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General Equilibrium Model with Horizontal Innovations and Heterogeneous Products

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Abstract The paper develops a general equilibrium endogenous growth model involving consumption of heterogeneous goods by an age-structured population with uncertain life span and balanced life-time budget. The heterogeneity is introduced via weights which the individuals attribute in their utility function to consumption of different goods depending on the vintage of the good. The goods are produced by monopolistically competitive firms and the variety of available goods/technologies is determined endogenously through R&D investments. The general equilibrium is characterized by a system of functional equations and is analytically or numerically determined for several particular weight functions. The results exhibit the qualitative difference between the dynamics of the model with heterogeneous versus homogeneous variety of goods.

Keywords Horizontal innovation · Endogenous growth · Heterogeneous goods

JEL Classification O3 · O4

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1 Introduction

We consider an economy with overlapping generations of finitely living agents in continuous time, in the spirit of Cass and Yaari (1967). In our model, agents consume a continuum of goods, like in Judd (1985). The variety of goods increases with a speed proportional to the labor employed by perfectly competitive R&D firms, like in Romer (1990), with allowance for knowledge spilover. The consumption goods are produced by monopolistically competitive firms. Each firm possesses a permanent patent which the firm buys on R&D market. We consider any improvement in the quality of a good as the invention of a new good by some R&D firm that increases the variety of goods, thus implying only *horizontal innovations*.

We study the growth of lifetime discounted utility of generations. We do not specify how productivity of labor (and therefore per capita consumption) depends on R&D activity. Thus, we leave aside the problem of scale effects[1] pointed out by Jones (1995a, 1995b). However the proposed model allows for introduction of functional dependencies of labor productivity on the goods' variety frontier, thus describing the endogenous growth of production.

All agents are born with zero assets and should be insured from dying indebted. Agents can borrow money from other coexisting generations like in d'Albis, H. & Augeraud-Véron, E. (2011) which is a continuous time generalization of discrete pure exchange OLG models (e.g., Samuelson, 1958; Gale, 1973). In our model we introduce production without physical capital as in Sorger (2011), so that agents can invest only in patents. But, in contrast to Sorger (2011), the agents are not the only investors in the model. We assume that there is a competitive banking sector that gives loans to the startup firms for purchasing the patents under the pledge of these very patents[2]. Thus, the banks create additional liquidity equal to the change of the value of the patents in the economy. Hence, there are two asset markets in the economy. The one is the market of firms' loans balanced with the values of their patents. The other is the market of agents' savings balanced with their life insurances, according to which in the case of sudden death of an agent her debt is repayed or her deposit is taken by the insurance company. The banks keep deposits of agents under the same interest rate as loans for the firms because of the no-arbitrage condition.

A substantial novelty of the proposed model is that it is a hybrid of continuous time OLG model and growth model with a continuum of consumption goods which can be heterogeneous. We show that the heterogeneity of goods (discounting old goods) may bring qualitative difference in the model dynamics compared to the homogenous case. The growth of the life time discounted utility of agents' generations could be bounded in the heterogeneous case in contrast to the case of homogeneous goods, where growth of utility is unbounded. In the long run the general equilibrium with heterogeneous goods can be inefficient and the real interest rate can become negative as it may happen in OLG models (e.g., Blanchard, 1985; Diamond, 1965). While in the case of homogeneous goods we prove that the real interest

[1] Scale effect prediction in idea-based growth models is when the long growth rate depends on the population of the country (scale of the economy), which is strongly at odds with 20th-century empirical evidence, see (Jones, 1995b). We disentangle the concepts of productivity growth and growth of variety of goods. In other words, scientists increase the variety of products rather than the productivity of labor. The productivity can depend on the variety frontier, but finding such dependence that would mitigate the scale effect is beyond the scope of our paper.

[2] The banking sector plays a role of infinity living institution possessing the patents of the firms. In Sorger (2011) such infinitely living owners of the firms were the households, while in our model agents have limited lifetime and cannot make bequests. In contrast to Sorger (2011) we do not require the total value of patents to be equal to the aggregated savings of agents. Moreover, it will be seen that such requirement is not needed in our model for determining a general equilibrium and could make the model inconsistent. Because of the difference between savings and patents' value the economy needs money (liquidity) provided by the banks.

rate tends to zero which is the *biological interest rate* (the population growth rate is zero in the long run), see Samuelson (1958).

The paper is organized as follows. Section 1.1 introduces population and labor dynamics. Section 1.2 solves the agent's problem of finding her optimal consumption and investment profiles. Section 1.3 considers the monopolistically competitive production of continuum of goods and finds the price of goods in equilibrium. Section 1.4 introduces competitive R&D sector that increases the variety of goods. Section 1.5 finds the equilibrium conditions in financial sector (zero profit of banks) and in R&D sector (zero profit of R&D firms). Section 1.6 describes the market clearing including the full employment condition. Section 1.7 presents final succinct format of the general equilibrium equations. Section 2 studies the equations analytically with the use of aggregation in section 2.1. Section 2.2 finds simple formula for the life time aggregated utility of an agent. Section 2.3 analyzes simple cases of homogeneous goods and heterogeneous goods with exponential discounting. Section 3 discusses the model considering another source of heterogeneity in section 3.2 with introduction of heterogeneous production of labor that also allows for discussion on the economic growth in section 3.3. Section 3.1 introduces the government and shows that its fiscal policy is efficient in the heterogeneous case in contrast to the homogeneous one. Section 4 concludes the paper.

1.1 Population and labor

The population is time-varying and exogenous: $n(\tau,t)$ denotes the size at time t of the population cohort born at time τ. It is assumed that there is a maximal age ω, so that $n(t,\tau)=0$ for $t-\tau \geq \omega$, but $n(\tau,t)>0$ for $0 \leq t-\tau < \omega$. That is, there are always some people in every cohort until it reaches age ω.

For the population function $n(\cdot,\cdot)$ it is assumed that it is continuous in τ for any fixed t and is non-increasing and absolutely continuous in t for every fixed τ. Then the function

$$S(\tau,t) \equiv \frac{n(\tau,t)}{n(\tau,\tau)}$$

is the survival probability at time t of an individual of cohort τ. Survival probability defines mortality rate as

$$\mu(\tau,t) \equiv -\frac{\dot{S}(\tau,t)}{S(\tau,t)} = -\frac{\dot{n}(\tau,t)}{n(\tau,t)} \tag{1}$$

(the dot above a symbol denotes the derivative with respect to the time t). Thus, we have the expression

$$S(\tau,t) = e^{-\int_\tau^t \mu(\tau,\theta)\,\mathrm{d}\theta} \tag{2}$$

for the survival probability, thus

$$n(\tau,t) = n(\tau,\tau)\,e^{-\int_\tau^t \mu(\tau,\theta)\,\mathrm{d}\theta}. \tag{3}$$

The total population at t is obtained by the integration of n over all currently living cohorts

$$N(t) \equiv \int_{t-\omega}^{t} n(\tau,t)\,\mathrm{d}\tau.$$

We assume that each individual is endowed with $l(\tau,t)$ units of homogeneous labor per time, so that

$$L(t) \equiv \int_{t-\omega}^{t} l(\tau,t)\,n(\tau,t)\,\mathrm{d}\tau \tag{4}$$

is the total amount of available labor units in the economy at time t.

1.2 Consumption and savings

The evolution of the assets is determined by the consumption/saving decisions of the individuals. The assets are homogeneous, while the consumption goods are heterogeneous: products labeled by the numbers $q \in [0, Q(t)]$ are available at time t. These products are ordered according to the invention time of the respective technology q, so that $Q(t)$ is the newest product at time t – the one just created at t.

It is assumed that each individual has perfect foresight for the wage $w(t)$, the real return rate on asset $r(t)$, the available consumption goods $q \in [0, Q(t)]$, and their real prices $p(t, q)$, for all t within her life-horizon. To write everything in real terms we will use labor as *numeriare*, that is, we set the wage equal to unity[3].

A representative agent born at time τ chooses her consumption $c(\tau, t, q)$ of good $q \in [0, Q(t)]$ at time t so that her expected total discounted utility

$$u(\tau) \equiv \int_{\tau}^{\tau+\omega} e^{-\rho(t-\tau)} S(\tau,t) \int_{0}^{Q(t)} m(\tau,t,q,Q(t))\, c(\tau,t,q)^{\alpha}\, \mathrm{d}q\, \mathrm{d}t \tag{5}$$

is maximized subject to the dynamic budget constraint

$$\dot{a}(\tau,t) = l(\tau,t) + (r(t) + \mu(\tau,t))\, a(\tau,t) - \int_{0}^{Q(t)} p(t,q)\, c(\tau,t,q)\, \mathrm{d}q, \tag{6}$$

$$a(\tau,\tau) = 0, \tag{7}$$

$$a(\tau,\tau+\omega) = 0, \tag{8}$$

where $a(\tau,t)$ is the real amount of assets that the agent has at time $t \in [\tau, \tau + \omega]$, $\alpha \in (0, 1)$ is an elasticity parameter, $m(\tau,t,q,Q(t)) \geq 0$ is the weight function, with which an agent aggregates her utilities from consumption of different products. The weight function $m(\tau,t,q,Q(t))$ represents agent's preferences among available goods $q \in [0, Q(t)]$ and is assumed to be strictly positive at least on some subset of $[0, Q(t)]$ with positive measure. The weight function $m(\tau,t,q,Q(t))$ makes the goods heterogeneous if it depends on q. Notice that it can depend on the current technological frontier Q and on the age $t - \tau$ of the agent.

We assume that individuals are insured against the risk of dying with positive assets by a fair life-insurance company in the spirit of Yaari (1965) that redistributes wealth of individuals who died to those who are still alive in the same cohort[4]. Therefore the real rate of return $r(t)$ is augmented by the mortality rate $\mu(\tau,t)$. Thus, functional (5) is an extension of that in Judd (1985). The main novelty is the function $m(\tau,t,q,Q(t))$ which makes goods heterogeneous[5]. Since the insurance should ensure that each cohort ultimately consumes all its assets we add the end-point condition (8). The absence of bequest implies the initial condition (7).

For any fixed τ problem (5)–(8) has an optimal control $c(\tau, \cdot, \cdot)$ which can be obtained by a two-stage procedure similar to the one in Dixit and Stiglitz (1977).

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[3] The wage $w(t) \equiv 1$ is assumed to be equal for all jobs, which is reasonable in a model where qualification is not taken into account.

[4] The equal saving/debt redistribution of deceased agent only within her cohort satisfies the balance of fare insurance, because agents from the same cohort have the same assets and probability of death. This redistribution is possible due to the assumption that there are always some people in the cohort until it becomes ω years old ($n(\tau,t) > 0$ for $0 \leq t - \tau < \omega$).

[5] The functional form $m(\tau,t,q,Q(t))$ is given exogenously, thus it differs from the quality in *quality-adjusted Dixit-Stiglits consumption index* used in some models with *vertical innovations* (e.g., Dinopoulos & Thompson, 1998).

In the *inner* stage we define the total real expenditures for consumption at time t as

$$E(\tau,t) \equiv \int_0^{Q(t)} p(t,q)\, c(\tau,t,q)\,\mathrm{d}q. \tag{9}$$

Then we fix an arbitrary non-negative function $E(\cdot)$ and determine for fixed τ and t the optimal distribution $c(\tau,t,\cdot)$ that maximizes the inner integral in (5) subject to (9). The solution is

$$c(\tau,t,q) = \left(\frac{m(\tau,t,q,Q(t))}{p(t,q)}\right)^{\frac{1}{1-\alpha}} \frac{E(\tau,t)}{G(\tau,t)}, \tag{10}$$

where

$$G(\tau,t) \equiv \int_0^{Q(t)} \left(\frac{m(\tau,t,q,Q(t))}{p(t,q)}\right)^{\frac{1}{1-\alpha}} p(t,q)\,\mathrm{d}q. \tag{11}$$

The resulting value of the instantaneous utility (the inner integral in (5)) can be written as $(G(\tau,t))^{1-\alpha}(E(\tau,t))^{\alpha}$. Then, inserting expression (2) for the survival probability and solution (10) of the inner problem in (5), we obtain the following *outer* problem (for a fixed cohort τ)

$$u(\tau) = \int_\tau^{\tau+\omega} e^{-\rho(t-\tau)-\int_\tau^t \mu(\tau,\theta)\,\mathrm{d}\theta}\,(G(\tau,t))^{1-\alpha}\,(E(\tau,t))^{\alpha}\,\mathrm{d}t \to \max_{E(\tau,\cdot)} \tag{12}$$

subject to

$$\dot{a}(\tau,t) = l(\tau,t) + (r(t)+\mu(\tau,t))\,a(\tau,t) - E(\tau,t), \qquad E(\tau,t) \geq 0, \tag{13}$$

$$a(\tau,\tau) = 0, \quad a(\tau,\tau+\omega) = 0. \tag{14}$$

Using the Pontryagin maximum principle we obtain the following first-order conditions for problem (12)–(14):

$$\alpha\left(\frac{E(\tau,t)}{G(\tau,t)}\right)^{\alpha-1} - \lambda(\tau,t) = 0, \tag{15}$$

$$-\dot{\lambda}(\tau,t) = \lambda(\tau,t)\,(r(t)-\rho), \tag{16}$$

where $\lambda(\tau,t)$ is the adjoint variable representing the marginal utility per unit of income. Note that like in Sorger (2011) the adjoint equation (16) does not depend on the mortality rate μ because the agent has fair life insurance. Adjoint equation (16) has the solution

$$\lambda(\tau,t) = \lambda_0(\tau)\, e^{-\int_\tau^t (r(\eta)-\rho)\,\mathrm{d}\eta}, \tag{17}$$

which defines along with (15) the solution of the outer problem

$$E(\tau,t) = G(\tau,t)\left(e^{\int_\tau^t (r(\theta)-\rho)\,\mathrm{d}\theta}\,\frac{\alpha}{\lambda_0(\tau)}\right)^{\frac{1}{1-\alpha}}, \tag{18}$$

where $\lambda_0(\tau)$ is the initial value (at time $t=\tau$) of the adjoint variable that has to be adjusted to ensure the end-condition $a(\tau+\omega)=0$ in (14). In doing this it will be notationally convenient to define the discount factors

$$R_\mu(\tau,t) \equiv \exp\left(-\int_\tau^t (r(\theta)+\mu(\tau,\theta))\,\mathrm{d}\theta\right), \tag{19}$$

$$R_\rho(\tau,t) \equiv \exp\left(-\int_\tau^t \frac{\rho - r(\theta)}{1-\alpha}\,\mathrm{d}\theta\right). \tag{20}$$

Standard calculations using the Cauchy formula for the solution of (13) and notations (19)–(20) give the following expression for $\lambda_0(\tau)$, for which conditions (14) are satisfied

$$\left(\frac{\alpha}{\lambda_0(\tau)}\right)^{\frac{1}{1-\alpha}} = \frac{\int_\tau^{\tau+\omega} R_\mu(\tau,t)\, l(\tau,t)\,\mathrm{d}t}{\int_\tau^{\tau+\omega} R_\rho(\tau,t) R_\mu(\tau,t)\, G(\tau,t)\,\mathrm{d}t}. \tag{21}$$

Then expression (18) with the use of (20) and (21) takes the explicit form

$$E(\tau,t) = G(\tau,t)\frac{R_\rho(\tau,t)\int_\tau^{\tau+\omega} R_\mu(\tau,s) l(\tau,s)\,\mathrm{d}s}{\int_\tau^{\tau+\omega} R_\rho(\tau,s) R_\mu(\tau,s) G(\tau,s)\,\mathrm{d}s}. \tag{22}$$

Using (10) and (22), one can obtain the consumption of product q at time t by an agent born at τ:

$$c(\tau,t,q) = \left(\frac{m(\tau,t,q,Q(t))}{p(t,q)}\right)^{\frac{1}{1-\alpha}} \frac{R_\rho(\tau,t)\, h(\tau)}{\int_\tau^{\tau+\omega} R_\rho(\tau,s) R_\mu(\tau,s) G(\tau,s)\,\mathrm{d}s}. \tag{23}$$

It is proportional to the agent's *human wealth* defined as in Blanchard (1985)

$$h(\tau) \equiv \int_\tau^{\tau+\omega} R_\mu(\tau,s) l(\tau,s)\,\mathrm{d}s, \tag{24}$$

which is the present value of agent's income flow from labor. The propensity to consume the product q decreases with its relative price $\frac{p(t,q)}{m(\tau,t,q,Q(t))}\left(\int_\tau^{\tau+\omega} R_\rho(\tau,s) R_\mu(\tau,s) G(\tau,s)\,\mathrm{d}s\right)^{1-\alpha}$ and (due to the multiplier $R_\rho(\tau,t)$ in (23)) with the age of the agent $t-\tau$, provided that $\rho > r(\cdot)$ in (20). There is no *nonhuman wealth* in the model because other factors of production (besides labor, e.g. physical capital available for agents to invest into) are absent. Thus, agents' assets only redistribute values among generations in time.

Note that the optimal consumption profile (23) is completely defined by the price $p(\cdot,\cdot)$, the real interest rate $r(\cdot)$, and the frontier of the product variety $Q(\cdot)$. This is the key difference from the models with infinitely living households (like in Sorger, 2011) or OLG models with bequests, where consumption also depends on some free variable ($\lambda_0(\tau)$ in Sorger, 2011) which is to be defined in equilibrium.

1.3 Production sector

Further we assume that each product $q \in [0, Q(t)]$ available at time t is produced by a single firm, that is, we implement the concept of monopolistic competition. Moreover, we assume, similarly as Sorger (2011), that the production of each good involves only labor, and that the production of one unit of any good requires one unit of labor.

Here we denote by $C(t,q)$ the production of good q and use the fact that at equilibrium it is equal to the aggregated consumption of good q:

$$C(t,q) \equiv \int_{t-\omega}^{t} c(\tau,t,q)\, n(\tau,t)\,\mathrm{d}\tau. \tag{25}$$

The firm holding permanently the patent for product q sets the price $p = p(t,q)$. Inserting (23) into (25) one can express the demanded quantity at price p as

$$C(t,q) = \frac{F(t,q)}{p^{\frac{1}{1-\alpha}}}, \tag{26}$$

where the function

$$F(t,q) = \int_{t-\omega}^{t} \frac{(m(\tau,t,q,Q(t)))^{\frac{1}{1-\alpha}} R_\rho(\tau,t)\, h(\tau)}{\int_\tau^{\tau+\omega} R_\rho(\tau,s) R_\mu(\tau,s) G(\tau,s)\, \mathrm{d}s}\, n(\tau,t)\, \mathrm{d}\tau$$

does not depend on the price $p(t,q)$ of any particular good q due to the assumption of monopolistic competition. It follows from the definition of $G(\tau,t)$ in (11) that it depends on the prices of the goods in an integrated form, so that no single firm may influence $F(t,q)$. The q-th firm produces quantity $C(t,q)$, the production cost of which in real terms is $C(t,q)$, according to the assumption that the production of one unit of any good requires one unit of labor. Then the real operating profit of the q-th firm at time t is

$$\pi(t,q) = pC(t,q) - C(t,q). \tag{27}$$

Due to (26) the firm's profit takes the form $\pi(t,q) = (p-1)\, p^{\frac{-1}{1-\alpha}} F(t,q)$, which attains its maximum with respect to the price p at

$$p(t,q) = \frac{1}{\alpha}. \tag{28}$$

Thus, all goods have the same price (28), although the consumption of these goods (23), (25) can be different. For the optimal operating profit of the q-th firm at time t (with $q \in [0, Q(t)]$) we obtain from (27) and (28) the expression

$$\pi(t,q) = \frac{1-\alpha}{\alpha} C(t,q), \tag{29}$$

relating this profit with the production $C(t,q)$, which is equal to the aggregated consumption according to (25).

1.4 R&D sector

The R&D sector produces new technologies increasing in this way the variety $[0, Q(t)]$ of available consumer's goods. The R&D industry sells patents for new productions. The R&D sector is a perfectly competitive industry which requires only labor and shares the labor market with the production sector, so that the wage at time t is $w(t) = 1$. Similarly as Sorger (2011) we assume that the dynamics of the technological frontier $Q(t)$ is given by the equation

$$\dot{Q}(t) = \beta\, (Q(t))^{\varphi}\, L_Q(t), \tag{30}$$

where $L_Q(t)$ is the total labor employed in R&D, $\beta > 0$ is a productivity parameter, φ is the parameter determining how productivity depends on $Q(t)$, that is on the already existing technologies. Thus, we allow for knowledge spillover if $\varphi > 0$ [6]. With $\varphi > 0$, formulation (30) implies increasing returns to scale in R&D, when previous inventions raise the productivity of current research effort. Alternately, with $\varphi < 0$, the formulation allows for diminishing returns in R&D, as if past inventions make it more difficult to find new ideas. For examples of models with different values of the parameter φ see Jones (1999) and the references therein.

[6] Different extension of equation (30) are possible, including the dependence on a weight average of the distribution of labor across different technologies, which is reasonable (see e.g. Sorger, 2011). This would cause only some technical burden, therefore below we focus only on dynamics (30).

1.5 Financial sector

The firm willing to establish the production of a new good takes a loan from a bank at real interest rate $r(t)$ and buys a patent of infinite life from the R&D sector. The bank cannot make higher interest rate for the firm than that on agents' deposits because in this case the firm can attract investments from other bank (because of the free entry condition in the bank sector) or directly from people offering them the same rate $r(t)$ of return as the bank does. Banks can "create money" taking patents as security, thus giving more loans than the total deposits of people. Actually, there is no restriction in the model on how much loans a bank can give since the firms for sure will repay their debts in the future. These debts are balanced in the bank accounting by the values of patents that firms put in pledge.

Similar to Sorger (2011) we denote the present value of the real profit flow for the firm producing product $q \in [0, Q(t)]$ over interval $[t, +\infty)$ as

$$v(t,q) = \int_t^\infty \exp\left(-\int_t^s r(\theta)\,\mathrm{d}\theta\right)\pi(s,q)\,\mathrm{d}s, \tag{31}$$

which, on the other hand, is the debt that the q-th firm owes to the bank.

Since the market for patents is competitive we assume that the R&D industry as a whole makes no profit, and the firm who buys a patent makes zero net profit[7] too, because of the free entry assumption. This idea of *intertemporal zero net profit constraint* (as in Romer, 1990) is taken from Grossman and Helpman (1989). Thus, the market price $v(t, Q(t))$, of the patents produced at time t is determined by the zero-profit condition

$$v(t,Q(t))\,\dot{Q}(t) = L_Q(t), \tag{32}$$

which results from the fact that the R&D industry creates $\dot{Q}(t)$ patents in a unit of time at t and the cost is what is paid for the involved labor $L_Q(t)$.

The time-derivative of expression (31) for the price of a new patent $q = Q(t)$ yields the following no-arbitrage condition

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}v(t,Q(t))}{v(t,Q(t))} + \frac{\pi(t,Q(t))}{v(t,Q(t))} = r(t) + \frac{\dot{Q}(t)}{v(t,Q(t))}\int_t^\infty e^{-\int_t^s r(\theta)\,\mathrm{d}\theta}\,\frac{\partial\pi}{\partial q}(s,q)\bigg|_{q=Q(t)}\mathrm{d}s. \tag{33}$$

It states that the rate of return on the ownership of a firm — consisting of the rate of capital gain $\frac{\mathrm{d}}{\mathrm{d}t}v(t,Q(t))/v(t,Q(t))$ and the profit rate $\pi(t,Q(t))/v(t,Q(t))$ — has to equal the real interest rate $r(t)$ plus the *rate of novelty premium* (as we call the last term). It differs from the no-arbitrage condition in the literature (e.g. Romer, 1990; Grossman & Helpman, 1991, Chapter 3) by the rate of novelty premium that appears because of the heterogeneity of goods.

In special cases the improper integral in (33) can be expressed via $v(t,Q(t))$. Then, no-arbitrage condition (33) yields an explicit expression for the real interest rate $r(t)$. Such expression is more convenient for calculating the general equilibrium, than the original integral equation (31) in zero-profit condition (32).

[7] The *net profit* of the firm is its *operating profit* π minus taxes (that will be introduced later) and minus interest on debt.

1.6 Market clearing

We have already taken into account the clearing on product market (25) where production equals consumption. In view of the technological assumption in Section 1.3 we can express the total labor in production as

$$L_P(t) = \int_0^{Q(t)} C(t,q)\,\mathrm{d}q. \tag{34}$$

The labor market is cleared by the full employment condition

$$L_P(t) + L_Q(t) = L(t), \tag{35}$$

where the total labor $L(t)$ is given exogenously by (4).

Financial market is cleared by intertemporal zero-profit condition (32) for the firms, meaning that all invented technologies are purchased by the firms whose debts to the banks are balanced with the values of their patents. The banks have zero profit too, because of the absence of entry barriers for new banks. Thus, the real interest rate $r(t)$ on firms' debts is the same as that on agents' deposits.

1.7 Succinct format of the general equilibrium equations

With the use of (31) and (29) the zero-profit condition (32) in the R&D sector takes the form

$$L_Q(t) = \beta(Q(t))^{\varphi} L_Q(t) \int_t^{\infty} e^{-\int_t^s r(\theta)\,\mathrm{d}\theta}\,\frac{1-\alpha}{\alpha}\,C(s,Q(t))\,\mathrm{d}s. \tag{36}$$

Due to (34) the labor balance equation (35) becomes

$$L_Q(t) = L(t) - \int_0^{Q(t)} C(t,q)\,\mathrm{d}q. \tag{37}$$

The dynamics of the variety frontier Q is as in (30)

$$\dot{Q}(t) = \beta\,(Q(t))^{\varphi}\,L_Q(t), \qquad Q(0) = Q^0 > 0. \tag{38}$$

The above three equations involve the total consumption $C(t,q)$ given by (25), which in view of (23) takes the form

$$C(t,q) = \alpha \int_{t-\omega}^{t} \frac{(m(\tau,t,q,Q(t)))^{\frac{1}{1-\alpha}}}{g(\tau)} R_{\rho}(\tau,t)\,h(\tau)\,n(\tau,t)\,\mathrm{d}\tau. \tag{39}$$

Here

$$g(\tau) \equiv \int_{\tau}^{\tau+\omega} R_{\rho}(\tau,s) R_{\mu}(\tau,s) M(\tau,s)\,\mathrm{d}s \tag{40}$$

and $M(\tau,s)$ is a substitution for $G(\tau,s) = \alpha^{\frac{\alpha}{1-\alpha}} M(\tau,s)$, so that

$$M(\tau,s) \equiv \int_0^{Q(s)} (m(\tau,s,q,Q(s)))^{\frac{1}{1-\alpha}}\,\mathrm{d}q. \tag{41}$$

The integral $M(\tau,s)$ represents how much the agent in cohort τ enjoys the variety of goods at time $s \geq \tau$, and $g(\tau)$ is the present discounted value of this propensity for a newborn at τ. So we have the three equations (36)–(38) for three unknown functions, Q, L_Q, and r which determine the general equilibrium. Further we use these equations for analytical and numerical investigation of the economy.

2 Analytical study

Notice that because of the assumption that for any time t there is a subset of goods in $[0,Q(t)]$ with nonzero measure, such that $m(\tau,t,q,Q(t))>0$, the integrals $M(\tau,t)$ and $g(\tau)$ are strictly positive for all τ. Thus, the integral in (37) of consumption (39) is also strictly positive since functions R_ρ, h and n are strictly positive in the integration domain. Hence, there are always workers in the production sector, meaning $L_Q(t)<L(t)$.

We also mention that expression (22) for the consumption expenditure can be written in terms of h, g, and M defined in (24), (40), and (41)

$$E(\tau,t)=R_\rho(\tau,t)\,h(\tau)\,\frac{M(\tau,t)}{g(\tau)}. \tag{42}$$

2.1 Aggregated state variables

For better understanding of the model dynamics we introduce aggregated state variables and derive corresponding aggregated equations. Integration of the profit expression (27) over all existing products $[0,Q(t)]$ yields

$$\Pi(t)=I(t)-L_P(t), \tag{43}$$

where we introduce the total profit Π and income I of the firms

$$\Pi(t)\equiv\int_0^{Q(t)}\pi(t,q)\,\mathrm{d}q, \tag{44}$$

$$I(t)\equiv\int_0^{Q(t)}p(t,q)\,C(t,q)\,\mathrm{d}q. \tag{45}$$

Thus, (43) reads as the total profit of the firms equals their total income minus the total labor expenses in production.

At equilibrium the total income of all firms in the economy is equal to the aggregated expenditure for consumption,

$$I(t)=\int_{t-\omega}^{t}E(\tau,t)\,n(\tau,t)\,\mathrm{d}\tau, \tag{46}$$

where the expenditure $E(\tau,t)$ of each cohort τ is defined in (42).

It follows from (34), (45), and (28) that the labor in production $L_P(t)$ has the following simple relation with the total firms' income $I(t)$

$$L_P(t)=\alpha I(t), \tag{47}$$

and defines the total profit of firms

$$\Pi(t)=\frac{1-\alpha}{\alpha}\,L_P(t). \tag{48}$$

Now let us aggregate the assets of individuals at time t,

$$A(t)\equiv\int_{t-\omega}^{t}a(\tau,t)\,n(\tau,t)\,\mathrm{d}\tau, \tag{49}$$

that are on deposits in the bank.

Recalling that newborns have no initial assets, see (7), we have the following formula for the derivative of $A(t)$ obtained by time-differentiation of (49) with the use of expression (2)

$$\dot{A}(t) = \int_{t-\omega}^{t} \dot{a}(\tau,t)\, n(\tau,t)\, \mathrm{d}\tau - \int_{t-\omega}^{t} a(\tau,t)\, \mu(\tau,t)\, n(\tau,t)\, \mathrm{d}\tau. \tag{50}$$

The aggregation of the budget constraints (13) over age cohorts gives

$$\int_{t-\omega}^{t} \dot{a}(\tau,t)\, n(\tau,t)\, \mathrm{d}\tau = \int_{t-\omega}^{t} [l(\tau,t) + (r(t)+\mu(\tau,t))\, a(\tau,t) - E(\tau,t)]\, n(\tau,t)\, \mathrm{d}\tau.$$

Then equations (4), (46), and (50) yield the following asset balance equation:

$$\dot{A}(t) = r(t)A(t) + L(t) - I(t). \tag{51}$$

Thus, the aggregated assets A rise at the amount of interest rA and what is left from the labor income L after the consumption expenses (equal to the total income of all firms I).

We also introduce the aggregated value of all patents in the economy,

$$V(t) \equiv \int_{0}^{Q(t)} v(t,q)\, \mathrm{d}q, \tag{52}$$

where $v(t,q)$ is given by (31). Since the value $V(t)$ is equal to the firms' debt to the bank we have the following differential equation for $V(t)$:

$$\dot{V}(t) = r(t)V(t) + L_Q(t) - \Pi(t), \tag{53}$$

meaning that the debt (besides interest rV) raises with the loan for research L_Q and falls with the current firms' profit Π payed to the banks. Equation (53) can also be obtained by differentiation of (52) with respect to time, taking into account expression (31), definition (43), and the R&D zero-profit condition (32).

Proposition 1 *The total value of patents $V(t)$ and the total assets of the individuals $A(t)$ are governed by the same differential equation*

$$\dot{A}(t) = r(t)A(t) + L(t) - \frac{1}{\alpha} L_P(t), \tag{54}$$

and are related as follows: $V(t) = A(t) + \psi e^{\int_0^t r(s)\,\mathrm{d}s}$, where ψ is a constant.

Proof See Appendix A.1. □

Although the dynamics of the aggregated assets and the firms' debts is described by the same differential equation (54) they are not equal in general ($\psi \neq 0$), as it will be shown by the next proposition.[8]

It shows that the aggregated real assets, $A(t)$, are bounded due to the constant wages, and the no-bequest scenario of the model.

[8] We do not require the aggregated assets of agents to be equal to the total debts of firms like it is done in (Sorger, 2011), because in our model the optimal consumption profile and investments of a finitely living agent are completely defined via initial condition (7) and terminal conditions (8), like in (Cass & Yaari, 1967; d'Albis, H. & Augeraud-Véron, E., 2011), which is not the case for infinitely living households in (Sorger, 2011), where he needs additional condition $V(t) \equiv A(t)$ to specify general equilibrium.

Proposition 2 *If the real interest rate r and total labor endowment L are bounded (for all t, $|r(t)| < \bar{r}$ and $L(t) \leq \bar{L}$), then the aggregated assets of the individuals are also bounded: $|A(t)| < \bar{A}$.*

Proof See Appendix A.2. □

Corollary 1 *If the value of all patents in the economy V is bounded ($V(t) \leq \bar{V}$ for all t) and the real interest rate converges to a constant, $r(t) \to \hat{r}$ with $t \to \infty$, then $\hat{r} \leq 0$ or $V(t) = A(t)$ for all t.*

Proof See Appendix A.3. □

2.2 Utility of an agent

We are to analyze the growth of the expected life-time aggregated utility (12) of a representative individual in cohort τ (see Appendix B.1)

$$u(\tau) = (\alpha h(\tau))^{\alpha} (g(\tau))^{1-\alpha}, \tag{55}$$

where h and g are defined in (24) and (40). This expression shows that $u(\tau)$ depends positively on the human wealth of the agent and on the discounted integral M defined in (41) representing how much the agent values the variety of goods.

2.3 Analysis of some special cases

Let us consider the case of $m(\tau,t,q,Q) = m_0(\tau,t,Q)\, m_1(q)$. From (29), (34), and (39) we can derive the expressions (see Appendix B.2)

$$\frac{\partial \pi}{\partial q}(s, Q(t)) = \frac{\pi(s,Q(t))}{1-\alpha} \frac{\frac{\partial m_1}{\partial q}(Q(t))}{m_1(Q(t))}, \tag{56}$$

$$\pi(s, Q(t)) = \frac{1-\alpha}{\alpha} \frac{m_1(Q(t))^{\frac{1}{1-\alpha}}}{M_1(Q(s))} L_P(s), \tag{57}$$

where we introduce notation

$$M_1(Q) = \int_0^Q m_1(q)^{\frac{1}{1-\alpha}}\, \mathrm{d}q, \tag{58}$$

so that the integral defined in (41) takes the form $M(\tau,t) = m_0(\tau,t,Q(t))^{\frac{1}{1-\alpha}} M_1(Q(t))$. We also assume that there is R&D activity $L_Q(t) > 0$. Then, from (30) and (32) we obtain the price of a new patent

$$v(t, Q(t)) = \frac{1}{\beta\, (Q(t))^{\varphi}}. \tag{59}$$

Due to (56), (57), and (59) no-arbitrage condition (33) takes the form

$$r(t) = \frac{\beta}{(Q(t))^{1-\varphi}} \left(Q(t) \left(\frac{1-\alpha}{\alpha} \frac{m_1(Q(t))^{\frac{1}{1-\alpha}}}{M_1(Q(t))} L_P(t) - \frac{L_Q(t)}{1-\alpha} \frac{\frac{\partial m_1}{\partial q}(Q(t))}{m_1(Q(t))} \right) - \varphi L_Q(t) \right). \tag{60}$$

This condition is easier to use instead of zero-profit condition (36) in the R&D sector. Although condition (60) does not depend explicitly on the function $m_0(\tau,t,Q)$, it is still affected by m_0 through expressions (39), (40), and (41). However, if the function m_0 depends only on τ, then it does not influence the solution ($L_Q(t)$, $r(t)$, $Q(t)$) of system ((37), (38), (60)) and matters only for the utilities of the agents.

The expression for the real interest rate (60) allows to construct a simple iterative procedure for numerical calculation of a general equilibrium in the case of interior solution ($L_Q(t) > 0$ for all t), which is presented below.

2.3.1 Iterative procedure to calculate general equilibrium

In order to solve system (37), (38), (60) first, we take the initial labor in research $L_Q[1](t)$ in the time interval $[0,T]$, where $T \gg \omega$. With $L_Q[1](t)$ we calculate the product variety domain $Q[1](t)$ in $[0,T]$ from (38). Then, we calculate the real interest rate $r[1](t)$ in $[0,T]$ with the use of (60). The functions $Q[1](t)$ and $r[1](t)$ determine the aggregated consumption of agents $C[1](t,q)$ in $[\omega, T-\omega]$ by (39). Knowing $C[1](t,q)$ we obtain the labor in R&D for the next iteration, $L_Q[2](t)$, $t \in [\omega, T-\omega]$ with the formula (37) and extrapolate it linearly to the whole interval $[0,T]$. Then we continue in the same way.

If the sequence of so defined iterations converges, then as a result we have a numerical approximation of the general equilibrium with two allowances: finiteness of time horizon T and linear extrapolation to the ends of the time interval.

There are two cases to be considered separately. The first case is when consumers evaluate all goods homogenously ($\frac{\partial m_1}{\partial q}(q) \equiv 0$) and the second case is when consumers value newest goods more ($\frac{\partial m_1}{\partial q}(q) > 0$ for all $q \in [0, Q(t)]$).

2.3.2 Homogeneous goods

Let us consider the usually studied in the literature (Baldwin, etc.) case $m(\tau,t,q,Q) \equiv 1$ meaning that all existing products are equally appreciated by the consumer. In this case $M_1(Q(t)) = Q(t)$ so we have

$$\begin{cases} r(t) = \dfrac{\beta}{(Q(t))^{1-\varphi}} \left(\dfrac{1-\alpha}{\alpha} L_P(t) - \varphi L_Q(t) \right), & \text{when} \quad L_Q(t) > 0 \\ L(t) = L_P(t), & \text{when} \quad L_Q(t) = 0. \end{cases} \tag{61}$$

For an internal solution $L_Q(t) > 0$ we can calculate the value of patents in the economy with the use of (32) and (30) as

$$V(t) = \int_0^{Q(t)} v(t,q)\,\mathrm{d}q = Q(t)\,v(t,Q(t)) = Q(t)\,\frac{L_Q(t)}{\dot{Q}(t)} = \frac{(Q(t))^{1-\varphi}}{\beta}, \tag{62}$$

where we take into account that, because of the homogeneity ($m(\tau,t,q,Q) \equiv 1$), the consumption $C(t,q)$ in (39), the profit $\pi(s,q)$ in (29), and, consequently, the patent value $v(t,q)$ defined in (31), do not depend on q.

For the sake of simplicity we consider the stationary population and labor endowment.

Definition 1 We say that an economy has *stationary population and labor endowment* if the functions $n(\tau,t)$ and $l(\tau,t)$ depend only on the age $t-\tau \in [0,\omega]$ of the agent.

Equivalently we can say that the number of newborns $n(\tau,\tau) \equiv n_0$ is constant and the mortality rate $\mu(\tau,t) \equiv \hat{\mu}(t-\tau)$ and the labor endowment $l(\tau,t) \equiv \hat{l}(t-\tau)$ depend only on the age $t-\tau$.

Case $\varphi = 1$*:* Here, if $L_Q(t) > 0$, the patents' value (62) is constant

$$V(t) = \frac{1}{\beta} \tag{63}$$

and expression (61) takes the form

$$r(t) = \beta \left(\frac{L_P(t)}{\alpha} - \hat{L} \right). \tag{64}$$

The next proposition finds a "balanced growth path" corresponding to the exponential growth of the variety of goods in the economy.

First we find relation between labor endowment L defined in (4) and human wealth h defined in (24) when $r(t) \equiv 0$.

Lemma 1 *Let the population and labor endowment be stationary (see Defenition 1). If the real interest rate is zero (*$r(t) \equiv 0$*) in the equilibrium, then the human wealth, given by (24), is constant* $h(t) \equiv \hat{h}$ *and the total labor endowment is also constant and equals* $L(t) \equiv \hat{L} = n_0 \hat{h}$*, where* n_0 *is the number of newborns.*

Proof See Appendix A.4. □

Proposition 3 *In the economy with stationary population and labor endowment if goods are homogeneous (*$m(\tau, t, q, Q) \equiv 1$*) and the knowledge spillover parameter is equal to one (*$\varphi = 1$*), then:*

(i) the triple $L_P(t) \equiv \alpha \hat{L}$*,* $r(t) \equiv 0$*,* $Q(t) = Q_0 \, e^{\beta(1-\alpha)\hat{L}t}$ *is a steady state solution of the system (36)–(38),*

(ii)

$$g(\tau) = Q_0 \, e^{\beta(1-\alpha)\hat{L}\tau} \hat{g}, \tag{65}$$

where $\hat{g} = \int_0^{\omega} e^{\left(\beta(1-\alpha)\hat{L} - \frac{\rho}{1-\alpha}\right)\varsigma - \int_0^{\varsigma} \hat{\mu}(\vartheta)\,\mathrm{d}\vartheta} \, \mathrm{d}\varsigma$.

Proof See Appendix A.5. □

It follows from (55) and (65) that for $\varphi = 1$ the growth of the life-time aggregated utility of individuals is unbounded:

$$u(\tau) = \left(\alpha \hat{h}\right)^{\alpha} \left(Q_0 \hat{g}\right)^{1-\alpha} e^{\beta(1-\alpha)^2 \hat{L} \tau} \to \infty.$$

Aggregated assets A cannot grow infinitely because of the Proposition 2, thus converge to a constant $\hat{A}$.

Case $\varphi < 1$*:* First, let us assume unbounded growth of the variety frontier $Q(t) \to \infty$. Then, due to the first equation in (61), we have that $r(t) \to 0$. Thus, from equation (54) we obtain the convergence

$$L_P(t) \to \alpha \hat{L}, \qquad L_Q(t) \to (1-\alpha)\hat{L}. \tag{66}$$

The product variety frontier, $Q(t)$, grows asymptotically as driven by the equation

$$\dot{Q}(t) = \beta(1-\alpha)\hat{L}\,(Q(t))^{\varphi},$$

which implies the unbounded growth of patents' value (62):

$$V(t) = \frac{(Q(t))^{1-\varphi}}{\beta} \to \infty.$$

Hence, $V(t) > A(t)$ for all t since the assets A are bounded due to Proposition 2 and the patents' value V cannot intersect A transversely because both V and A are governed by the same ODE (54), see Proposition 1.

The growth of the life-time aggregated utility $u(\tau) = (\alpha h(\tau))^{\alpha} (g(\tau))^{1-\alpha}$ is also unbounded since $g(\tau) \sim Q(\tau) \to \infty$ due to (40)–(41) and $h(\tau) \to \check{h} \in (0, \hat{h}]$ due to (24).

The unbounded growth of the utility $u(\tau)$ means that the future generations will live unrestrictedly better than their presently living ancestors. This happens because in equilibrium the agent enjoys equal consumption of all goods, and her instantaneous utility can be described via the common consumption level $c(t) = c(\tau, t, q)$ and variety frontier $Q(t)$ as $Q(t)\,c(t)^{\alpha}$. It is clear, that the $Q(t)$-elasticity of the instantaneous utility is greater than its $c(t)$-elasticity, since $1 > \alpha$. Thus, the gain in utility from the increase of the variety frontier $Q(t)$ overweighs the loss from the decrease of the consumption level $c(t)$ so that the utilities of agents grow proportionally to $(Q(\tau))^{1-\alpha}$.

Case of any φ: If we assume bounded growth of the variety frontier $Q(t) \to \bar{Q}$, then due to (38) we have $L_Q(t) \to 0$ and, from the first equation in (61), strictly positive limit of real interest rate $r(t) \to \hat{r} > 0$. Then, due to bounded V in (62) and $\hat{r} > 0$ we have from Corollary 1 that $A(t) \equiv V(t)$.

In the corner solution, where $L_Q(t) = 0$ for all $t \geq \bar{t}$, there is no growth of $Q(t)$ and R&D is absent. So, instead of equation (32) one needs to use $L_P(t) = L(t)$. Let us calculate V assuming convergence of the real interest rate $r(t) \to \hat{r} \neq 0$ (see Appendix B.3):

$$V(t) \to \frac{1-\alpha}{\alpha} \frac{\hat{L}}{\hat{r}}. \tag{67}$$

The same limit we get from equation (54) for the assets A

$$A(t) \to \frac{1-\alpha}{\alpha} \frac{\hat{L}}{\hat{r}}$$

Hence, according to Corollary 1, $A(t) \equiv V(t) = \frac{1-\alpha}{\alpha} \frac{\hat{L}}{\hat{r}}$ or $A(t) - V(t) \to 0$ with $\hat{r} < 0$. But, by definition, $V(t) \geq 0$, hence $\hat{r} > 0$. Thus, we have the only option that $A(t) \equiv V(t) = \frac{1-\alpha}{\alpha} \frac{L}{r}$.

Note that both the corner solution $L_Q(t) \equiv 0$ and the bounded growth $Q(t) \to \bar{Q}$ imply the equivalence $A(t) \equiv V(t)$, while the unbounded growth $Q(t) \to \infty$ with $\varphi < 1$ results in $A(t) < V(t) \to \infty$. Such unbounded growth is demonstrated by numerical calculation of the economy dynamics for $\varphi = 0.5$ shown in Fig. 1.

From now on we will consider only unbounded growth of Q.

2.3.3 Heterogeneous goods

We assume that at equilibrium $Q(t) \to \infty$ and agents have heterogeneous preferences, such that $m(\tau, t, q, Q) = e^{-\gamma(Q-q)}$, where $\gamma > 0$ is the parameter of heterogeneity[9]. This corresponds to the special case, where $m_0(\tau, t, Q) = e^{-\gamma Q}$ and $m_1(q) = e^{\gamma q}$. Thus, we can calculate the real interest rate r from (60) as follows

$$r(t) = \frac{\beta}{(Q(t))^{1-\varphi}} \left(Q(t) \left(\frac{(1-\alpha)}{\alpha \tilde{M}(Q(t))} L_P(t) - \frac{\gamma}{1-\alpha} L_Q(t) \right) - \varphi L_Q(t) \right), \tag{68}$$

[9] The case of $\gamma = 0$ corresponds to homogeneous goods studied above.

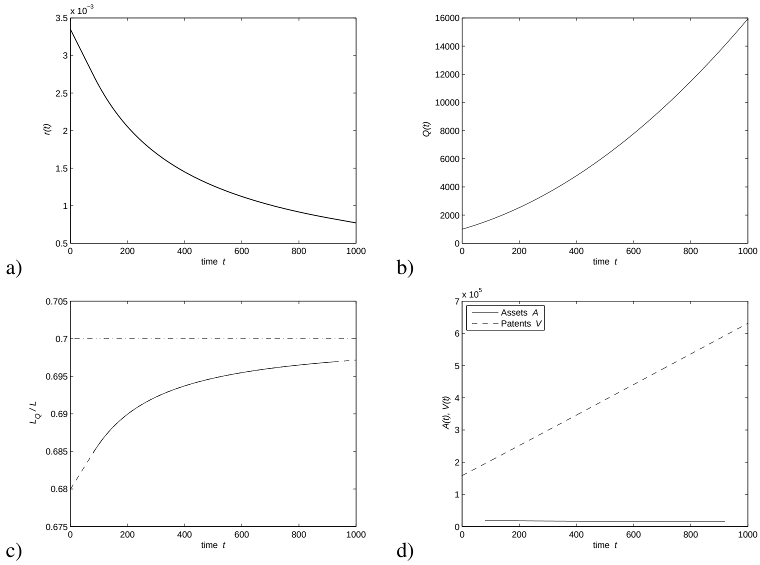

Fig. 1 The case of homogeneous goods $m \equiv 1$. Labor endowment is $l(\tau,t) = 0.3$ for $t-\tau \le 45$ and $l(\tau,t) = 0$ after retirement ($t-\tau > 45$), where we consider the agent's life from her adulthood. Mortality $\mu(\tau,t) = 0$ for $t-\tau < \omega$, life horizon $\omega = 80$, age concentration of people $n(\tau,t) = n_0 = 100$, initial frontier of the products' variety $Q_0 = 1000$, $\beta = 0.0002$, $\varphi = 0.5$, individual discounting $\rho = 0.01$. a) Real interest rate r. b) The variety frontier Q. c) The relative labor employed in R&D L_Q/L (solid line with dashed line depicting the linear extrapolation) and its limit value $1-\alpha$ (dashed-dotted line). d) Dynamics of total assets A (solid line) and value of patents V (dashed line).

where $\tilde{M}(Q(t)) = M_1(Q(t))/m_1(Q(t))^{\frac{1}{1-\alpha}} = \frac{1-\alpha}{\gamma}(1-e^{\frac{-\gamma}{1-\alpha}Q(t)})$.

The value of patents can be expressed in the following way, with the use of (32) and (30) (see Appendix B.4)

$$V(t) = \frac{1-\alpha}{\gamma\beta}\,\frac{1-e^{\frac{-\gamma}{1-\alpha}Q(t)}}{Q(t)^{\varphi}}. \tag{69}$$

If we assume convergence to a steady state $r(t) \to \hat{r}$, then it follows from (68) that $\frac{(1-\alpha)}{\alpha\tilde{M}(Q(t))}L_P(t) - \frac{\gamma}{1-\alpha}L_Q(t) \to 0$, because otherwise $r(t) \to \infty$ as $Q(t) \to \infty$. From which we have $L_P(t) - \frac{\alpha}{1-\alpha}L_Q(t) \to 0$ and, taking into account labor balance (37), we obtain $L_P(t) - \alpha L(t) \to 0$. The value of patents in the economy (69) tends to zero $V(t) \to 0$ as time $t \to \infty$. According to Corollary 1, $A(t) \equiv V(t)$ or $\hat{r} \le 0$. We study the second type of solution, $\hat{r} \le 0$, that is confirmed by the numerical calculation depicted in Fig. 2.

The life-time utility of an agent $u(\tau) = (\alpha\, h(\tau))^{\alpha}\,(g(\tau))^{1-\alpha}$ in (55) is bounded, because $h(\tau) \to const$ and $\tilde{M}(Q(t)) \to \frac{1-\alpha}{\gamma}$ (therefore $g(\tau) \to const$). The intuition for the boundedness of the expected life time utility u is as follows. When the variety frontier Q increases the consumption profile, roughly speaking, shifts to the newer goods, thus exponentially (in Q) decreasing the consumption of the older goods, because of the function $m(q,Q) = e^{-\gamma(Q-q)}$ in (39). Exponential decrease in consumption of the older goods prevails over the increase of their variety Q. As a result the instantaneous utility converges to a limitconstant so, consequently, does the life time utility $u(\tau)$.

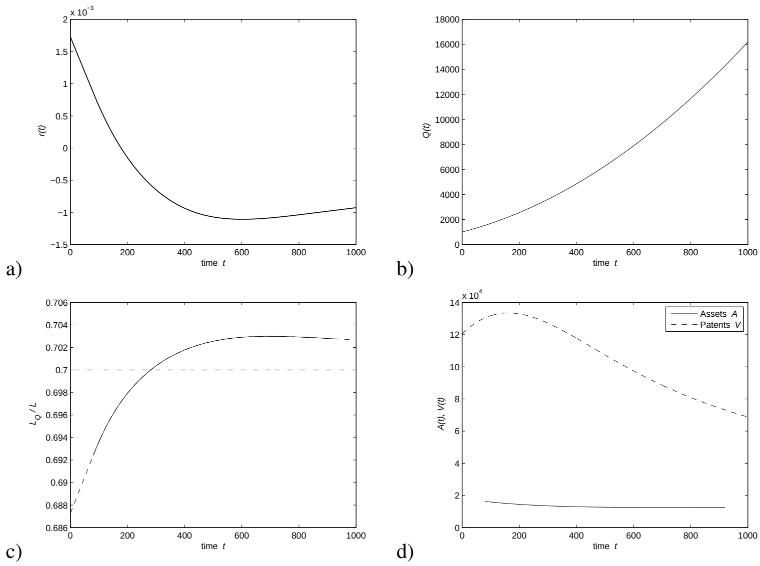

Fig. 2 The case of heterogeneous goods $\gamma = 0.0004$. Labor endowment is $l(\tau,t) = 0.3$ for $t-\tau \leq 45$ and $l(\tau,t) = 0$ after retirement ($t-\tau > 45$), where we consider the agent's life from her adulthood. Mortality $\mu(\tau,t) = 0$ for $t-\tau < \omega$, life horizon $\omega = 80$, age concentration of people $n(\tau,t) = n_0 = 100$, initial frontier of the products' variety $Q_0 = 1000$, $\beta = 0.0002$, $\varphi = 0.5$, individual discounting $\rho = 0.01$. a) Real interest rate r. b) The variety frontier Q. c) The relative labor employed in R&D L_Q/L (solid line with dashed line depicting the linear extrapolation) and its limit value $1-\alpha$ (dashed-dotted line). d) Dynamics of total assets A (solid line) and value of patents V (dashed line).

We have qualitatively different behavior of V and u in the homogeneous ($m \equiv 1$) and the heterogeneous ($m \equiv e^{-\gamma(Q-q)}$) cases. Mathematically, this difference is conditioned by the properties of the integral $\tilde{M}(Q)$. Intuitively speaking, the reason for vanishing V and bounded u in the heterogeneous case is the abandonment of older goods because of the agent's preferences.

3 Discussion of the model and extensions

3.1 Efficiency and fiscal policy

In this subsection we investigate how the government can improve the utility of the agents in the long run by taxing the firms and paying subsidies and pensions to the agents. We assume that there is a constant tax rate $\delta < 1$ payed by each firm from its profit $\pi(t,q)$ and the sum of the collected taxes $\delta\,\Pi(t)$ are distributed among the currently living agents of different generations in the form of lump subsidies. For the sake of simplicity, we assume that the collected taxes are divided equally among the newborns and put on their deposits. The equation for government balance reads as

$$a(\tau,\tau)\,n(\tau,\tau) = \delta\,\Pi(\tau), \tag{70}$$

that, taking from the expression (48) for the total profit $\Pi(\tau)$ we obtain the following initial assets of the agents (instead of (7)):

$$a(\tau,\tau) = \delta\,\frac{1-\alpha}{\alpha}\,\frac{L_P(\tau)}{n(\tau,\tau)}. \tag{71}$$

The value $v(t,q)$ in (31) of the patent to produce product q decreases to

$$v(t,q) = (1-\delta)\int_t^\infty \exp\left(-\int_t^s r(\theta)\,\mathrm{d}\theta\right)\pi(s,q)\,\mathrm{d}s, \tag{72}$$

while the agent's *human wealth* $h(\tau)$ in (24) is augmented by *nonhuman wealth* from the subsidies $a(\tau,\tau)\geq 0$. Thus, in all the previous formulas we should replace $h(\tau)$ with the *total wealth* $a(\tau,\tau)+h(\tau)$. Taking into account these changes, the aggregated equations (51) and (53) take the form

$$\dot{A}(t) = r(t)A(t) + L(t) - I(t) + \delta\,\Pi(t), \tag{73}$$
$$\dot{V}(t) = r(t)V(t) + L_Q(t) - (1-\delta)\,\Pi(t), \tag{74}$$

similarly as in (54) we have

$$\dot{A}(t) = r(t)A(t) + L(t) - \frac{1-\delta\,(1-\alpha)}{\alpha}\,L_P(t). \tag{75}$$

Following the same arguments as before we can state that if $L_Q(t)>0$ and $L(t)\to\hat{L}$, the real interest rate can be calculated as follows.

Homogeneous case ($m(\tau,t,q,Q)\equiv 1$):

$$r(t) = \frac{\beta}{(Q(t))^{1-\varphi}}\left(\frac{(1-\alpha)\,(1-\delta)}{\alpha}\,L_P(t) - \varphi L_Q(t)\right). \tag{76}$$

Heterogeneous case ($m(\tau,t,q,Q)\equiv e^{-\gamma(Q-q)}$ with $\gamma>0$):

$$r(t) = \frac{\beta}{(Q(t))^{1-\varphi}}\left(Q(t)\left(\frac{(1-\alpha)\,(1-\delta)}{\alpha\tilde{M}(Q(t))}L_P(t) - \frac{\gamma}{1-\alpha}L_Q(t)\right) - \varphi L_Q(t)\right), \tag{77}$$

where $\tilde{M}(Q(t)) = \frac{1-\alpha}{\gamma}(1-e^{\frac{-\gamma}{1-\alpha}Q(t)})$.

In both homogenous and heterogeneous cases we have

$$L_P(t)\to\frac{\alpha}{1-\delta\,(1-\alpha)}\hat{L}, \qquad L_Q(t)\to\frac{(1-\delta)\,(1-\alpha)}{1-\delta\,(1-\alpha)}\hat{L}, \tag{78}$$

generalizing (66) to the case of taxation and distribution of subsidies. Hence, the tax increases the amount of labor in production and decreases it in R&D, so that the product variety frontier $Q(t)$ grows asymptotically slower with the tax $\delta>0$ then without it $\delta=0$, which can be seen from its asymptotical differential equation

$$\dot{Q}(t) = \beta\,\frac{(1-\delta)\,(1-\alpha)}{1-\delta\,(1-\alpha)}\,\hat{L}\,(Q(t))^{\varphi}.$$

Using the explicit formula for the solution of this equation we can write

$$\frac{Q(t)|_{\delta=0}}{Q(t)|_{\delta>0}} \to \begin{cases} \left(1+\frac{\delta\alpha}{1-\delta}\right)^{\frac{1}{1-\varphi}} &, \quad \varphi<1 \\ \exp\left(\frac{\beta\hat{L}(1-\alpha)\alpha\delta}{1-\delta(1-\alpha)}\,t\right), & \varphi=1 \end{cases}. \tag{79}$$

Now we can ask the question: Can the fiscal policy improve the expected life-time aggregated utility (55) of a representative agents in a long run? The utility of cohort τ takes the following form:

$$u(\tau)=\alpha^{\alpha}\left(a(\tau,\tau)+h(\tau)\right)^{\alpha}\left(g(\tau)\right)^{1-\alpha} \tag{80}$$

, by substitution the expression for nonhuman capital (71).

We see that $u(\tau)$ depends positively on the human wealth of the agent and discounted integral M defined in (41). Further we assume that population is stationary and labor is fixed, i.e. density $n(\tau,t)$ and labor endowment $l(\tau,t)$ depend only on the age $t-\tau$. Hence, the total labor endowment is constant, so we can take $L(t)=n_0\,\bar{l}$, where $n_0=n(\tau,\tau)$ is the constant number of newborns, and $\bar{l}$ is the constant effective labor of a newborn. Thus, when $\tau\to\infty$ we have from (71) and (78) that the initial assets converge as follows

$$a(\tau,\tau)\to\frac{\delta\,(1-\alpha)}{1-\delta\,(1-\alpha)}\,\bar{l}. \tag{81}$$

In order to answer the question of efficiency of fiscal policy we consider asymptotics of the utility $u(\tau)$ in (80) separately in homogeneous and heterogeneous cases.

3.1.1 Homogeneous goods

In the homogeneous case $r(t)\to 0$ when $t\to\infty$, hence, according to Lemma 1, $h(\tau)\to\hat{h}=\bar{l}$ when $\tau\to\infty$. Thus, with the use of (79) and (81) we can write for $\varphi\in[0,1)$ the following limit for ratio of utilities with tax and without tax (see Appendix B.5)

$$\frac{u(\tau)|_{\delta=0}}{u(\tau)|_{\delta>0}}\to\frac{1-\delta\,(1-\alpha)}{(1-\delta)^{1-\alpha}}\geq 1. \tag{82}$$

When $\varphi=1$ this utility ratio tends to infinity with $t\to\infty$, due to the ratio in (79). Thus, in the homogeneous case the fiscal policy cannot improve agents' utilities in the long run[10].

3.1.2 Heterogeneous goods

Below we show that in contrast with the homogeneous case an appropriate fiscal policy of the government may improve the agents' utility in the long run. We take constant labor endowment l until the retirement age $\kappa<\omega$ and zero mortality ($\mu=0$) until the maximal age ω. It is shown in Section 2.3.3 that $r(t)\to\hat{r}\leq 0$. We consider the most difficult case when $\hat{r}<0$, so that the human wealth can be calculated as follows:

$$h(\tau)=l\int_{\tau}^{\tau+\kappa}R_{\mu}(\tau,s)\,\mathrm{d}s\to l\int_{\tau}^{\tau+\kappa}e^{-\hat{r}(s-\tau)}\,\mathrm{d}s=\frac{l}{\hat{r}}\left(1-e^{-\hat{r}\kappa}\right), \tag{83}$$

[10] Note that for $\varphi<0$ even small tax rate $\delta>0$ improves agents' utilities in the long run, because the derivative $\frac{\mathrm{d}}{\mathrm{d}\delta}\frac{u(\tau)|_{\delta=0}}{u(\tau)|_{\delta>0}}\Big|_{\delta=0}=\alpha\varphi\frac{1-\alpha}{1-\varphi}$ becomes negative. A value $\varphi<0$ means that past inventions make it more difficult to find new ideas, which we think to be unlikely.

and the other integral in (80) can be also taken explicitly:

$$g(\tau) \to \frac{1-\alpha}{\gamma} \int_{\tau}^{\tau+\omega} e^{-\frac{\rho-\alpha\hat{r}}{1-\alpha}(t-\tau)}\,\mathrm{d}t = \frac{(1-\alpha)^2}{\gamma(\rho-\alpha\hat{r})}\left(1-e^{-\frac{\rho-\alpha\hat{r}}{1-\alpha}\omega}\right). \tag{84}$$

Then, we set δ and ρ to be small parameters and assume that $\hat{r}$ has the same order of smallness ($\delta \sim \rho \sim \hat{r} \ll 1$), so we can make the following approximations of the asymptotic expected utility[11] (see Appendix B.6)

$$u(\tau) \to (\alpha\,\kappa\,l)^{\alpha}\left(\frac{(1-\alpha)\,\omega}{\gamma}\right)^{1-\alpha}\left(1+\delta\,(1-\alpha)\,\alpha-\frac{\omega}{2}\,\rho+\frac{\alpha}{2}\,(\omega-\kappa)\,\hat{r}\right)+o(\delta). \tag{85}$$

As we see, when the agent works all her life ($\kappa = \omega$), then her expected utility $u(\tau)$ does not depend on the interest rate $\hat{r}$ in the first approximation. This is intuitively clear because the agent with constant income and very small personal discounting has almost no intention to save. In this case, as it is seen from (85), the implementation of the fiscal policy ($\delta > 0$) increases the agents' expected life-time aggregated utilities in the long run.

The fiscal policy can improve well-being of the future generations consuming heterogeneous goods which is not the case with homogeneous goods.

3.2 Sources of heterogeneity

There are two, mathematically equivalent, sources of heterogeneity of goods. The first is that we have already described, where goods are weighted heterogeneously in the agent's utility function. The other sourse of heterogeneity is the potential difference in productivity of labor allocated to different firms. That can be described by the same model after renormalization. Indeed, let $\zeta(t,q,Q(t))$ be the amount of physical units of the good q that can be produced with one unit of labor, then equation for the profit (27) would take the form

$$\pi(t,q) = \tilde{p}(t,q)\,\tilde{C}(t,q) - \frac{\tilde{C}(t,q)}{\zeta(t,q,Q(t))}, \tag{86}$$

where $\tilde{p}(t,q)$ is the price of one physical unit of the good q, $\tilde{C}(t,q)$ is the aggregated production (and consumption) in physical units of the good q. Let the agent value all goods equally maximizing her following expected lifetime utility

$$u(\tau) = \int_{\tau}^{\tau+\omega} e^{-\rho(t-\tau)-\int_{\tau}^{t}\mu(\tau,\theta)\,\mathrm{d}\theta} \int_{0}^{Q(t)} \tilde{c}(\tau,t,q)^{\alpha}\,\mathrm{d}q\,\mathrm{d}t, \tag{87}$$

subject to the dynamic budget constraint

$$\dot{a}(\tau,t) = l(\tau,t) + (r(t)+\mu(\tau,t))\,a(\tau,t) - \int_{0}^{Q(t)} \tilde{p}(t,q)\,\tilde{c}(\tau,t,q)\,\mathrm{d}q, \tag{88}$$

$$a(\tau,\tau) = 0, \qquad a(\tau,\tau+\omega) = 0, \tag{89}$$

where $\tilde{c}(\tau,t,q)$ is her consumption in physical units. Then, if we change the variables $\tilde{p}(t,q) = p(t,q)/\zeta(t,q,Q(t))$, $\tilde{c}(\tau,t,q) = c(\tau,t,q)\,\zeta(t,q,Q(t))$ (so that $\tilde{C}(t,q) = C(t,q)\,\zeta(t,q,Q(t))$), equation (86) for the profit will coincide with (27) and problem (87)–(89) will take the form (5)–(8), where $m \equiv (\zeta(t,q,Q(t)))^{\alpha}$. Thus, the performed analysis is also applicable to the problem with heterogeneous productivity of labor, but with a different meanings of the function m.

[11] In the case of $\hat{r} = 0$ the result can be easily obtained for all $\delta \in (0,1)$ and $\rho > 0$.

3.3 Economic growth

So far we have studied only the growth of agents' utilities as indicator of prosperity. However, measuring goods in physical units as we did in Section 3.2 we can also discuss economic growth in terms of production and consumption. If we assume limited labor endowment L the economy can grow only due to increase in productivity of labor $\zeta(t,q,Q(t))$, introduced in Section 3.2. The growth can be endogenous if the productivity of labor depends on the state variable $Q(t)$.

We can consider a function m in problem (5) in the form

$$m(\tau,t,q,Q(t)) \equiv (\zeta(t,q,Q(t)))^{\alpha}\,\tilde{m}(\tau,t,q,Q(t)), \tag{90}$$

where the function $\tilde{m}$, inducing the actual agent's preferences, is bounded, while the function $\zeta(q,t,Q(t))$, describing the dependence of the productivity upon the variety frontier $Q(t)$, may grow unboundedly when $Q(t) \to \infty$. Thus, we would have an unbounded growth of per capita consumption (in physical units) i.e. an infinite economic growth.

The functional form of $\zeta(t,q,Q(t))$ is supposed to be chosen to fit an observed path of the per capita consumption (Jones, 1995b), but this is beyond the scope of the present paper.

4 Conclusions and prospects

We suggest an endogenous growth model of an economy, where technological growth is promoted by the entrepreneurial activity of new firms. The liquidity for these new enterprisers is provided by the banking sector that *de jure* owns all intellectual property. We interpret the technological growth as a growth of the variety of goods. The question of how productivity of labor depends on the variety of available technologies (goods) still needs to be answered. One can try to determine this dependence empirically, but we believe that the productivity of labor should be related with qualification, hence education should be explicitly included in the model as a new decision variable of the individuals.

Heterogeneity of products brings two qualitative effects. The first one is that the growth of the expected life-time aggregated utilities of agents becomes bounded. The second is that the general equilibrium can loose its dynamical efficiency. So that it becomes possible to introduce a fiscal policy that improves all agents' utilities in the long run.

In the case of constant labor endowment and stationary population consuming homogeneous goods we have found the simple steady steady state solution with zero real interest rate and ($r \equiv 0$) and constant share of labor in production ($L_P/L \equiv \alpha$) with knowledge spillover parameter $\varphi = 1$. This solution sustain exponential growth of goods' variety. The same interest rate and labor distribution happen to be an attractor for the solutions in homogeneous cases with $\varphi < 1$ and heterogeneous cases with $\varphi < 0$.

Since the general equilibrium can be numerically calculated without any steady state assumptions and with arbitrary exogenous population dynamics the model allows to investigate the effect of different shocks including demographical changes.

Appendix

A Proofs

A.1 Proposition 1

Proof Let us substitute profit expression (43) into differential equation (53) for V and use labor balance (35). Thus, we obtain exactly the same equation (51) as for the aggregated assets A. One can check that the substitution of the relation $V(t) = A(t) + \psi e^{\int_0^t r(s)\,\mathrm{d}s}$ into equation (53) gives equation (51). Moreover, due to (47) equation (51) can be written in terms of labor, (54). □

A.2 Proposition 2

Proof Expression (49), with $a(\tau,t)$ taken from the solution of personal budget constraints (13) same as in (Cass & Yaari, 1967), has the following form

$$
\begin{aligned}
A(t) &= \int_{t-\omega}^{t} \frac{n(\tau,t)}{R_\mu(\tau,t)} \int_\tau^t R_\mu(\tau,s)\,(l(\tau,s)-E(\tau,s))\,\mathrm{d}s\,\mathrm{d}\tau \\
&= \int_{t-\omega}^{t} n(\tau,t) \int_\tau^t e^{-\int_t^s (r(\theta)+\mu(\tau,\theta))\,\mathrm{d}\theta}\,(l(\tau,s)-E(\tau,s))\,\mathrm{d}s\,\mathrm{d}\tau \\
&= \int_{t-\omega}^{t} n(\tau,t) \int_\tau^t e^{\int_s^t r(\theta)\,\mathrm{d}\theta}\,\frac{n(\tau,s)}{n(\tau,t)}\,(l(\tau,s)-E(\tau,s))\,\mathrm{d}s\,\mathrm{d}\tau \\
&= \int_{t-\omega}^{t} \int_\tau^t e^{\int_s^t r(\theta)\,\mathrm{d}\theta} n(\tau,s)\,(l(\tau,s)-E(\tau,s))\,\mathrm{d}s\,\mathrm{d}\tau \\
&= \int_{t-\omega}^{t} e^{\int_s^t r(\theta)\,\mathrm{d}\theta} \int_{t-\omega}^{s} n(\tau,s)\,(l(\tau,s)-E(\tau,s))\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned} \tag{91}
$$

Since $l(\tau,s) \geq 0$ and $E(\tau,s) \geq 0$ we have the following chain of inequalities

$$
\begin{aligned}
|A(t)| &\leq e^{\omega\bar{r}} \int_{t-\omega}^{t} \int_{t-\omega}^{s} n(\tau,s)\,(l(\tau,s)+E(\tau,s))\,\mathrm{d}\tau\,\mathrm{d}s \\
&\leq e^{\omega\bar{r}} \int_{t-\omega}^{t} \int_{t-\omega}^{t} n(\tau,s)\,(l(\tau,s)+E(\tau,s))\,\mathrm{d}\tau\,\mathrm{d}s \\
&= e^{\omega\bar{r}} \int_{t-\omega}^{t} (L(s)+I(s))\,\mathrm{d}s \\
&= e^{\omega\bar{r}} \int_{t-\omega}^{t} \left(L(s)+\frac{L_P(s)}{\alpha}\right)\mathrm{d}s \\
&\leq e^{\omega\bar{r}} \int_{t-\omega}^{t} \left(\bar{L}+\frac{\bar{L}}{\alpha}\right)\mathrm{d}s \\
&\leq \omega e^{\omega\bar{r}}\,\frac{1+\alpha}{\alpha}\,\bar{L},
\end{aligned}
$$

where we use expression (4) for total labor, balance equation (46) and relation (47). □

A.3 Corollary 1

Proof Proposition 2 claims that the aggregated assets A in the economy are always bounded. Since both A and V are bounded, the term with exponent in the relation $V(t) = A(t) + \psi e^{\int_0^t r(s)\,\mathrm{d}s}$ from Proposition 1 is also bounded. This happens only if constant $\psi = 0$ (then $V(t) = A(t)$ for all t) or when the exponent $\exp\left(\int_0^t r(s)\,\mathrm{d}s\right)$ is bounded. But we have that $r(t) \to \hat{r}$, hence boundedness of $\int_0^t r(s)\,\mathrm{d}s$ occurs only if $\hat{r} \leq 0$. □

A.4 Lemma 1

Proof First we show that the human wealth h defined in (24) is constant:

$$\begin{aligned} h(\tau) &\equiv \int_{\tau}^{\tau+\omega} R_{\mu}(\tau,s)\, l(\tau,s)\,\mathrm{d}s \\ &= \int_{\tau}^{\tau+\omega} e^{-\int_{\tau}^{s}\hat{\mu}(\theta-\tau)\,\mathrm{d}\theta}\,\hat{l}(s-\tau)\,\mathrm{d}s \\ &= \int_{\tau}^{\tau+\omega} e^{-\int_{0}^{s-\tau}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\hat{l}(s-\tau)\,\mathrm{d}s \\ &= \int_{0}^{\omega} e^{-\int_{0}^{\varsigma}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\hat{l}(\varsigma)\,\mathrm{d}\varsigma = \hat{h}, \end{aligned} \tag{92}$$

where we used the definition of $R_{\mu}(\tau,s)$ in (19) along with $r(t)\equiv 0$. Then, from the definition of L in (4) and expression (3) we have

$$\begin{aligned} L(t) &\equiv \int_{t-\omega}^{t} l(\tau,t)\, n(\tau,t)\,\mathrm{d}\tau \\ &= \int_{t-\omega}^{t} \hat{l}(t-\tau)\, n(\tau,\tau)\, e^{-\int_{\tau}^{t}\hat{\mu}(\theta-\tau)\,\mathrm{d}\theta}\,\mathrm{d}\tau \\ &= n_0 \int_{t-\omega}^{t} \hat{l}(t-\tau)\, e^{-\int_{0}^{t-\tau}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\mathrm{d}\tau \\ &= n_0 \int_{0}^{\omega} \hat{l}(\varsigma)\, e^{-\int_{0}^{\varsigma}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\mathrm{d}\varsigma = n_0\,\hat{h}. \end{aligned} \tag{93}$$

□

A.5 Proposition 3

Proof We need to check if $L_P(t)\equiv\alpha\hat{L}$, $r(t)\equiv 0$, and $Q(t)=Q_0\,e^{\beta(1-\alpha)\hat{L}t}$ satisfy equations (36)–(38). It is easy to check that (36) is satisfied due to the no-arbitrage condition in (61). The solution of (38) is obviously $Q(t)=Q_0\,e^{\beta(1-\alpha)\hat{L}t}$.

Let us check (37). From (41) it follows that $M(\tau,s)=Q(s)=Q_0\,e^{\beta(1-\alpha)\hat{L}s}$. Then from (40) we obtain expression (65) for $g(\tau)$ as follows

$$\begin{aligned} g(\tau) &\equiv \int_{\tau}^{\tau+\omega} R_{\rho}(\tau,s) R_{\mu}(\tau,s) M(\tau,s)\,\mathrm{d}s \\ &= Q_0\,e^{\beta(1-\alpha)\hat{L}\tau} \int_{\tau}^{\tau+\omega} e^{\left(\beta(1-\alpha)\hat{L}-\frac{\rho}{1-\alpha}\right)(s-\tau)-\int_{0}^{s-\tau}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\mathrm{d}s \\ &= Q_0\,e^{\beta(1-\alpha)\hat{L}\tau} \int_{0}^{\omega} e^{\left(\beta(1-\alpha)\hat{L}-\frac{\rho}{1-\alpha}\right)\varsigma-\int_{0}^{\varsigma}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\mathrm{d}\varsigma \\ &= Q_0\,e^{\beta(1-\alpha)\hat{L}\tau}\hat{g}, \end{aligned}$$

where the last integral is a constant denoted by $\hat{g}$. Thus, from (39) we have

$$\begin{aligned} C(t,q) &= \alpha \int_{t-\omega}^{t} \frac{h(\tau)}{g(\tau)} R_{\rho}(\tau,t)\, n(\tau,t)\,\mathrm{d}\tau \\ &= \alpha\hat{h} \int_{t-\omega}^{t} \frac{1}{g(\tau)}\, e^{-\frac{\rho}{1-\alpha}(t-\tau)}\, n(\tau,\tau)\, e^{-\int_{0}^{t-\tau}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\mathrm{d}\tau \\ &= \alpha\hat{h}n_0 \int_{t-\omega}^{t} \frac{1}{g(\tau)}\, e^{-\frac{\rho}{1-\alpha}(t-\tau)}\, e^{-\int_{0}^{t-\tau}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\mathrm{d}\tau. \end{aligned} \tag{94}$$

After substitution of $g(\tau)$ from (65) and canceling the constant we obtain the following expression with the use of the relation $\hat{L}=\hat{h}n_0$ from Lemma 1

$$C(t,q) = \frac{\alpha\hat{h}n_0}{Q_0\,\hat{g}} \int_{t-\omega}^{t} e^{-\beta(1-\alpha)\hat{L}\tau}\, e^{-\frac{\rho}{1-\alpha}(t-\tau)-\int_{0}^{t-\tau}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\mathrm{d}\tau$$

$$
\begin{aligned}
&= \frac{\alpha \hat{h} n_0}{Q_0\, \hat{g}}\, e^{-\beta(1-\alpha)\hat{L}t} \int_{t-\omega}^{t} e^{\beta(1-\alpha)\hat{L}(t-\tau) - \frac{\rho}{1-\alpha}(t-\tau) - \int_0^{t-\tau}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\mathrm{d}\tau \\
&= \frac{\alpha \hat{h} n_0}{Q_0\, e^{\beta(1-\alpha)\hat{L}t}\, \hat{g}} \int_0^{\omega} e^{\left(\beta(1-\alpha)\hat{L} - \frac{\rho}{1-\alpha}\right)\varsigma - \int_0^{\varsigma}\hat{\mu}(\vartheta)\,\mathrm{d}\vartheta}\,\mathrm{d}\varsigma = \frac{\alpha \hat{h} n_0}{Q_0\, e^{\beta(1-\alpha)\hat{L}t}} \\
&= \frac{\alpha \hat{h} n_0}{Q(t)} = \frac{\alpha \hat{L}}{Q(t)} = \frac{L_P(t)}{Q(t)} \\
&= \frac{\hat{L} - L_Q(t)}{Q(t)}.
\end{aligned} \tag{95}
$$

Thus (37) is also satisfied. □

B Calculations

B.1 Expression (55)

With the use of consumption (23) and relation $G(\tau,s) = \alpha^{\frac{\alpha}{1-\alpha}} M(\tau,s)$

$$
\begin{aligned}
u(\tau) &= \int_{\tau}^{\tau+\omega} e^{-\rho(t-\tau) - \int_{\tau}^{t}\mu(\tau,\theta)\,\mathrm{d}\theta} \int_0^{Q(t)} m(\tau,t,q,Q(t))\, c(\tau,t,q)^{\alpha}\,\mathrm{d}q\,\mathrm{d}t \\
&= \int_{\tau}^{\tau+\omega} e^{-\rho(t-\tau) - \int_{\tau}^{t}\mu(\tau,\theta)\,\mathrm{d}\theta} (G(\tau,t))^{1-\alpha}\,(E(\tau,t))^{\alpha}\,\mathrm{d}t \\
&= \int_{\tau}^{\tau+\omega} e^{-\rho(t-\tau) - \int_{\tau}^{t}\mu(\tau,\theta)\,\mathrm{d}\theta} (\alpha^{\frac{\alpha}{1-\alpha}} M(\tau,s))^{1-\alpha} \left(R_\rho(\tau,t)\, h(\tau)\, \frac{M(\tau,t)}{g(\tau)}\right)^{\alpha} \mathrm{d}t \\
&= \left(\alpha\, \frac{h(\tau)}{g(\tau)}\right)^{\alpha} \int_{\tau}^{\tau+\omega} R_\rho(\tau,t)\, R_\mu(\tau,t)\, M(\tau,t)\,\mathrm{d}t \\
&= (\alpha\, h(\tau))^{\alpha}\, (g(\tau))^{1-\alpha},
\end{aligned}
$$

we have (55).

B.2 Expressions (56) and (57)

Let us rewrite expression (39) using the assumption $m(\tau,t,q,Q) = m_0(\tau,t,Q)\, m_1(q)$

$$
C(t,q) = \alpha\, (m_1(q))^{\frac{1}{1-\alpha}} \int_{t-\omega}^{t} \frac{(m_0(\tau,t,Q(t)))^{\frac{1}{1-\alpha}}}{g(\tau)}\, R_\rho(\tau,t)\, h(\tau)\, n(\tau,t)\,\mathrm{d}\tau = (m_1(q))^{\frac{1}{1-\alpha}} f(t). \tag{96}
$$

Then from (34) and (96) we have

$$
L_P(t) = \int_0^{Q(t)} C(t,q)\,\mathrm{d}q = (m_1(q))^{\frac{1}{1-\alpha}} f(t) = f(t) \int_0^{Q(t)} (m_1(q))^{\frac{1}{1-\alpha}}\,\mathrm{d}q = f(t)\, M_1(Q(t)). \tag{97}
$$

We can derive from (96) and (97) the following relations for $C(s,q)$ and, due to (29), for $\pi(s,q)$:

$$
C(s,q) = \frac{(m_1(q))^{\frac{1}{1-\alpha}}}{M_1(Q(s))} L_P(s), \quad \pi(s,q) = \frac{1-\alpha}{\alpha}\, C(s,q) = \frac{1-\alpha}{\alpha}\, \frac{(m_1(q))^{\frac{1}{1-\alpha}}}{M_1(Q(s))} L_P(s) \tag{98}
$$

Thus, we have the expression for the derivative of $C(s,q)$ with respect to q

$$
\frac{\partial C}{\partial q}(s,q)\bigg|_{q=Q(t)} = \frac{(m_1(Q(t)))^{\frac{\alpha}{1-\alpha}}\, \frac{\partial m_1}{\partial q}(Q(t))}{(1-\alpha)\, M_1(Q(s))} L_P(s) = \frac{C(s,Q(t))}{1-\alpha}\, \frac{\frac{\partial m_1}{\partial q}(Q(t))}{m_1(Q(t))},
$$

and, due to (29), the derivative of $\pi(s,q)$:

$$
\frac{\partial \pi}{\partial q}(s,q)\bigg|_{q=Q(t)} = \frac{1-\alpha}{\alpha}\, \frac{\partial C}{\partial q}(s,q)\bigg|_{q=Q(t)} = \frac{\pi(s,Q(t))}{1-\alpha}\, \frac{\frac{\partial m_1}{\partial q}(Q(t))}{m_1(Q(t))}, \tag{99}
$$

with the use of which along with equation (98) for $C(t,Q(t))$

B.3 Expression (67)

$$\begin{aligned}
V(t) &= \int_0^{Q(\bar{t})} v(t,q)\,\mathrm{d}q \\
&= \int_0^{Q(\bar{t})} \int_t^{\infty} e^{-\int_t^s r(\theta)\,\mathrm{d}\theta} \pi(s,q)\,\mathrm{d}s\,\mathrm{d}q \\
&= \int_t^{\infty} e^{-\int_t^s r(\theta)\,\mathrm{d}\theta} \frac{1-\alpha}{\alpha} \int_0^{Q(\bar{t})} C(s,q)\,\mathrm{d}q\,\mathrm{d}s \\
&= \int_t^{\infty} e^{-\int_t^s r(\theta)\,\mathrm{d}\theta} \frac{1-\alpha}{\alpha} L_P(s)\,\mathrm{d}s \\
&\to \frac{1-\alpha}{\alpha} \hat{L} \int_t^{\infty} e^{-\hat{r}(s-t)\,\mathrm{d}\theta}\,\mathrm{d}s = \frac{1-\alpha}{\alpha} \frac{\hat{L}}{\hat{r}}.
\end{aligned}$$

B.4 Expression (69)

$$\begin{aligned}
V(t) &= \int_0^{Q(t)} v(t,q)\,\mathrm{d}q \\
&= \int_0^{Q(t)} \int_t^{\infty} e^{-\int_t^s r(\theta)\,\mathrm{d}\theta} \pi(s,q)\,\mathrm{d}s\,\mathrm{d}q \\
&= \int_0^{Q(t)} \int_t^{\infty} e^{-\int_t^s r(\theta)\,\mathrm{d}\theta} \left(\frac{m(q,Q(s))}{m(Q(t),Q(s))} \right)^{\frac{1}{1-\alpha}} \pi(s,Q(t))\,\mathrm{d}s\,\mathrm{d}q \\
&= \int_0^{Q(t)} \int_t^{\infty} e^{-\int_t^s r(\theta)\,\mathrm{d}\theta} e^{-\frac{\gamma}{1-\alpha}(Q(t)-q)} \pi(s,Q(t))\,\mathrm{d}s\,\mathrm{d}q \\
&= \int_t^{\infty} e^{-\int_t^s r(\theta)\,\mathrm{d}\theta} \pi(s,Q(t)) \int_0^{Q(t)} e^{-\frac{\gamma}{1-\alpha}(Q(t)-q)}\,\mathrm{d}q\,\mathrm{d}s \\
&= \tilde{M}(Q(t)) \int_t^{\infty} e^{-\int_t^s r(\theta)\,\mathrm{d}\theta} \pi(s,Q(t))\,\mathrm{d}s = \tilde{M}(Q(t))\, v(t,Q(t)) \\
&= \tilde{M}(Q(t)) \frac{L_Q(t)}{\dot{Q}(t)} = \frac{\tilde{M}(Q(t))}{\beta Q(t)^{\varphi}} = \frac{1-\alpha}{\gamma\beta} \frac{1-e^{\frac{-\gamma}{1-\alpha}Q(t)}}{Q(t)^{\varphi}}.
\end{aligned}$$

B.5 Expression (82)

$$\begin{aligned}
\frac{u(\tau)|_{\delta=0}}{u(\tau)|_{\delta>0}} &\to \left(\frac{\bar{l}}{\frac{\delta(1-\alpha)}{1-\delta(1-\alpha)}\bar{l}+\bar{l}} \right)^{\alpha} \left(1+\frac{\delta\alpha}{1-\delta}\right)^{\frac{1-\alpha}{1-\varphi}} \\
&= (1-\delta(1-\alpha))^{\alpha} \left(1+\frac{\delta\alpha}{1-\delta}\right)^{\frac{1-\alpha}{1-\varphi}} \\
&> (1-\delta(1-\alpha))^{\alpha} \left(1+\frac{\delta\alpha}{1-\delta}\right)^{1-\alpha} \\
&= \frac{1-\delta(1-\alpha)}{(1-\delta)^{1-\alpha}} \geq 1.
\end{aligned}$$

B.6 Expression (85)

$$
\begin{aligned}
u(\tau) \to\ & \alpha^{\alpha} \left(\frac{\delta\,(1-\alpha)}{1-\delta\,(1-\alpha)}\,\kappa l + \frac{l}{\hat{r}} \left(1 - e^{-\hat{r}\kappa}\right) \right)^{\alpha} \left(\frac{(1-\alpha)^2}{\gamma(\rho - \alpha\hat{r})} \left(1 - e^{-\frac{\rho-\alpha\hat{r}}{1-\alpha}\omega}\right) \right)^{1-\alpha} \\
=\ & (\alpha\,\kappa l)^{\alpha} \left(\frac{\delta\,(1-\alpha)}{1-\delta\,(1-\alpha)} + \frac{1-e^{-\hat{r}\kappa}}{\hat{r}\kappa} \right)^{\alpha} \left(\frac{(1-\alpha)\,\omega}{\gamma} \right)^{1-\alpha} \left(\frac{1-e^{-\frac{\rho-\alpha\hat{r}}{1-\alpha}\omega}}{\frac{\rho-\alpha\hat{r}}{1-\alpha}\,\omega} \right)^{1-\alpha} \\
\approx\ & (\alpha\,\kappa l)^{\alpha} \left(\frac{1}{1-\delta\,(1-\alpha)} - \frac{\kappa}{2}\,\hat{r} \right)^{\alpha} \left(\frac{(1-\alpha)\,\omega}{\gamma} \right)^{1-\alpha} \left(1 - \frac{\omega}{2}\,\frac{\rho-\alpha\hat{r}}{1-\alpha} \right)^{1-\alpha} \\
\approx\ & (\alpha\,\kappa l)^{\alpha} \left(1 + \delta\,(1-\alpha) - \frac{\kappa}{2}\,\hat{r} \right)^{\alpha} \left(\frac{(1-\alpha)\,\omega}{\gamma} \right)^{1-\alpha} \left(1 - \frac{\omega}{2}\,\frac{\rho-\alpha\hat{r}}{1-\alpha} \right)^{1-\alpha} \\
\approx\ & (\alpha\,\kappa l)^{\alpha} \left(\frac{(1-\alpha)\,\omega}{\gamma} \right)^{1-\alpha} \left(1 + \delta\,(1-\alpha)\,\alpha - \frac{\alpha\kappa}{2}\,\hat{r} \right) \left(1 - \frac{\omega}{2}\,(\rho - \alpha\hat{r}) \right) \\
\approx\ & (\alpha\,\kappa l)^{\alpha} \left(\frac{(1-\alpha)\,\omega}{\gamma} \right)^{1-\alpha} \left(1 + \delta\,(1-\alpha)\,\alpha - \frac{\omega}{2}\,\rho + \frac{\alpha}{2}\,(\omega - \kappa)\,\hat{r} \right).
\end{aligned}
$$

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