sect. VII) modeled binges in the framework of a *two*-state dynamic optimization model. Dockner and Feichtinger (1993) put their model on a firm ground by using an Hopf approach; see also Feichtinger (1992); Feichtinger and Wirl (1994).

The purpose of the present paper may be seen similarly. Becker and Murphy (1988) were well aware on the multiplicity of steady states resulting from their usual approach to deal with addiction. They stress that with two long-run equilibria, '*consumption diverges from the unstable state toward zero or toward the sizable steady-state level*' (compare also Fig. 1 in Becker and Murphy, 1988, p.681). As illustrated in the appendix much progress has been made since the early days of Skiba points to clarify the occurrence of multiple steady states.

Our aim in this paper was to illustrate how saddle point equilibria are separated by Skiba thresholds. For explanatory reasons we have selected a simple linear-quadratic scenario. For more complex models, the analysis proceeds essentially in the same manner. If we consider, e.g., models considering two state variables, not only limit cycles can be established but also Skiba curves. By starting in a point at such an indifference curve we are indifferent whether to converge in the long run to strong addictive behavior or to abstention.

# Acknowledgments

The authors like to thank Dieter Grass for his helpful comments. This research was supported by the Austrian Science Fund (FWF) under Grant P21410-G16.

# References

G. S. Becker. Habits, addictions, and traditions. *Kyklos*, 45(3):327–346, 1992.

G. S. Becker and K. M. Murphy. A theory of rational addiction. *Journal of Political Economy*, 96(4):675–700, 1988.

N. Braun and P. Vanini. On habits and addictions. *Journal of Institutional and Theoretical Economics*, 159:603–626, 2003.

W. A. Brock. Pricing, predation and entry barriers in regulated industries. In D. S. Evans, editor, *Breaking Up Bell*, pages 191–229. North-Holland, New York, 1983.

W. A. Brock and W. D. Dechert. Dynamic Ramsey pricing. *International Economic Review*, 26(3):569–591, 1985.

W. A. Brock and A. G. Malliaris. *Differential Equations, Stability and Chaos in Dynamic Economics*. North-Holland, Amsterdam, 1989.

W. A. Brock and D. Starrett. Nonconvexities in ecological managment problems. *Environmental and resources economics*, 26(4):575–624, 2003.

R. Bultmann, G. Feichtinger, and G. Tragler. Stochastic skiba sets: An example from models of illicit drug consumption. In I. Lirkov, S. Margenov, and J. Wasniewski, editors, *Large-Scale Scientific Computing*, pages 239–246. Springer, Heidelberg, 2010.

D. Cass. Optimum growth in an aggregative model of capital accumulation. *Review of Economic Studies*, 32(3):233–240, 1965.

J. P. Caulkins, G. Feichtinger, D. Grass, R. F. Hartl, P. M. Kort, and A. Seidl. Skiba points in free end time problems: the option to sell the firm, 2010. In Submission.

C. W. Clark. *Mathematical bioeconomics, the optimal management of renewable resources*. Wiley-Interscience, New York, 1976.

C.W. Clark. Economically optimal policies for the utilization of biologically renewable resources. *Mathematical Biosciences*, 12(3–4):245–260, 1971.

W. D. Dechert. Increasing returns to scale and the reverse flexible accelerator. *Economics Letters*, 13(1):69–75, 1983.

W. D. Dechert and K. Nishimura. A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *Journal of Economic Theory*, 31(2):332–354, 1983.

C. Deissenberg, G. Feichtinger, W. Semmler, and F. Wirl. Multiple equilibria, history dependence, and global dynamics in intertemporal optimization models. In W. A. Barnett, C. Deissenberg, and G. Feichtinger, editors, *Economic Complexity: Non-linear Dynamics, Multi-agents Economies and Learning*, pages 91–122. Elsevier Science, Amsterdam, 2004.

E. J. Dockner and G. Feichtinger. Cyclical consumption patterns and rational addiction. *The American Economic Review*, 83(1):256–263, 1993.

G. Feichtinger. Hopf bifurcation in an advertising diffusion model. *Journal of Economic Behavior & Organization*, 17(3):401–411, 1992.

G. Feichtinger and A. Steindl. DNS curves in a production/inventory model. *Journal of Optimization Theory and Applications*, 128(2):295–308, February 2006.

G. Feichtinger and F. Wirl. On the stability and potential cyclicity of corruption in governments subject to popularity constraints. *Mathematical Social Sciences*, 28(2):113–131, 1994.

C. Gavrila, G. Feichtinger, G. Tragler, R. F. Hartl, and P. M. Kort. History-dependence in a rational addiction model. *Mathematical Social Sciences*, 49(3):273–293, May 2005.

D. Grass, J. P. Caulkins, G. Feichtinger, G. Tragler, and D. A. Behrens. *Optimal Control of Nonlinear Processes: With Applications in Drugs, Corruption and Terror*. Springer, Heidelberg, 2008.

R. F. Hartl. A simple proof of the monotonicity of the state trajectories in autonomous control problems. *Journal of Economic Theory*, 40(1):211–215, 1987.

R. F. Hartl, P. M. Kort, G. Feichtinger, and F. Wirl. Multiple equilibria and thresholds due to relative investment costs. *Journal of Optimization Theory and Applications*, 123(1): 49–82, October 2004.

J. L. Haunschmied, P. M. Kort, R. F. Hartl, and G. Feichtinger. A DNS-curve in a two-state capital accumulation model: A numerical analysis. *Journal of Economic Dynamics and Control*, 27(4):701–716, February 2003.

L. R. Iannaccone. Addiction and satiation. *Economic Letters*, 21(1):95–99, 1986.

M. I. Kamien and N. L. Schwartz. *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*. North-Holland, Amsterdam, 2nd edition, 1991.

T. C. Koopmans. On the concept of optimal economic growth. *Pontificiae Academiae Scientiarum Scripta Varia*, 28(1):225–300, 1965.

D. Léonard. Market behaviour of rational addicts. *Journal of Economic Psychology*, 10(1): 117–144, 1989.

T. R. Lewis and R. Schmalensee. Optimal use of renewable resources with nonconvexities in production. In L. J. Mirman and D. F. Spulber, editors, *Essays in the Economics of Renewable Resources*, pages 95–111. North-Holland, Amsterdam, 1982.

R. E. Lucas. On the mechanics of economic development. *Journal of Monetary Economics*, 22(1):3–42, 1988.

K. G. Mäler. Development, ecological resources and their management: a study of complex dynamic systems. *European Economic Review*, 44(4–6):645–665, 2000.

A. Orphanides and D. Zervos. Optimal consumption dynamics with non-concave habit-forming utility. *Economic Letters*, 44(1–2):67–72, 1994.

A. Orphanides and D. Zervos. Rational addiction with learning and regret. *The Journal of Political Economy*, 103(4):739–758, 1995.

A. Orphanides and D. Zervos. Myopia and addictive behaviour. *Economic Journal*, 108 (446):75–91, 1998.

H. E. Ryder and G. M. Heal. Optimal growth with intertemporally dependent preferences. *Review of Economic Studies*, 40:1–33, 1973.

S. P. Sethi. Nearest feasible paths in optimal control problems: Theory, examples, and counterexamples. *Journal of Optimization Theory and Applications*, 23(4):563–579, December 1977.

S. P. Sethi. Optimal advertising policy with the contagion model. *Journal of Optimization Theory and Applications*, 29(4):615–627, December 1979.

A. K. Skiba. Optimal growth with a convex-concave production function. *Econometrica*, 46 (3):527–539, 1978.

G. J. Stigler and G. S. Becker. De gustibus non est disputandum. *American Economic Review*, 67(2):76–90, 1977.

F. O. O. Wagener. Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. *Journal of Economic Dynamics and Control*, 27(9):1533–1561, July 2003.

F. Wirl and G. Feichtinger. History dependence in concave economies. *Journal of Economic Behavior and Organization*, 57(4):390–407, 2005.

# 6 Appendix: Multiple Equilibria, Points of Indifference, and Thresholds

In the sixties of the last century optimal control theory started its applications to economic analysis. A common feature of those early intertemporal optimization models is the existence of a *unique* long-run equilibrium. Or, to put it more precisely: the necessary optimality conditions resulting from Pontryagin's maximum principle, i.e. the canonical system, exhibits a *unique* steady state.

A well-known illustration is the golden rule of Ramsey-type optimal growth model. The neoclassical growth model of Cass (1965) and Koopmans (1965) predicts that countries will converge to a common standard of living.

In the seventies, however, this scenario has been enriched by the *multiplicity* of equilibria. This means that for given initial states (e.g., capital endowment) there exist *multiple* optimal solutions, i.e. the decision maker is *indifferent* about which to chose. The possibility of multiple equilibria provide a basis for the empirically observed heterogeneity of growth patterns.

By using a convex-concave production function, Skiba (1978) extended the Ramsey model and obtained an unstable steady state separating two saddle point equilibria. In the book of Brock and Malliaris (1989, Chap. 6) the threshold separating the basins of attraction of the two saddles was denoted as '*Skiba point*'.

However, there are forerunners. The first reference describing such a situation seems to be Clark (1971) dealing with a renewable resource model; see also Clark (1976). Sethi (1977, 1979) was another pioneer in this field. A first existence proof of a Skiba point was given by Dechert and Nishimura (1983).

A first wave of applications appeared in the eighties, e.g., Lewis and Schmalensee (1982) on renewable resources, Brock (1983) on lobbying, Dechert (1983) on regulated firms, Brock and Dechert (1985) on dynamic Ramsey pricing.

In the endogenous growth literature multiple equilibria have been used to explain the occurrence of development miracles and poverty traps (see,e.g., Lucas, 1988).

More recent applications may be found in environmental economics. Here the literature on the so-called shallow lake model is particularly interesting; see Mäler (2000); Brock and Starrett (2003); Wagener (2003).

For a survey of further applications compare Grass et al. (2008, p.272–276). Deissenberg et al. (2004) examine different economic mechanisms that can generate multiple equilibria.[3]

In policy making, it may be important to recognize whether a given problem exhibits multiple equilibria. In *one-state*[4] two saddle point steady states are separated by an unstable equilibrium. The latter gives rise to a threshold separating basins of attraction surrounding the saddles. A such a threshold, a rational agent is indifferent between moving toward one or the other steady state. Small movements away from this threshold will lead to different optimal courses of actions depending on the direction of the slight change.

It is this history-dependence (sometimes also denoted as path-dependence) which has the discussion of various economic problems considerably enriched.

To summarize: the optimal long-run stationary solution toward which and optimally controlled system converges can depend on the initial conditions.

Let us now put this scenarios to a formal ground.

**Definition 1** (Solution sets)**.** Let $x^*(\cdot)$ be the optimal state trajectory starting at $x_0$ then

$$\mathcal{S}(x_0) = \{x^*(\cdot) : x^*(0) = x_0\},$$

is called the *solution set*, and

$$\mathcal{S}^\infty(x_0) = \{x(x_0, \infty) : x(\cdot) \in \mathcal{S}(x_0)\},$$

where $x(x_0, \infty)$ is the limit set, is called the *asymptotic solution set*.

**Definition 2** (Indifference point)**.** Let $\mathcal{S}(x_0)$ be the solution set of an optimal control problem. If there exist $x_1^*(\cdot)$, $x_2^*(\cdot) \in \mathcal{S}(x_0)$ and $t \in [0, T]$ satisfying

$$x_1^*(t) \neq x_2^*(t),$$

then $x_0$ is called an *indifference point*. If the solution set $\mathcal{S}(x_0)$ has cardinality $k < \infty$, then $x_0$ is called of *order* $k$; otherwise it is of *infinite order*.

Wagener's conjecture: In an optimal control model with $n$ states, the maximal order of an indifference point is $n + 1$. An example for all threefold indifference point for a two-state production-inventory model is given in Feichtinger and Steindl (2006).

[3]There, the nice medieval story of Buridan's donkey is mentioned. A donkey stands at equal distance from two identical and equidistant bales of hay, unable to decide toward which bale to go. As rational economic agent the donkey is indifferent between moving toward the one or other bale, i.e. two optimal long-run stationary solutions.

[4]There exist also a few *two-state* models showing multiple equilibria, separated by 'Skiba-curves'; see, e.g., Haunschmied et al. (2003); Feichtinger and Steindl (2006).

**Definition 3** (Threshold point)**.** Let us consider an infinite time horizon problem with $x(0) = x_0$. Then $x_0$ is called a *threshold point* if for every neighborhood $U$ of $x_0$ there exist $x_{10},\, x_{20} \in U$ satisfying

$$\mathcal{S}^\infty(x_{10}) \cap \mathcal{S}^\infty(x_{20}) = \emptyset, \quad \mathcal{S}^\infty(x_{i0}) \neq \emptyset, \quad i = 1, 2.$$

**Definition 4** (Indifference-threshold point)**.** Let us consider an infinite time horizon problem with $x(0) = x_0$. Then $x_0$ is called an *indifference-threshold point* if it is both an indifference point and a threshold point.

*Remark* 1. At the moment there exists no canonical nomenclature of these points. However what is denoted here as asymptotic indifference point (AIP) is also known as Skiba or DNSS point; see Grass et al. (2008).

*Remark* 2. Note that in a one-dimensional infinite time horizon model the occurrence of an unstable focus plays a crucial role regarding the existence of an asymptotic indifference point described in Def. 4. An unstable node might be a threshold point. In this case the control is continuous, unlike the first case where it jumps at the indifference point.

The above discussion on multiple equilibria and Skiba points refers to *deterministic* optimal control problems with *infinite* time horizon. The multiplicity of optimal solutions may occur in models with *finite* time horizon. For a recent work in that context see Caulkins et al. (2010).

Finally, there are interesting extensions of multiple equilibria and threshold behavior to a stochastic framework; see, e.g., Bultmann et al. (2010).