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Minimizing the Dependency Ratio in a Population with Below-Replacement Fertility through Immigration

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Abstract

Many industrialized countries face fertility rates below replacement level, combined with declining mortality especially in older ages. Consequently, the populations of these countries have started to age. One important indicator of age structures is the dependency ratio which is the ratio of the nonworking age population to the working age population. In this work we find the age-specific immigration profile that minimizes the dependency ratio in a stationary population with below-replacement fertility. We consider two alternative policies. First, we fix the total number of people who annually immigrate to a country. Second, we prescribe the size of the receiving country's population. For both cases we provide numerical results for the optimal immigration profile, for the resulting age-structure of the population, as well as for the dependency ratio.

Keywords: optimal control theory, stationary population, immigration, below-replacement fertility

1. Introduction

In many industrialized countries fertility rates are below-replacement. Additionally, these countries face mortality decline, in particular at ages after

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retirement. Since fertility decline is very often the dominating effect, without immigration the population of these countries would decline. Moreover, the age structure of these countries' population is changing, i.e. growth in the number of elderly people and decline in the number of young people.

One important indicator of age structures is the so-called dependency ratio, which is the ratio of persons of nonworking age to persons of working age, usually the 20 to 65-year-olds. A low dependency ratio is desirable because it indicates that there are proportionally more adults of working age that can support the young and the elderly of the population. This in turn is advantageous for the countries' health-care system and pension schemes. A downfall of the relative number of working people in a population also has negative impacts on the growth path of the economy. Possible ways to counter the risks of these demographic changes are to increase the labor force participation by encouraging women and the elderly to work, and to step up immigration.

Similar to (Arthur and Espenshade, 1988; Mitra, 1990; Schmertmann, 1992; Wu and Li, 2003) , in this work we consider a population with constant immigration, fertility, and mortality rates. These studies already have shown that such populations eventually converge to stationary populations. Following (Schmertmann, 1992) from now on we will denote this kind of population as SI, meaning *stationary through immigration* [1]. Here, fertility is below-replacement level, meaning that without immigration the population would converge to zero. In our model we assume that the age-specific fertility rates of immigrants equal those of the natives.

In this work, we aim to find the age-specific immigration profile that minimizes the dependency ratio in a stationary population. We do so by applying optimal control theory which is a rather new methodology in demographic research, see for example (Feichtinger and Veliov, 2007). As typical in population dynamics, we assume that in our model the population only consists of females. We therefore formulate an optimal control problem where the age-specific immigration profile is the control variable and the population's age structure is the state variable.

A similar question to the one posed here is asked in (United Nations, 2001) where the authors study whether replacement migration can hinder aging and in further consequence, decline of the population. They examine

[1] The abbreviation SI was introduced by C. Schmertmann, cf. (Schmertmann, 1992)

the situation of eight industrialized countries during the time period from 1995 to 2050.

In the following, we consider two alternative policies in order to investigate their impact on the optimal immigration profile.

Policy 1: We fix the total number of people who annually immigrate to a country.

Policy 2: We prescribe the (stationary) total size of the receiving country's population.

Moreover, we assume that there are age-specific upper limits for immigration. Since the problem is linear in the control, the optimal immigration profile for both policies is of bang-bang type, meaning that the optimal solution jumps from one age-specific limit to the other. We prove that for the optimal solution of the problem with a fixed volume of immigration it holds that besides immigration at young and middle ages also in the vicinity of the maximal attainable age immigration takes place. Such bizarre old age immigration does not happen when we fix the size of the population rather than the total number of immigrants. Under reasonable assumptions on the vital rates and the age-specific immigration bounds, the optimal solution of the second policy equals the upper bound for immigration on not more than two separate intervals to the left of the retirement age and it is on the lower bound elsewhere.

In (Schmertmann, 2011) the question is raised of how age-targeted immigration policy can be used to increase the relative number of working people in a population. There, the total number of annual immigrants is fixed and the problem is reduced to a static optimization problem. It is shown that the highest relative number of workers can be achieved if all immigrants arrive at one single age. However, it is assumed that at each age an arbitrary high number of immigrants can be recruited. Consequently, in this case the bizarre old age immigration does not occur. However, the paper (Schmertmann, 2011) leaves open the question of how the optimal age-specific immigration profile looks like if not all immigrants are admitted at one single age. This issue among others is tackled in what follows.

From a mathematical point of view, a similar linear optimal control problem is considered in (Dawid et al., 2009). There, the authors determine the optimal recruitment policy of a stationary learned society using the example of the Austrian Academy of Sciences that minimizes the average age of the organization for a fixed number of recruits. In (Feichtinger and Veliov, 2007) their study was extended to the transitory case. Remarkably, the optimal recruitment is the same as in the stationary case. That is why we also start with the stationary case, hoping that its optimal immigration age profile is also the optimal solution to the transitory problem, as it could be shown for the similar problem in (Feichtinger and Veliov, 2007) for stationary (independent of time) data. However, for our particular problem the transitory case still needs to be studied.

The optimal control approach enables us to determine the marginal value of an immigrant of a certain age in terms of the dependency ratio, cf. (Wrzaczek et al., 2010), by interpretation of the so-called adjoint variable, cf. (Grass et al., 2008), whose clear meaning will be defined in Section 3. As a consequence we are able to decide which age-specific immigration profile is optimal for minimizing the dependency ratio. Moreover, the impact of an a-year-old immigrant on the dependency ratio can be represented as a sum of two components. The first component, which is referred to as the *direct* effect, accounts for a female's expected life time in the work force, and the fact that she eventually becomes part of the fraction of old people in the population after retiring. The second component, known as the *indirect* effect on the dependency ratio accounts for her expected number of children. Clearly, when an immigrant arrives towards the end of childbearing age the *indirect* effect is low. However, the expected remaining time in the working population is then also reduced, meaning that she is also relatively longer dependent. This represents the trade-off that drives our optimization problem.

The numerical analysis presented in this paper is restricted to the Austrian case. Following (Schmertmann, 2011) we do not account for emigration.

The rest of the paper (Schmertmann, 2011) is organized as follows. In Section 2 we state the problem. The optimal age-specific immigration profile for a fixed annual number of immigrants is characterized in Section 3. There, we also present numerical results for the case study of the Austrian population based on demographic data from 2008. In Section 4 we consider the total population size as fixed and provide also some numerical results for the optimal immigration profile and the dependency ratio in the case of the Austrian population. Moreover, in Section 5 we determine an a-year-old's value in units of the objective function, by using the adjoint variable's interpretation as shadow price. In Sections 6 and 7 we discuss the obtained results and indicate points of future work. In Appendix A and Appendix

C we apply Pontryagin's Maximum Principle to obtain necessary conditions for the optimal solution. The proof of the result that arbitrarily close to the maximum age there are ages where immigration takes place for the problem with a fixed number of immigrants is given in Appendix B.

2. Model Description and Definitions

In the following, α and β denote the lower and upper age limits determining the working age population and ω is the maximum attainable age of an individual. We aim to minimize the dependency ratio given as

$$D(M(\cdot)) := \frac{\int_0^\alpha N(a)\,\mathrm{d}a + \int_\beta^\omega N(a)\,\mathrm{d}a}{\int_\alpha^\beta N(a)\,\mathrm{d}a}, \quad 0 < \alpha < \beta < \omega,$$

by choosing the age distribution of immigrants $M(\cdot)$ in an optimal way. Therefore, $D(M(\cdot))$ denotes the dependency ratio that results when realizing the immigration profile $M(\cdot)$. Here, $N(a)$ denotes the number of females in the population of age a.

We come up with the following optimal control problem:

$$\min_{M(a)} D, \tag{2.1}$$

subject to

$$\dot{N}(a) = -\mu(a)N(a) + M(a), \tag{2.2}$$

$$N(0) = \int_0^\omega f(a)N(a)\,\mathrm{d}a, \tag{2.3}$$

$$0 \leq M(a) \leq \bar{M}(a). \tag{2.4}$$

Here, the age a is considered as a continuous variable and $\dot{N}(a)$ denotes the derivative of $N(\cdot)$. The immigration age profile $M(\cdot)$ is usually referred to as *control*, since it is the controlable input to the optimization problem. By $f(a)$ and $\mu(a)$, we denote age-specific fertility and mortality rates which do not change with time and are continuous functions in a. Additionally, we assume that $\int_0^\omega \mu(a)\,\mathrm{d}a = +\infty$, cf. (Anita, 2000), which ensures that $N(\omega) = 0$ holds. The exogenous data $\bar{M}(a)$ denoting the age-specific immigration bounds is supposed to be continuous.[2]

In this work, we consider two alternative policies:

[2]From a mathematical point of view the reason for imposing these age-specific bounds

Policy 1. We prescribe the total number of immigrants M_{tot}

$$M_{\mathrm{tot}} = \int_0^\omega M(a)\,\mathrm{d}a. \tag{2.5}$$

Policy 2. We prescribe the stationary population size N_{tot}

$$N_{\mathrm{tot}} = \int_0^\omega N(a)\,\mathrm{d}a. \tag{2.6}$$

These policies represent constraints on the immigration profile $M(\cdot)$ and the age structure $N(\cdot)$ in the population.

We define $l(a) := \mathrm{e}^{-\int_0^{\mathrm{a}} \mu(\mathrm{x})\ \mathrm{dx}}$ which is the probability that a female survives at least a years.

We recall the *reproductive value* of an a-year-old female, introduced by Fisher (Fisher, 1930) (see also Keyfitz (Keyfitz, 1977)), which is the expected number of future daughters of an individual from her current age onward, given that she has survived to this age as

$$v(a) = \int_a^\omega \frac{l(x)}{l(a)} f(x)\,\mathrm{d}x.$$

Accordingly, the *population's net reproduction rate* in a stationary population is the average number of daughters a female will have,

$$R = \int_0^\omega l(a) f(a)\,\mathrm{d}a.$$

The support of $f(\cdot)$ is a subset $[a_{\min}, a_{\max}] \subset [0, \omega]$, where $a_{\min}$ and $a_{\max}$ denote the youngest and oldest age of childbearing, respectively. Presumably, fertility $f(a)$ is below-replacement, which means it is not high enough to replace the current population. This property of below-replacement fertility fertility (and mortality) can be expressed in terms of the population's reproduction rate, meaning that $R < 1$ must hold.

is the applicability of Pontryagin's Maximum Principle. However, more practically spoken these bounds are justifiable because they may reflect the fact that age is not the only factor that should be taken into account when determining the optimal immigration policy. Therefore, policy makers are restricted in a way that they cannot get arbitrarily high immigration at any single age.

3. The optimal immigration profile for a fixed number of immigrants

In this section we analyze problem (2.1)–(2.5) by making use of optimal control theory. Our aim is to find the optimal immigration profile $M^*(a)$ that minimizes (2.1). For a given age interval $[\alpha, \beta] \subseteq [0, \omega]$ the function $\mathbb{I}_{[\alpha,\beta]}(\cdot)$ is defined as

$$\mathbb{I}_{[\alpha,\beta]}(a) = \begin{cases} 1 & \text{if } a \in [\alpha, \beta], \\ 0 & \text{otherwise.} \end{cases}$$

We introduce the adjoint variable $\xi(a)$ and interpret it as the shadow price of an individual of a certain age a [3]. Function $\xi(a)$ reflects the shift of the dependency ratio that results from changing the population's age structure by one a-year-old individual. Therefore, in analogy to its interpretation in capital theory we also call $\xi(a)$ the shadow price of an a-year-old. Specifically, it measures the marginal worth in units of the dependency ratio, of an increment in $N(a)$ at age a. Moreover, we introduce the constants λ_1, λ_2 and refer to them as *Lagrange multipliers*. The Lagrange multiplier λ_1 reflects the marginal worth of an additional newborn. The constant λ_2 may be interpreted as the marginal change in the dependency ratio when adding an additional immigrant per year.

Theorem 1. *If $(N^*(\cdot), M^*(\cdot))$ is an optimal solution of problem (2.1)–(2.5), then there are Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ and an adjoint variable $\xi(\cdot)$, such that:*

(i) the continuous function $\xi(\cdot)$ on $[0, \omega]$ satisfies

$$\begin{aligned} \dot{\xi}(a) &= \mu(a)\xi(a) - \lambda_1 f(a) - \frac{(D(M^*(\cdot)) + 1)^2}{N_{\text{tot}}(M^*(\cdot))}\mathbb{I}_{[\alpha,\beta]}(a) + \frac{(D(M^*(\cdot)) + 1)}{N_{\text{tot}}(M^*(\cdot))}, \\ \xi(0) &= \lambda_1, \quad \xi(\omega) = 0, \end{aligned} \tag{3.7}$$

[3]The term shadow price results from the interpretation of the adjoint variable (here $\xi(a)$) in optimal control problems arising from capital theory. There, it is interpreted as the highest hypothetical - therefore also called - shadow price a rational decision-maker would be willing to pay for owning an additional unit of the corresponding state variable at time a measured by the discounted future profit. See (Grass et al., 2008) for a more detailed discussion of the economic interpretation of the maximum principle.

(ii) and the following maximum principle holds for almost every $a \in (0, \omega)$:

$$(\xi(a) - \lambda_2)M^*(a) = \max_{0 \le M \le \bar{M}(a)} (\xi(a) - \lambda_2)M. \tag{3.8}$$

Proof 1. *For the proof of Theorem 1 see Appendix A.*

Theorem 1 provides necessary conditions (3.7),(3.8) for the solution of problem (2.1)–(2.5), meaning that they constitute requirements that the optimal solution has to fulfill. The existence of an optimal solution follows from a general argument. From (3.8) it can be seen that the optimal control is of bang-bang type, jumping from one boundary to the other. Therefore, function $\xi(\cdot) - \lambda_2$ is usually referred to as *switching function* because the change of its sign determines the ages a at which the optimal control switches from one boundary to the other in consequence of (3.8):

$$M^*(a) = \begin{cases} \bar{M}(a) & \text{if } \xi(a) > \lambda_2, \\ \text{not determined} & \text{if } \xi(a) = \lambda_2, \\ 0 & \text{if } \xi(a) < \lambda_2. \end{cases} \tag{3.9}$$

We assume that equality $\xi(a) = \lambda_2$ happens only in isolated points, so that the values $M^*(a)$ at these points have no effect on the dependency ratio.[4]

To obtain the optimal immigration profile it remains to determine $\xi(\cdot)$ and λ_2. The right hand side of the differential equation (3.7) is discontinuous at ages $a = \alpha$ and $a = \beta$ and therefore the solution ξ has two kinks at each of these ages, see for example Figure 3.2.
The solution of (3.7) can be expressed in terms of demographic variables, providing a possibility to give a clear interpretation of $\xi(a)$ as shadow price of $N(a)$.

We note that (3.7) is a boundary value problem for a linear differential equation. Finally, by using the Cauchy formula for (3.7) we obtain the solution,

$$\xi(a) = \lambda_1 v(a) + \frac{(D(M^*(\cdot)) + 1)}{N_{\text{tot}}(M^*(\cdot))} \Big(D(M^*(\cdot) + 1) e_{[\alpha,\beta]}(a) - e_{[0,\omega]}(a) \Big). \tag{3.10}$$

[4]This assumption holds if the fertility and mortality rates are such that for any $d \in \mathbb{R}$ meas$\{a \in \Omega : \lambda_2 \mu(a) - \lambda_1 f(a) = d\} = 0$.

Using the boundary condition $\xi(0) = \lambda_1$ and noting that $R = v(0)$ we obtain that

$$\lambda_1 = \frac{\frac{(D(M^*(\cdot))+1)}{N_{\text{tot}}(M^*(\cdot))}\Big(D(M^*(\cdot)+1)e_{[\alpha,\beta]}(0) - e_{[0,\omega]}(0)\Big)}{1-R} \tag{3.11}$$

holds. Here, $e_{[0,\omega]}(a) = \int_a^\omega \frac{l(x)}{l(a)}\,\mathrm{d}x$ is the life expectancy in $[0,\omega]$ at age a. Similarly, $e_{[\alpha,\beta]}(a) = \int_a^\omega \frac{l(x)}{l(a)}\mathbb{I}_{[\alpha,\beta]}(x)\,\mathrm{d}x$ is the working life expectancy of an a-year-old, reflecting the expected number of years an a-year-old would spend working. Clearly, $e_{[\alpha,\beta]}(a) = 0$ for $a \geq \beta$. With (3.9) and expressions (3.10)–(3.11) we are now able to obtain the optimal immigration profile $M^*(\cdot)$, where the Lagrange multiplier λ_2 has to be determined in such a way, that (2.5) holds for the resulting solution.

Moreover, under an additional regularity condition precisely formulated in Appendix B, the optimal immigration profile $M^*(a)$ is such that arbitrarly close to the maximum age ω there are ages where immigration is optimal. The additional 'regularity' condition that is needed for the theorem below is such, that the contribution of an additional a-year-old immigrant to the number of workers in the resulting SI population (measured in years) must not be proportional to its contribution to the whole population for almost every age $a \in [0,\omega]$. An individual's contribution consists of her own expected years lived in the host country and the analog contribution of all her future descendants. The precise formulation of this result reads as follows.

Theorem 2. *Let $M(\cdot)$ be an arbitrary immigration profile which fulfills* (2.4),(2.5) *and additionally $M(a) < \bar{M}(a)$ for $a \in [\omega-\delta,\omega]$ and some $\delta > 0$. Then there is an immigration profile $\tilde{M}(\cdot)$ which satisfies* (2.4),(2.5) *such that*

$$\tilde{D}(\tilde{M}) < D(M).$$

Proof 2. *For the proof see Appendix B.*

From Theorem 2 we may conclude that for the optimal solution it holds that arbitrarily close to ω there exist ages $\bar{a}$, with immigration on its upper bound, $M^*(\bar{a}) = \bar{M}(\bar{a})$. Since we aim to minimize the relative number of dependent people in the population, the fact that immigration at the end of the life horizon is optimal seems to be counterintuitive. This bizarre property of the optimal solution is due to the age-specific immigration bounds, (2.4), that are introduced in this model. If they are removed, as done in (Schmertmann, 2011), this effect cannot be observed anymore.

We also overcome this counterintuitive result in Section 4 by considering a second policy, where we fix the size of the stationary population, see Equation (2.6), and not the volume of immigrants, (2.5).

3.1. A case study: the Austrian case

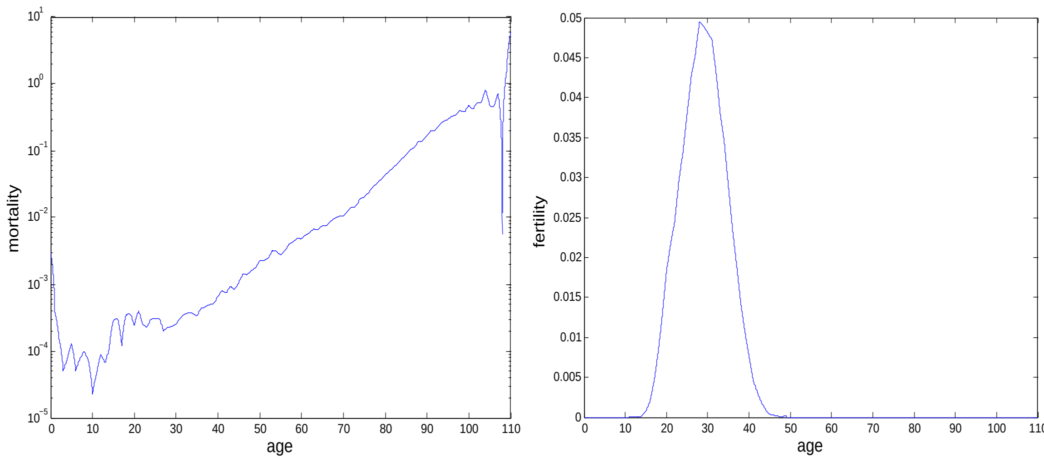

Figure 3.1: Female mortality rate $\mu(a)$ (left, logarithmic scale) and fertility rate $f(a)$ (right); Austria 2008.

The numerical results for the optimal immigration profile and the dependency ratio obtained in this section are based on the analytical derivations above. In the following, we will assume that $\alpha = 20$, $\beta = 65$, and $\omega = 110$. For the computations we initialize the age structure of demographic variables referring to Austrian data as of 2008, cf. Figure 3.1, and interpolate these data piecewise linearly to obtain continuous representations of the vital rates. The actual age-specific immigration numbers of 2008 are denoted by $M_{\text{act}}(a)$.

Scenario 1. We set

$$\bar{M}(a) = 2M_{\text{act}}(a), \quad \forall a \in [0, \omega],$$

which corresponds to a doubling of the number of immigrants at all ages compared to the 2008 level. For M_{tot} we prescribe a total volume of 80000 females.

The resulting age profile that fulfills the maximization condition (3.8) is

$$M^*(a) = \begin{cases} \bar{M}(a) & \text{if } a \in [11, 49] \cup [82, 110], \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

This can be concluded from the values of the adjoint variable $\xi(a)$ at ages a depicted in Figure 3.2. The solid line in Figure 3.2 corresponds to the λ_2-level. Consequently, for ages where $\xi(a)$ has values larger than λ_2 immigration is at its upper bound and for ages where $\xi(a)$ is smaller than λ_2 the optimal immigration profile is zero. As one can see in Figure 3.2, the adjoint variable $\xi(a)$ exhibits two kinks at ages $\alpha = 20$ and $\beta = 65$, due to the discontinuity of the right hand side of the differential equation (3.7). For a detailed explanation of the shape of the adjoint variable as a function of a see Section 5. On the right side of Figure 3.2, the age structure of the optimal SI population is depicted. As for a closed stationary population, the age structure of an SI population exhibits a flat line at young ages due to the low mortality at these ages. The resulting minimal dependency ratio is 75.14%[5], which corresponds to about 75 dependents per 100 workers. The resulting total SI (stationary through immigration) population size is 13.0 million females.

Scenario 2. We also performed the calculations with $M_{\text{tot}} = 50000$ which is close to the actual total number of (female) immigrants for Austria in 2008. The age-specific upper bound was set to $\bar{M}(a) = 20M_{\text{act}}(a)$ which corresponds to a high supply of immigrants at all ages. From the switching function depicted in Figure 3.3, we can conclude that the optimal immigration profile reads

$$M^*(a) = \begin{cases} \bar{M}(a) & \text{if } a \in [33, 36] \cup [109, 110], \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

The resulting minimal dependency ratio is 72.24%. This corresponds to a share of 58.1% workers in the population. The resulting total size of the female SI population is 4.1 million.

The right-hand side of Figure 3.3 represents the age structure of the optimal SI population. What is striking is that it is optimal to let people immigrate at the end of the age interval, although they are part of the economically dependent population. This can be intuitively explained by the fact that (2.5) has to be fulfilled.

[5]Typically, the dependency ratio is expressed as percentage, i.e. $100 \times D$.

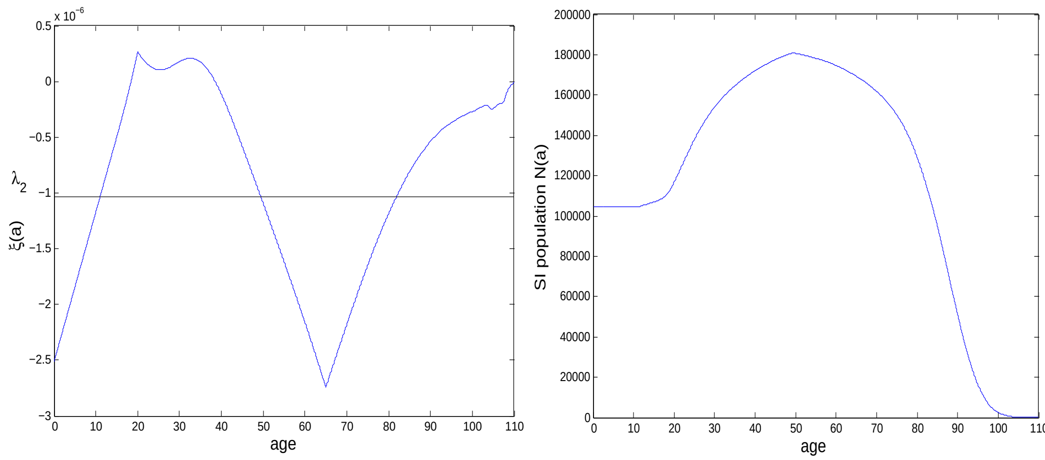

Figure 3.2: The adjoint variable $\xi(\cdot)$ determining the optimal immigration policy of problem (2.1)–(2.5) for $M_{\text{tot}} = 80000$ and $\bar{M}(a) = 2M_{\text{act}}(a)$. The horizontal line indicates the λ_2 level. The optimal age structure of the SI population $N(a)$ is presented in the right plot.

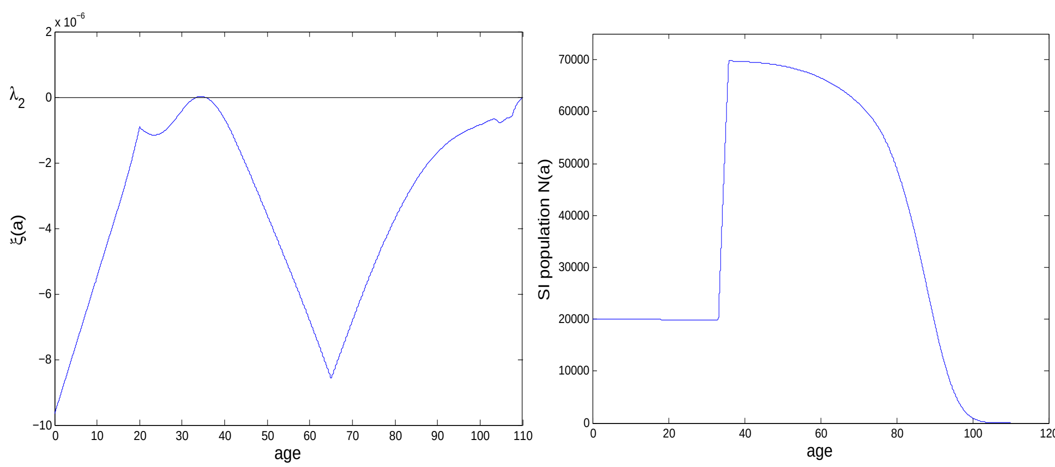

Figure 3.3: The adjoint variable $\xi(\cdot)$ determining the optimal immigration policy of problem (2.1)–(2.5) for $M_{\text{tot}} = 50000$ and $\bar{M}(a) = 20M_{\text{act}}(a)$. The horizontal line indicates the λ_2 level. The optimal age structure of the SI population $N(a)$ is presented to the right.

4. The optimal immigration profile for a fixed population size

We slightly change the model and instead of fixing the volume of immigrants (Policy 1), we require that the number of people in the population equals a prescribed value (Policy 2), i.e. we consider problem (2.1)–(2.4) with the additional constraint (2.6). Theorem 2 below states necessary conditions for the optimal solution.

Theorem 3. *If* $(\tilde{N}^*(\cdot), \tilde{M}^*(\cdot))$ *is an optimal solution of problem* (2.1)–(2.4) & (2.6)*, then there are Lagrange multipliers* $\tilde{\lambda}_1, \tilde{\lambda}_2$*, and an adjoint variable* $\tilde{\xi}(\cdot)$ *such that:*

x *the continuous function* $\tilde{\xi}(\cdot)$ *on* $[0,\omega]$ *satisfies*

$$\begin{aligned} \dot{\tilde{\xi}}(a) &= \mu(a)\tilde{\xi}(a) - \tilde{\lambda}_1 f(a) - \mathbb{I}_{[\alpha,\beta]}(a) + \tilde{\lambda}_2, \\ \xi(0) &= \tilde{\lambda}_1, \quad \xi(\omega) = 0, \end{aligned} \tag{4.14}$$

x *and the maximum principle holds for almost every* $a \in (0,\omega)$

$$\tilde{\xi}(a)\tilde{M}^*(a) = \max_{0 \le M \le \bar{M}(a)} \tilde{\xi}(a) M. \tag{4.15}$$

Proof 3. *For the proof see Appendix C.*

The optimal immigration profile is again of bang-bang type,

$$\tilde{M}^*(a) = \begin{cases} \bar{M}(a) & \text{if } \tilde{\xi}(a) > 0, \\ \text{not determined} & \text{if } \tilde{\xi}(a) = 0, \\ 0 & \text{if } \tilde{\xi}(a) < 0, \end{cases} \tag{4.16}$$

and it remains to determine $\tilde{\xi}(\cdot)$. Similar calculations as in Section 3 give

$$\tilde{\xi}(a) = \tilde{\lambda}_1 v(a) + e_{[\alpha,\beta]}(a) - \tilde{\lambda}_2 e_{[0,\omega]}(a), \tag{4.17}$$

where, using the boundary condition $\xi(0) = \tilde{\lambda}_1$, we obtain

$$\tilde{\lambda}_1 = \frac{e_{[\alpha,\beta]}(0) - \tilde{\lambda}_2 e_{[0,\omega]}(0)}{1-R}. \tag{4.18}$$

Note, that $\tilde{\xi}(\cdot)$ is independent of the optimal solution $(\tilde{N}^*(\cdot), \tilde{M}^*(\cdot))$ and can therefore be calculated separately for each $\tilde{\lambda}_2$. In order to determine the optimal solution $(\tilde{N}^*(\cdot), \tilde{M}^*(\cdot))$, the Lagrange multiplier $\tilde{\lambda}_2$ in (4.14) has to be chosen in such a way that condition (2.6) is fulfilled. Therefore, the value of $\tilde{\lambda}_2$ depends on the choice of the prescribed value N_{tot}.

4.1. Analytical study of the optimal immigration profile

In the following, we derive general results for the optimal immigration profile for given age-specific fertility $f(a)$ and mortality $\mu(a)$. We show that the optimal immigration profile attains its upper bound $\bar{M}(a)$ on no more than two separate intervals.

Since the change of the sign of the adjoint variable $\tilde{\xi}(a)$ determines the switches of the optimal solution from one limit to the other, we count how many times the switching function (4.17) can cross its switching level $\tilde{\xi}(a) = 0$. To estimate this number from above we count how many times the derivative in (4.14) can change its sign at level $\tilde{\xi}(a) = 0$ from positive to negative

$$\dot{\tilde{\xi}}(a)\Big|_{\tilde{\xi}=0} = -\tilde{\lambda}_1 f(a) - \mathbb{I}_{[\alpha,\beta]}(a) + \tilde{\lambda}_2 = 0. \tag{4.19}$$

Assumption 1. *The upper limit $\bar{M}(a)$ is such that if $M(a) \equiv \bar{M}(a)$, then $\int_0^\omega N(a)\,\mathrm{d}a > N_{\mathrm{tot}}$.*

Assumption 2. *If $M(a) \equiv 0$, then $\int_0^\omega N(a)\,\mathrm{d}a = 0$.*

That means that below-replacement fertility holds.

Corollary 1. *There should be at least one interval with $\xi(a) > 0$.*

Otherwise the optimality condition (4.16) requires $M(a) = 0$ for almost every a. This, however, leads to the contradiction between Assumption 2 and $N_{\mathrm{tot}} > 0$ in (2.6). ■

Proposition 1. $\tilde{\lambda}_2 \in [0, 1]$.

Indeed, $\tilde{\lambda}_2 < 0$ leads to $\tilde{\lambda}_1 > 0$ in (4.18) and both lead to $\dot{\tilde{\xi}}(a)\Big|_{\tilde{\xi}=0} < 0$ in (4.19) for all $a \in [0, \omega)$ so that $\xi(a) > 0$ on $a \in [0, \omega)$ which contradicts Assumption 1. If $\tilde{\lambda}_2 > 1$ then $\tilde{\lambda}_1 < 0$ because $e_{[\alpha,\beta]}(0) < e_{[0,\omega]}(0)$, thus the derivativein (4.19) has the following property $\dot{\tilde{\xi}}(a)\Big|_{\tilde{\xi}=0} = -\tilde{\lambda}_1 f(a) - \mathbb{I}_{[\alpha,\beta]}(a) + \tilde{\lambda}_2 > -1 + \tilde{\lambda}_2 > 0$ for all $a \in [0, \omega]$, since $\min\{f(a)\} = 0$. But to satisfy terminal condition $\tilde{\xi}(\omega) = 0$, for the adjoint variable it should hold, that $\tilde{\xi}(a) < 0$ for $a \in [0, \omega)$. That contradicts Assumption 2 and $N_{\mathrm{tot}} > 0$ in (2.6). ■

Proposition 2.

a) $\tilde{\xi}(a) < 0$ *if* $\tilde{\lambda}_2 > 0$ *for all* $a \in [\beta, \omega)$,

b) $\tilde{\xi}(a) = 0$ *if* $\tilde{\lambda}_2 = 0$ *for all* $a \in [\beta, \omega]$.

Indeed, since $e_{[\alpha,\beta]}(a) = 0$ and $v(a) = 0$ holds for all $a \in [\beta, \omega]$ it follows from (4.17) and Proposition 1 that $\tilde{\xi}(a) = -\tilde{\lambda}_2 e_{[0,\omega]}(a) \le 0$, $a \in [\beta, \omega]$. Thus, **b)** is obvious and **a)** follows from the inequality $e_{[0,\omega]}(a) > 0$ for all $a \in [0, \omega)$ provided that $l(a) > 0$ for all $a \in [0, \omega)$. ■

Assumption 3. *The fertility schedule* $f(a)$ *is single peaked with support to the left from* β *and to the right from* 0, *i.e.* $a_{min} < \alpha < a_{max} \le \beta$.

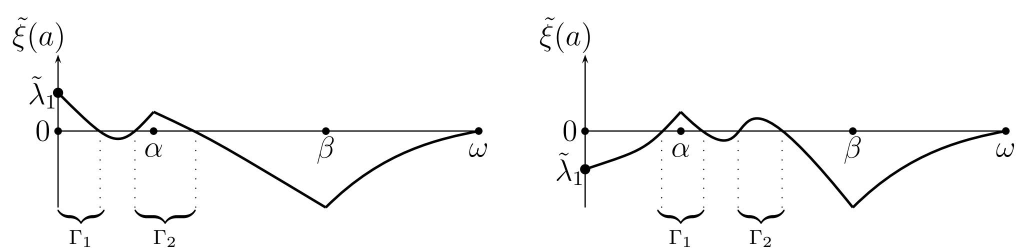

Figure 4.4: The adjoint variable $\tilde{\xi}(\cdot)$ determining the optimal immigration for two separate age intervals Γ_1 and Γ_2 in two cases: **a)** $\tilde{\lambda}_1 > 0$ (left) and **b)** $\tilde{\lambda}_1 < 0$ (right).

Let us denote the maximal fertility age as $a_{fmax} = \arg\max(f(a))$.

Lemma 1. *The maximal number of separate intervals on which* $\tilde{\xi}(a) > 0$ *is two and denoting these intervals* Γ_1 *and* Γ_2 *so that for all* $a_1 \in \Gamma_1$ *and* $a_2 \in \Gamma_2$ *the inequality* $a_1 < a_2$ *holds, we have (see Fig. 4.4):*

a) *if* $a_{fmax} < \alpha$ *then* $\alpha \in \Gamma_2$,

b) *if* $a_{fmax} > \alpha$ *then* $\alpha \in \Gamma_1$.

It follows from Proposition 2 that $\Gamma_1, \Gamma_2 \subset [0, \beta]$.

The derivative (4.19) can change its sign from plus to minus only at $a = \alpha$ because of the jump of the function $\mathbb{I}_{[\alpha,\beta]}(a)$ or/and at $a = a_0$, where a_0 is such a root of the equation $\dot{\tilde{\xi}}(a_0)\Big|_{\tilde{\xi}=0} = 0$ that $\ddot{\tilde{\xi}}(a_0)\Big|_{\tilde{\xi}=0} = -\tilde{\lambda}_1 \dot{f}(a_0) < 0$. It follows from Proposition 1 and Assumption 3 that equation $\dot{\tilde{\xi}}(a_0)\Big|_{\tilde{\xi}=0} = 0$ cannot have more than two roots all located either in $[0, \alpha)$ or in $[\alpha, \beta]$ depending on the sign of $\tilde{\lambda}_1$.

If $\tilde{\lambda}_1 > 0$ then equation $\dot{\tilde{\xi}}(a_0)\Big|_{\tilde{\xi}=0} = 0$ can only have roots in $[0, \alpha)$, where a_0 is the first root, if any, of the equation $-\tilde{\lambda}_1 f(a) + \tilde{\lambda}_2 = 0$.

If $\tilde{\lambda}_1 < 0$ then equation $\dot{\tilde{\xi}}(a_0)\Big|_{\tilde{\xi}=0} = 0$ can have roots only in $[\alpha, \beta]$ so a_0 is the second root, if any, of the equation $-\tilde{\lambda}_1 f(a) - 1 + \tilde{\lambda}_2 = 0$, which can happen only when $a_{fmax} > \alpha$.

Thus, it follows from the continuity of the function $\tilde{\xi}(a)$ that it can be positive on not more than two separate intervals. It is also easy to see graphically in Fig. 4.4 that if the function $\tilde{\xi}(a)$ is positive on two separate intervals $\Gamma_1, \Gamma_2 \subset [0, \beta]$, these intervals must contain both points a_0 and α where derivative (4.19) changes its sign, so that $a_0 \in \Gamma_1$, $\alpha \in \Gamma_2$ when $\tilde{\lambda}_1 > 0$ and $\alpha \in \Gamma_1$, $a_0 \in \Gamma_2$ when $\tilde{\lambda}_1 < 0$. ∎

4.2. A case study: the Austrian case

For the calculations we initialize again the fertility and mortality profiles with Austrian data as of 2008, cf. Figure 3.1. The Lagrange multiplier $\tilde{\lambda}_2$ is calculated such that condition (2.6) is fulfilled by the optimal solution. For the total population size we prescribe the resulting sizes from Section 3, i.e. $N_{\text{tot}} = 13.0$ million and $N_{\text{tot}} = 4.1$ million, respectively.

Scenario 1. Therefore, by setting $N_{\text{tot}} = 13.0$ million and $\bar{M}(a) = 2M_{act}(a)$, we achieve a corresponding dependency ratio $D = 74.73\%$ which is slightly smaller than the one we obtain above and the resulting volume of immigrants is 72000. The corresponding optimal immigration profile reads as

$$\tilde{M}^*(a) = \begin{cases} \bar{M}(a) & \text{if } a \in [9, 41], \\ 0 & \text{otherwise}, \end{cases} \tag{4.20}$$

which is determined according to (4.16). Figure 4.5 shows the corresponding adjoint variable $\tilde{\xi}(\cdot)$ and the optimal age structure $\tilde{N}^*(\cdot)$.

Scenario 2. We also calculate the optimal immigration profile for $N_{\text{tot}} = 4.1$ million females and $\bar{M}(a) = 20M_{act}(a)$,

$$\tilde{M}^*(a) = \begin{cases} \bar{M}(a) & \text{if } a \in [33, 36], \\ 0 & \text{otherwise}. \end{cases} \tag{4.21}$$

Figure 4.6 shows the adjoint variable $\tilde{\xi}(\cdot)$ and the optimal age structure $\tilde{N}^*(\cdot)$. For these parameter values we achieve a corresponding dependency ratio $D = 72.24\%$ and the resulting volume of immigrants is 50000. We observe that although the optimal immigration profiles differ, we obtain the same numerical results for the dependency ratio D and the total number of immigrants, for problem (2.1)–(2.5) and problem (2.1)–(2.6) for these numerical values. This is because the upper bound $\bar{M}(a)$ is zero for ages $a > 95$.

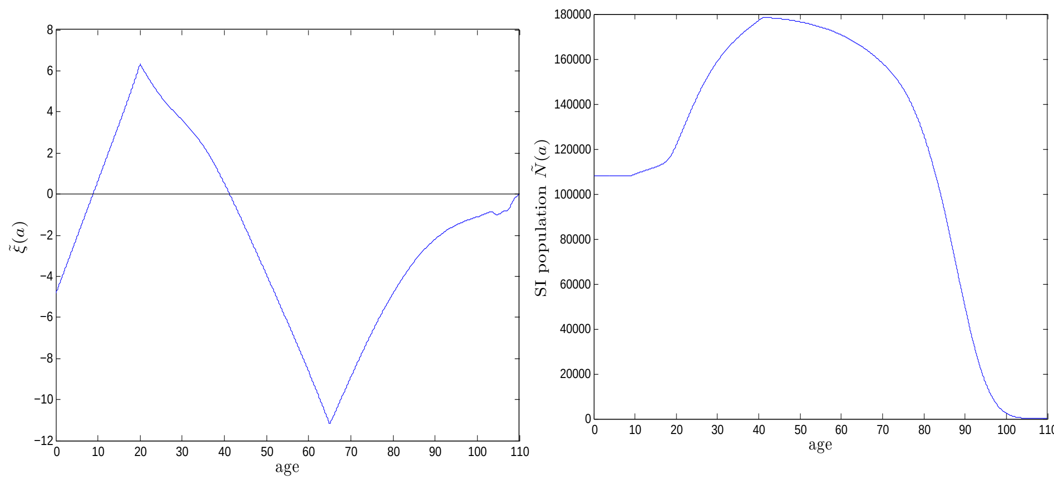

Figure 4.5: The adjoint variable $\tilde{\xi}(\cdot)$ determining the optimal immigration policy of problem (2.1)–(2.4) with constraint (2.6) $N_{\text{tot}} = 13.0$ million and $\bar{M}(a) = 2M_{act}(a)$ (left). The black line indicates the zero line. The age structure of the SI population is presented to the right.

From the switching functions in Figure 4.5 and Figure 4.6 we also see that it is not optimal that people immigrate towards the end of the life cycle.

5. Direct and indirect effect of an additional individual

The adjoint variables $\xi(a)$ and $\tilde{\xi}(a)$ may also be interpreted as shadow price of $N(a)$ and $\tilde{N}(a)$, meaning that they reflect the change of the dependency ratio, when the population's age structure is marginally increased at age a, roughly speaking, when the population is increased by one a-year-old. Moreover, this shadow price, see Equation (3.10) and (4.17), can be written as the sum of two parts

$$\xi(a) = \xi^d(a) + \lambda_1 v(a). \tag{5.22}$$

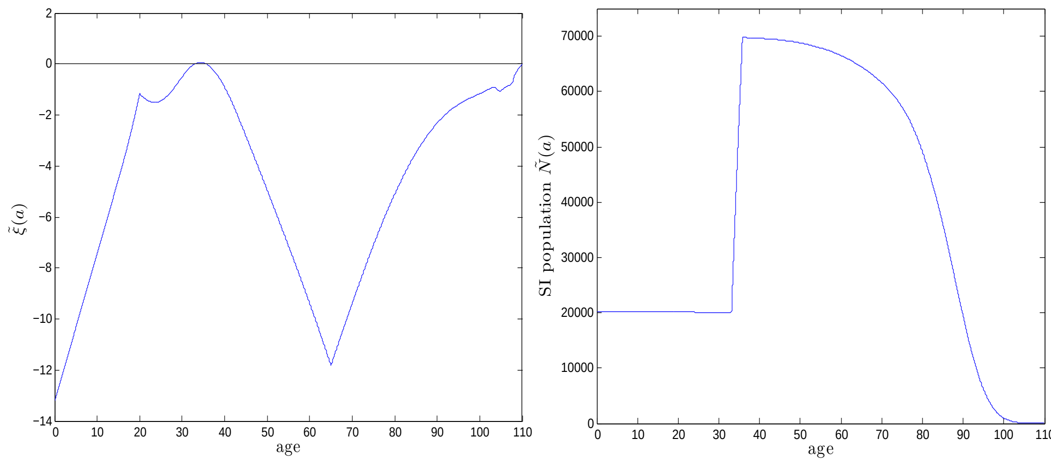

Figure 4.6: The adjoint variable $\tilde{\xi}(\cdot)$ determining the optimal immigration policy of problem (2.1)–(2.4) with constraint (2.6) for $N_{\text{tot}} = 4.1$ million and $\bar{M}(a) = 20M_{act}(a)$ (left). The black line indicates the zero line. The age structure of the SI population is presented to the right.

As pointed out in a more general setting in (Wrzaczek et al., 2010), the *direct* effect $\xi^d(a)$ determines the marginal value of an individual of age a who is currently alive. The *indirect* effect of an a-year-old, $\lambda_1 v(a)$, is the reproductive value, i.e. the number of expected future daughters, weighted by the shadow price of newborns, λ_1, since $\xi(0) = \lambda_1$. Therefore, the indirect effect can be interpreted as the value of expected future births of an a-year-old in units of the dependency ratio. This is a generalization of the interpretation of the reproductive value, cf. (Fisher, 1930; Wrzaczek et al., 2010). Note, that the indirect effect can also be negative, namely when an additional newborn is negatively valued for the population. The corresponding interpretation holds for $\tilde{\lambda}_1$ in (4.17).

The remaining terms in (3.10) and correspondingly (4.17) therefore represent the direct effect, because they aggregate the individual's own contribution to a minimal dependency ratio if she were to survive.
The direct effect accounts positively for her expected remaining years in $[\alpha, \beta]$ and negatively for her remaining life expectancy in $[0, \alpha]$ (for $a \leq \alpha$) and $[\beta, \omega]$.

The Lagrange multiplier λ_2 may also be interpreted as the marginal effect on the dependency ratio when changing the total number of immigrants M_{tot}.

Similarly, $\tilde{\lambda}_2$ measures the effect of a marginal change of the prescribed population size N_{tot} on the dependency ratio.
In Figure 5.7 we plotted the direct and indirect effect of an additional a-year-old separately. We consider again the Austrian case for problem (2.1)–(2.5), where we set $M_{\text{tot}} = 50000$ and $\bar{M}(a) = 20M_{act}(a)\ \forall a$. The dotted line corresponds to the weighted reproductive value, representing the indirect effect. The dashed line corresponds to the direct effect. The sum of these two lines, by definition, exhibits $\xi(\cdot)$, which is depicted by the solid line.
As it can be seen in Figure 5.7, the indirect effect reduces the absolute value of the adjoint variable $\xi(\cdot)$ in early ages, preventing these ages to be optimal. Furthermore, this effect is zero for ages older than the maximum age of childbearing. Therefore, after this age the direct effect and the adjoint variable coincide. Moreover, we see in Figure 5.7, that the direct effect reaches its maximum at age 20, since these individuals spend their whole working life in the receiving country, and then falls monotonically until age 65. However, the sharp increase in the indirect effect between ages $[20, 40]$ shifts the optimal age away from 20 and further to the right. The increase of the direct effect after age 65 is due to the fact that the remaining life expectancy in $[0, \omega]$, which is the only term left in equation (3.10), is decreasing with age, and therefore the burden represented by these females on the dependency ratio is reduced. We also see from equation (3.10) that the direct effect always increases until age 20. This is due to the fact, that the remaining life expectancy decreases, implying a higher value of this individual in units of the dependency ratio and also because the ratio between number of person-years lived in the working ages, $\int_\alpha^\beta l(x)\,\mathrm{d}x$, and the individual's probability to survive until age a, $l(a)$, increases with a.

6. Discussion

The aim of the present paper is to determine the age-specific immigration policy that minimizes the dependency ratio in a population with below-replacement fertility assuming that the vital rates remain constant over time. We apply optimal control theory which is a rather new approach in demographic research.

We consider two alternative policies. First, we prescribe the total number of immigrants. Secondly, we fix the total population size while the rest of the model remains the same. We note that in both cases the bang-bang behavior

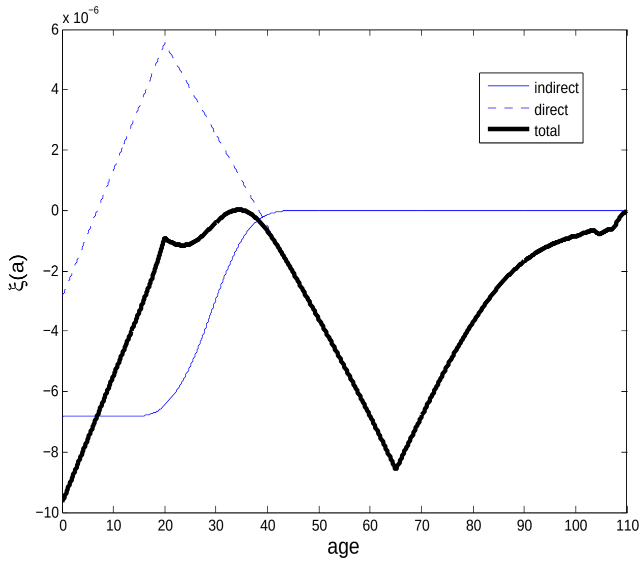

Figure 5.7: The direct (dashed line), indirect (solid blue line) and total (solid black line) effect of an additional a-year-old.

of the solution depends on the sign of the so-called switching function whose shape with varying age a is determined by the adjoint variable.

In the model with a fixed total number of immigrants, it is shown that for the optimal solution it holds that there are ages in the vicinity of the maximum attainable age where immigration occurs. When we fix the total population size of the receiving country, the optimal solution is such that immigration happens at not more than two separate age intervals, which lie to the left of the retirement age and thus may exhibit a two-peeked optimal immigration profile. The qualitative shape of the optimal immigration profile can be studied by investigating the shape of the adjoint variable as was done in Section 4.

We present numerical results for a case study of the Austrian population based on demographic data from 2008 which underline our theoretical findings.

Moreover, by analyzing the shape of the switching function or equivalently the adjoint variable, and interpreting it as shadow price, we determine the marginal value of an a-year-old individual in terms of the objective function.

7. Extensions

Having in mind the statement by John M. Keynes: 'In the long run we are all dead', in future work, we also aim to study the transitory case, where we consider time varying fertility, mortality and immigration rates. Similar as in (Feichtinger and Veliov, 2007), the resulting problem is a distributed control problem, which can be formalized by an infinite horizon optimal control model for a first order partial differential equation, which is of McKendrick-type (Keyfitz, 1977; Keyfitz and Keyfitz, 1997). Although, the similarity in the structure of the problem indicates that as in (Feichtinger and Veliov, 2007) it holds that for stationary data, i.e. fertility and mortality rates, the optimal solution is also stationary, this result does not follow immediately and needs some deeper mathematical involvement. Also in the transitory case, similar analysis of the adjoint variables which again can be interpreted as shadow price can be carried out, cf. (Wrzaczek et al., 2010). Therefore, optimality conditions for this distributed parameter control model have to be derived in order to obtain necessary conditions for the optimal solutions. These optimality conditions obtain partial differential equations, which have to be solved numerically.

8. Acknowledgements

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Appendix A. Proof of Theorem 1

We shall apply Pontryagin's maximum principle in the form obtained in (Alekseev et al., 1987). Therefore, in addition to $N(a)$ we introduce the auxiliary state variables $X(a)$, $Y(a)$, being continuous functions of a. The corresponding state equations read

$$\dot{X}(a) = N(a), \quad X(0) = 0, \tag{A.1}$$

$$\dot{Y}(a) = \mathbb{I}_{[\alpha,\beta]}(a)N(a), \quad Y(0) = 0. \tag{A.2}$$

Equivalently, it holds that

$$X(a) = \int_0^a N(\tau)\,\mathrm{d}\tau \quad \text{and} \quad Y(a) = \int_\alpha^a \mathbb{I}_{[\alpha,\beta]}(\tau)N(\tau)\,\mathrm{d}\tau.$$

State variable $X(a)$ accounts for the people in the population that are younger or equal than a years old and $Y(a)$ accounts for those whose age is between or equal α and $a \leq \beta$.
In this way we can express the objective function (2.1) by evaluating functions $X(\cdot)$ and $Y(\cdot)$ at the terminal value ω. Therefore, solving problem (2.1)–(2.4) with the additional constraint (2.5) is equivalent to solving

$$\min_{M(a)} \frac{X(\omega)}{Y(\omega)}, \tag{A.3}$$

subject to

$$\begin{aligned}
&\dot{N}(a) = -\mu(a)N(a) + M(a), && \text{(A.4)}\\
&\dot{Y}(a) = \mathbb{I}_{[\alpha,\beta]}(a)N(a), \quad Y(0) = 0, && \text{(A.5)}\\
&\dot{X}(a) = N(a), \quad X(0) = 0, && \text{(A.6)}\\
&N(0) = \int_0^\omega f(a)N(a)\,\mathrm{d}a, && \text{(A.7)}\\
&M_{\text{tot}} = \int_0^\omega M(a)\,\mathrm{d}a, && \text{(A.8)}\\
&0 \leq M(a) \leq \bar{M}(a). && \text{(A.9)}
\end{aligned}$$

Introducing Lagrange multipliers λ_1, λ_2, and the adjoint variables $\xi(\cdot)$, $\zeta(\cdot)$ and $\eta(\cdot)$ we define the Hamiltonian as

$$\begin{aligned}
H = {} & \xi(a)(-\mu(a)N(a) + M(a)) + \zeta(a)N(a) + \eta(a)\mathbb{I}_{[\alpha,\beta]}(a)N(a) && \text{(A.10)}\\
& - \lambda_0 \frac{X(\omega)}{Y(\omega)} - \lambda_1 f(a)N(a) - \lambda_2 M(a). && \text{(A.11)}
\end{aligned}$$

In (Alekseev et al., 1987), p.218 a version of Pontryagin's maximum principle is formulated.

These conditions provide necessary conditions for the optimal solution (N^*, X^*, Y^*, M^*) of problem (B.4)–(B.6) which can be summarized by the following expressions

$$(\xi(a) - \lambda_2)M^*(a) = \max_{0 \leq M \leq \bar{M}(a)} H = \max_{0 \leq M \leq \bar{M}(a)} (\xi(a) - \lambda_2)M, \tag{A.12}$$

$$\dot{\xi}(a) = \mu(a)\xi(a) - \lambda_1 f(a) - \frac{X(\omega)}{Y^2(\omega)}\mathbb{I}_{[\alpha,\beta]}(a) + \frac{1}{Y(\omega)}, \quad \xi(0) = \lambda_1, \quad \xi(\omega) = 0, \tag{A.13}$$

where $\eta = \frac{X(\omega)}{Y^2(\omega)} = \frac{(D+1)^2}{N_{\text{tot}}}$ and $\zeta = \frac{1}{Y(\omega)} = \frac{D+1}{N_{\text{tot}}}$.

Appendix B. Proof of Theorem 2

Note, that minimizing problem (2.1)–(2.5) is equivalent to maximizing

$$J := \frac{\int_\alpha^\beta N(a)\,\mathrm{d}a}{\int_0^\omega N(a)\,\mathrm{d}a},$$

subject to (2.2)–(2.5). From the Cauchy formula for equations (2.2), (2.3) we obtain the expressions

$$N(a) = l(a)N(0) + \int_0^a \frac{l(a)}{l(s)} M(s)\,\mathrm{d}s, \tag{B.1}$$

$$N(0) = \frac{1}{1-R}\int_0^\omega f(a)\int_0^a \frac{l(a)}{l(s)} M(s)\,\mathrm{d}s\,\mathrm{d}a, \tag{B.2}$$

$$= \frac{1}{1-R}\int_0^\omega M(s)v(s)\,\mathrm{d}s. \tag{B.3}$$

Therefore,

$$\begin{aligned}
\int_0^\omega N(a)\,\mathrm{d}a &= \int_0^\omega \frac{l(a)}{1-R}\,\mathrm{d}a \int_0^\omega M(s)v(s)\,\mathrm{d}s + \int_0^\omega \int_0^a \frac{l(a)}{l(s)} M(s)\,\mathrm{d}s\,\mathrm{d}a,\\
&= \frac{e_{[0,\omega]}(0)}{1-R}\int_0^\omega M(s)v(s)\,\mathrm{d}s + \int_0^\omega e_{[0,\omega]}(s)M(s)\,\mathrm{d}s,\\
&= \int_0^\omega \left(\frac{e_{[0,\omega]}(0)}{1-R}v(s) + e_{[0,\omega]}(s)\right) M(s)\,\mathrm{d}s,\\
&= \int_0^\omega G(s)M(s)\,\mathrm{d}s,
\end{aligned}$$

where $G(s) := \frac{e_{[0,\omega]}(0)}{1-R}v(s) + e_{[0,\omega]}(s)$. Analogously, we obtain

$$\begin{aligned}
\int_\alpha^\beta N(a)\,\mathrm{d}a &= \int_\alpha^\beta \frac{l(a)}{1-R}\,\mathrm{d}a \int_0^\omega M(s)v(s)\,\mathrm{d}s + \int_\alpha^\beta \int_0^a \frac{l(a)}{l(s)} M(s)\,\mathrm{d}s\,\mathrm{d}a, \\
&= \int_0^\omega M(s)\frac{e_{[\alpha,\beta]}(0)}{1-R}v(s)\,\mathrm{d}s + \int_0^\omega M(s)\int_s^\omega \mathbb{I}_{[\alpha,\beta]}(a)\frac{l(a)}{l(s)}\,\mathrm{d}a\,\mathrm{d}s, \\
&= \int_0^\omega M(s)\left(\frac{e_{[\alpha,\beta]}(0)}{1-R}v(s) + e_{[\alpha,\beta]}(s)\right)\mathrm{d}s, \\
&= \int_0^\omega F(s)M(s)\,\mathrm{d}s,
\end{aligned}$$

where $F(s) := \frac{e_{[\alpha,\beta]}(0)}{1-R}v(s) + e_{[\alpha,\beta]}(s)$.
Function $F(s)$ can be interpreted as an s-year-old immigrant's effect on the age structure of the population. The first term is then the contribution of all of her future native-born descendants to the age group $[\alpha, \beta]$, (counting children, grandchildren and so forth) in the resulting SI population and the second term may be viewed as the immigrant's own effect by being further alive. The analogous interpretation holds for function $G(s)$ above for the age interval $[0, \omega]$.

We can rewrite problem (2.1)–(2.5) as

$$\max_{M(a)} J(M(\cdot)), \tag{B.4}$$

subject to

$$M_{\text{tot}} = \int_0^\omega M(a)\,\mathrm{d}a, \tag{B.5}$$

$$0 \le M(a) \le \bar{M}(a), \tag{B.6}$$

where $J(M(\cdot)) = \frac{\int_0^\omega F(a)M(a)\ da}{\int_0^\omega G(a)M(a)\ da}$. Furthermore, we assume for problem (B.4)–(B.6) that the following assumption holds:

Regularity Assumption 1. *For any $c > 0$ it holds that*

$$F(a) \neq c\ G(a),$$

almost everywhere in $[0, \omega]$.

For the reformulated problem (B.4)–(B.6) Theorem 2 reads as

Theorem 4. *Let $M(\cdot)$ be an arbitrary immigration profile which fulfills* (B.5), (B.6) *and additionally $M(a) < \bar{M}(a)$ for $a \in [\omega - \delta, \omega]$ and some $\delta > 0$. Then there is an immigration profile $\tilde{M}(\cdot)$ which satisfies* (B.5), (B.6) *such that*

$$J(\tilde{M}(\cdot)) > J(M(\cdot)).$$

For the proof of Theorem 4 we need the following Lemma:

Lemma 2. *For any immigration profile $M(\cdot)$ satisfying* (B.5), (B.6) *there exists a set $\Gamma \subset [0, \omega]$, $meas(\Gamma) > 0$ such that $M(a) > 0$ on Γ and*

$$\frac{F(a)}{G(a)} < \frac{\int_0^\omega F(s)M(s)\,\mathrm{d}s}{\int_0^\omega G(s)M(s)\,\mathrm{d}s}, \quad \forall a \in \Gamma,$$

holds.

Assume that

$$\frac{F(a)}{G(a)} \geq \frac{\int_0^\omega F(s)M(s)\,\mathrm{d}s}{\int_0^\omega G(s)M(s)\,\mathrm{d}s}, \quad \forall a \in \{s : M(s) > 0\} =: \Gamma^0, \; meas(\Gamma^0) > 0.$$

Because of the regularity assumption the strict inequality

$$F(a) \int_0^\omega G(a)M(a)\,\mathrm{d}a > G(a) \int_0^\omega F(a)M(a)\,\mathrm{d}a,$$

holds on a subset $\Gamma \subset \Gamma^0$ of positive measure. Multiplying both sides by $M(a)$ and integrating on $[0, \omega]$ we obtain

$$\int_0^\omega F(a)M(a)\,\mathrm{d}a \int_0^\omega G(a)M(a)\,\mathrm{d}a > \int_0^\omega G(a)M(a)\,\mathrm{d}a \int_0^\omega F(a)M(a)\,\mathrm{d}a,$$

which gives a contradiction. ■

Proof of Theorem 4: Let Γ be the set from Lemma 2, and let $b \in \Gamma$ be a Lebesgue point. Recall that almost every point of Γ is such. Let us define an immigration profile $\tilde{M}(\cdot)$

$$\tilde{M}(a) := \begin{cases} M(a) & a \notin [b - \delta, b] \cup [\omega - \delta, \omega], \\ M(a) - h & a \in [b - \delta, b], \\ M(a) + h & a \in [\omega - \delta, \omega], \end{cases}$$

where $M(a) > 0$ and $M(a) \leq \bar{M}(a) - h$ holds. The corresponding objective value reads as

$$J(\tilde{M}(\cdot)) = \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a - h\int_{b-\delta}^b F(a)\,\mathrm{d}a + h\int_{\omega-\delta}^\omega F(a)\,\mathrm{d}a}{\int_0^\omega G(a)M(a)\,\mathrm{d}a - h\int_{b-\delta}^b G(a)\,\mathrm{d}a + h\int_{\omega-\delta}^\omega G(a)\,\mathrm{d}a}.$$

We define

$$H(\delta) := h\int_{x-\delta}^x F(a)\,\mathrm{d}a, \quad x = b, \omega,$$

where, by transformation of the independent variable, $H(\delta) = h\int_0^\delta F(x-t)\,\mathrm{d}t$ holds. By Taylor expansion around 0 we obtain

$$H(\delta) = h(H(0) + \delta H'(0) + \delta^2 H''(0) + o(\delta^2)) = h\delta F(b) + h\delta^2 F'(b) + ho(\delta^2).$$

By $o(\delta^2)$ we mean, that F'' grows slower than δ^2. Therefore, by neglecting higher order terms,

$$J(\tilde{M}(\cdot)) = \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a - \delta hF(b) + \delta hF(\omega)}{\int_0^\omega G(a)M(a)\,\mathrm{d}a - \delta hG(b) + \delta hG(\omega)}.$$

Note, that $G(\omega) = F(\omega) = 0$ and therefore it holds that

$$\begin{aligned}
& J(\tilde{M}(\cdot)) - J(M(\cdot)) > 0 \\
\Leftrightarrow \quad & \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a - \delta hF(b)}{\int_0^\omega G(a)M(a)\,\mathrm{d}a - \delta hG(b)} > \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a}{\int_0^\omega G(a)M(a)\,\mathrm{d}a} \\
\Leftrightarrow \quad & -F(b)\int_0^\omega G(a)M(a)\,\mathrm{d}a > -G(b)\int_0^\omega F(a)M(a)\,\mathrm{d}a \\
\Leftrightarrow \quad & \frac{F(b)}{G(b)} < \frac{\int_0^\omega F(a)M(a)\,\mathrm{d}a}{\int_0^\omega G(a)M(a)\,\mathrm{d}a}
\end{aligned}$$

which is fulfilled by the choice of $b \in \Gamma$ as was proven in Lemma 2.

Since problem (2.1)–(2.5) and problem (B.4)–(B.6) are equivalent we have thus proven Theorem 2. ■

Appendix C. Proof of Theorem 3

We consider problem (2.1)–(2.4) with the additional constraint (2.6). Note, that minimizing the dependency ratio D in a population with fixed size is equivalent to maximizing the number of working people

$$\max_{M(a)} \int_0^\omega \mathbb{I}_{[\alpha,\beta]}(a)N(a)\,\mathrm{d}a. \tag{C.1}$$

Again, we define the Hamiltonian as

$$H = \tilde{\xi}(a)(-\mu(a)N(a) + M(a)) + \mathbb{I}_{[\alpha,\beta]}(a)N(a) - \tilde{\lambda}_1 f(a)N(a) - \tilde{\lambda}_2 N(a), \tag{C.2}$$

and aim to apply Pontryagin's maximum principle presented in (Alekseev et al., 1987), p.218. The optimality conditions for (N^*, M^*) can be formulated by the following expressions

$$\tilde{\xi}(a)M^*(a) = \max_{0\leq M\leq \bar{M}(a)} H = \max_{0\leq M\leq \bar{M}(a)} \tilde{\xi}(a)M(a), \tag{C.3}$$

$$\dot{\tilde{\xi}}(a) = \mu(a)\tilde{\xi}(a) - \tilde{\lambda}_1 f(a) + \mathbb{I}_{[\alpha,\beta]}(a) + \tilde{\lambda}_2, \quad \tilde{\xi}(0) = \tilde{\lambda}_1, \quad \tilde{\xi}(\omega) = 0, \tag{C.4}$$

where $\tilde{\lambda}_1$ should be calculated in such a way that (2.6) is satisfied for the resulting optimal solution. ■

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