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Anticipation in Innovative Investment under Oligopolistic Competition

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Abstract

We study firms' optimal investment behavior in a dynamic duopoly framework. Embodied technological progress makes later generations more productive. The resulting model is a differential game combined with a vintage capital goods structure. Since such a framework has not been analyzed before, existing concepts have to be modified. Our numerical analysis first shows that a technological breakthrough generates equilibrium investment behavior that admits anticipation waves enhanced by competition. Second, the shape of these anticipation waves depends on the age of the underlying capital good: for younger capital goods the upward peaks are more pronounced, whereas for older ones this holds for the downward peaks. Third, we show that if a firm is able to anticipate on future technological developments, this results in a higher market share in the long run.

Keywords: vintage capital, embodied technological progress, differential games, optimal control theory, Maximum principle
JEL classification: C61, C73, D92, O33

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1 Introduction

This paper studies the effect embodied technological progress has on the firm's capital accumulation process. Embodied technological progress implies that capital goods of later date are more productive. This is an important topic, as illustrated by Greenwood et al. (1997), who found that embodied technological progress is the main driver of economic growth. In particular, they discovered that in the post-war period in the US about 60% of labor productivity growth was investment-specific. A main example is information technology, the efficiency of which increases much faster over time compared with more traditional types of capital (Yorokoglu (1989)).

This paper sets up a theoretical model to study the firms' optimal investment behavior in a dynamic competitive framework, in which technological progress makes capital goods more productive. To do so we adopt a vintage capital goods model. Such a model is particularly well suited to deal with a situation where capital goods become more productive over time. This is because such a set up enables us to make productivity of a capital good dependent on its age and the year it actually operates. In view of technological progress, productivity increases with time, for capital goods of a given age.

Feichtinger et al. (2006) already adopted a vintage capital goods framework to analyze the firm's dynamic investment policy subject to embodied technological progress. However, for simplicity reasons it was assumed there that the firm under consideration was a monopolist, implying that competition was not taken into account. This is an important shortcoming, however, since, due to the extensive process of deregulation combined with a wave of mergers and acquisitions, oligopolistic market structures arise in a large number of sectors nowadays (Pawlina (2003)).

The present paper circumvents this shortcoming by adding competition to Feichtinger et al. (2006)'s framework. To do so we analyze a vintage capital goods set up with two firms. Both firms can develop their own capital stock, so they invest over time, while taking into account the investment behavior of the other firm and the fact that due to embodied technological progress capital goods of a given age get more productive as time passes. As such a differential game arises with both firms as the two players.[2] However, as far as we know, until now a differential game framework has never been combined with a vintage capital goods model. Hence, to analyze such a setting, concepts of differential game theory have to be adapted, which will be done in section 3.

Besides the obvious result that competition lowers investments, which arises because competition reduces the output price, and thus the profitability of the investment, we also generate the following findings. *First*, we derive the open-loop Nash equilibrium and show that it is time-consistent. *Second*, in a scenario where a technological breakthrough arises at a given point in time, we show that firms anticipate on this by investing less during a time period just before the breakthrough time. It turns out that this period of investing less is in turn anticipated on by investing more in the period before. This

[2]Note that the resulting model on the other hand also generalizes Cellini and Lambertini (1998) by introducing a vintage structure in the stock of capital.

process results in a whole wave of anticipation periods. Such an anticipation wave was also found in Feichtinger et al. (2006), but we find here that it also occurs under competition, and, moreover, that competition makes the wave more pronounced. *Third*, in addition we show that the shape of the anticipation wave depends on the age of the capital good. Compared to older machines it holds that for newer machines the investment increase in the upward phase is larger, while the investment decrease in the downward phase is smaller. *Fourth*, we derive that technological know how within the firm that enables it to forecast future technological developments is valuable. Knowing that a technological breakthrough will take place at some future point in time enables a firm to anticipate on that with its investment policy. We derive that an anticipating firm increases its market share and its profits at the expense of a non-anticipating firm.

The paper is organized as follows. Section 2 presents the model, while Section 3 goes into the theoretical implications of analyzing a differential game with a vintage capital goods structure. Section 4 analyzes the model and presents some economic results. Section 5 sets up a numerical exercise, while Section 6 concludes.

2 The Model

Consider a market that consists of two firms. Both firms need a capital stock to produce goods. The capital stock consists of differently aged machines and is the only production factor. It is denoted by $K_i(a,t)$, which thus stands for the number of a-year old machines[3] at time t (of firm i). For clarification we refer to figure 1, which is well known in demographics as the Lexis diagram. As time and age develops at the same pace, an a-year old machine at time t is $a+s$ years old in s years. This corresponds to the motion along the 45-degree line.

[Figure 1 about here]

The capital stock of firm i can be increased by non-negative age-specific investments $I_i(a,t) \geq 0$, and decreases with an age-dependent depreciation rate $\delta(a)$. Thus the dynamics of the capital stock of firm i along the "life-path" of one machine (45 degree line in the Lexis diagram) equals

$$\Big(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\Big) K_i(a,t) = I_i(a,t) - \delta(a) K_i(a,t), \qquad K_i(a,t_0) = K_i^0(a), K_i(0,t) = 0, \tag{1}$$

where $K_i^0(a) \geq 0$ for $a \in [0,\omega]$ is the capital stock at the beginning of the time horizon t_0 (from now on we chose $t_0 = 0$ w.l.o.g.). $\omega < +\infty$ denotes the maximal age a machine can reach. This maximal age ω is no restriction to the model as it can be chosen arbitrarily

[3] By the term a-year old machines we mean that the machines use a-year old technology. We assume that machines are more productive the newer their technology is.

high. Note that a non-negativity constraint for the capital $K_i(a,t) \geq 0 \; \forall \; (a,t)$ is not necessary due to non-negativity of the initial capital stock and investments.

The capital stock of both firms is used for producing output. We assume that the productivity of machines differs according to two different effects. Firstly, $f_i(t-a)$ accounts for the effect of technological progress on the productivity of a machine produced at time $t-a$. Clearly, productivity is higher for newer vintages, i.e. $f_i(t-a)$ is assumed to be a positive non-decreasing stepwise continuous function. Secondly, $\nu(a)$ accounts for the "learning by doing"-effect, implying that the older the machines are, the more the workforce is used to it, which raises productivity. Hence, $\nu(a)$ is a positive increasing function. Total output of firm i then reads

$$Q_i(t) = \int_0^\omega f_i(t-a)\nu(a)K_i(a,t) \; da. \tag{2}$$

The firms operate on a heterogenous product market, implying that the product price of firm i at time t depends on the current production of both firms in a different way (see e.g. Spence (1979)), i.e.

$$p_i(t) = 1 - \gamma_i Q_i(t) - \beta Q_j(t) \geq 0.$$

Since the influence of quantity on its own price is always larger than on the price of the other product, we impose that $0 \leq \beta \leq \gamma_i$. The parameter β in the demand function taken from Spence (1979) measures the degree of substitutability between the two goods, i.e. it is an inverse measure of product differentiation. It follows that the larger is β, the more competition there is between the firms on the output market. The higher β the more the firms are competing. In the case of $\beta = 0$ there is no competition and the model is equal to Feichtinger et al. (2006), as will be obvious soon.[4] The revenue of firm i can be obtained straightforwardly:

$$R_i(t) = p_i(t)Q_i(t) = \big(1 - \gamma_i Q_i(t) - \beta Q_j(t)\big)Q_i(t). \tag{3}$$

The costs of the investments in machines consists of two terms and are equal for both firms, as they have access to the same market for machines. Both are age-specific since the older (less productive) technology will be cheaper and easier to implement. The acquisition costs (of a-year old machines) are equal to $b(a)I_i(a,t)$ where $b(a) \geq 0$ is a non-increasing function over a. The costs of successful implementation (of a-year old machines) are convex, i.e. $\frac{c(a)}{2}I_i^2(a,t)$, again with a non-increasing function $c(a) \geq 0$.

[4]For an alternative interpretation consider the two firms producing different but similar products (e.g. road bikes, mountain bikes, triathlon bikes, etc.). The more similar the products are the higher β should be due to a stronger competition. If the products are completely different $\beta = 0$ and if they are not distinguishable $\beta = \gamma_i$.

Both firms maximize their profit (revenue minus investment costs) intertemporally. Therefore, the model reads

$$
\begin{aligned}
\max_{I_1(a,t)\geq 0} \quad & \int_0^T e^{-\rho_1 t}\Big[R_1(t) - \int_0^\omega \Big(\frac{c(a)}{2} I_1^2(a,t) + b(a) I_1(a,t)\Big)\ da\Big]\ dt + \\
& + \int_0^\omega e^{-\rho_1 T} l_1(a, K_1(a,T))\ da, \\
\max_{I_2(a,t)\geq 0} \quad & \int_0^T e^{-\rho_2 t}\Big[R_2(t) - \int_0^\omega \Big(\frac{c(a)}{2} I_2^2(a,t) + b(a) I_2(a,t)\Big)\ da\Big]\ dt + \\
& + \int_0^\omega e^{-\rho_2 T} l_2(a, K_1(a,T))\ da, \\
\Big(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\Big) K_1(a,t) \quad &= \quad I_1(a,t) - \delta(a) K_1(a,t), \\
& \quad\ K_1(a,0) = K_1^0(a), K_1(0,t) = 0, \\
\Big(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\Big) K_2(a,t) \quad &= \quad I_2(a,t) - \delta(a) K_2(a,t), \\
& \quad\ K_2(a,0) = K_2^0(a), K_2(0,t) = 0, \\
Q_1(t) \quad &= \quad \int_0^\omega f_1(t-a)\nu(a) K_1(a,t)\ da, \\
Q_2(t) \quad &= \quad \int_0^\omega f_2(t-a)\nu(a) K_2(a,t)\ da. \qquad (4)
\end{aligned}
$$

where ρ_i is the time preference rate of firm i and $l_i(a, K_i(a,T)) \geq 0$ is a salvage value function at the end of the time horizon T.

Remark: Note that in general also the functions $c(a)$, $b(a)$, $\delta(a)$ and $\nu(a)$ may depend on time. However, we neglect this possibility in our paper, as the analysis would not change substantially.

3 Theoretical Background

The model (4) is an age-specific differential game. Differential games usually focus on the interaction between two (or more) players over time. For an overview of the assumptions, solution concepts and methods we refer to Dockner et al. (2000) or Basar and Olsder (1982). However, as far as we know the concept of differential games has not been applied to a framework of distributed parameter control (in our case age and time). Therefore, concepts have to be adapted and well known results cannot be applied. For a more detailed discussion on the theoretical background we refer to Wrzaczek (2011). Within this section we just present the concepts that are needed for the analysis of our model (4).

Firstly we define the dynamics of a distributed parameter control model (for a more general introduction we refer to Feichtinger et al. (2003)):

$$
\begin{aligned}
\Big(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\Big)y(a,t) &= g(a,t,y(a,t),q(t),u(a,t)),\\
& \quad y(a,0)=y_0(a), y(0,t)=\varphi(t,q(t)),\\
q(t) &= \int_0^\omega h(a,t,y(a,t),q(t),u(a,t))\ \ da,\\
& \quad u(a,t)\in U,
\end{aligned} \tag{5}
$$

where $y(a,t)$ is the (distributed) state variable, $u(a,t)$ is the distributed control variable, and $q(t)$ is an integral state variable (depending only on time). $y_0(a)$ and $\varphi(t,q(t))$ are initial and boundary conditions, respectively, and U is the set of admissible controls. For a detailed discussion on the conditions for all functions we again refer to Feichtinger et al. (2003).

The objective function of player i ($i=1,\dots,n$) is defined as

$$
\max_{u_i(a,t)\in U}\int_0^T\int_0^\omega L_i(a,t,y(a,t),q(t),u(a,t))\ \ da\ \ dt+\int_0^\omega l_i(a,y(a,T))\ \ da,
$$

where $L_i(\cdot)$ and $l_i(\cdot)$ is an instantaneous utility and salvage value function, respectively.

Next, we define the concept of time-consistency analogously to the definition presented in Dockner et al. (2000) for the time-specific case.[5,6] For the sake of simplicity we assume a 2-player game. However, the definition can be extended easily to a framework with $n\geq 2$ players.

Definition 3.1 *Let $(u_1(a,t),u_2(a,t))$ be an equilibrium for the above age-specific differential game and denote $(y_1(a,t),y_2(a,t))$ the state trajectory generated in this equilibrium. The equilibrium is called time-consistent if for all $\hat{t}\in[0,T]$ the subgame starting at $\hat{t}$ with the initial condition $\hat{y}(a,\hat{t})=y(a,\hat{t})\ \forall a\in[0,\omega]$ admits an equilibrium $(\hat{u}_1(a,t),\hat{u}_2(a,t))$ such that $\hat{u}_i(a,t)=u_i(a,t)$ holds for $i=1,2$ and $(a,t)\in[0,\omega]\times[\hat{t},T]$.*

For clarification we refer to figure 2 showing the Lexis diagram. The 'basic game' (starting at $t=0$) is played over the whole rectangle $[0,\omega]\times[0,T]$ (white and grey area). The subgame starting at $\hat{t}$ is played over the rectangle $[0,\omega]\times[\hat{t},T]$ (with the corresponding initial conditions), which is represented by the grey area. If the controls resulting from the optimization over the grey area are equal to that from the optimization over the whole area, the game is time-consistent.

[Figure 2 about here]

[5]For a theoretical discussion of this definition we refer to Wrzaczek (2011). In the same paper also the term age-consistency is defined, which uses the consistency idea along the age dimension.

[6]Note that the term time-consistency used in Dockner et al. (2000) corresponds to the term weak time-consistency in other contributions e.g. Basar and Olsder (1982).

In time-dependent differential game theory it is known that an open-loop Nash equilibrium is time-consistent, but not subgame perfect (see Basar and Olsder (1982)). However, the vintage structure makes things more involved, so that this specific result need not carry over. In fact, in Wrzaczek (2011) it is shown that the consistency concept fails along the age-direction, because of the new type of states $p(a,t)$ and $q(t)$. Beyond the numerous questions in differential game theory, Wrzaczek (2011) provides a formal definition of vintage differential games and a discussion on time- and age-consistency (including the definition of classes that are always time-consistent in the open-loop Nash equilibrium). For the model presented in this paper, time-consistency is proven in the next section.

4 Analysis of the Model and Economic Interpretation

In this section we study the proposed model in detail. In subsection 4.1 we derive the necessary optimality conditions as well as some technical results. The economic intuition will be provided in the subsequent subsection 4.2.

4.1 The Open-loop Nash Equilibrium

We analyze the problem of firm i, $i = \{1,2\}$. The other firm is firm j, $j = \{1,2\}$ while $j \neq i$. The Hamiltonian for firm i reads (from now on we omit the time and age argument whenever there can be no misunderstanding)

$$\begin{aligned} \mathcal{H}^i &= \frac{1}{\omega}\Big(1-\gamma_i Q_i - \beta Q_j\Big)Q_i - \Big(\frac{c}{2}I_i^2 + bI_i\Big) + \xi^{iK_i}(I_i - \delta K_i) + \xi^{iK_j}(I_j - \delta K_j) + \\ &\quad + \eta^{iQ_i} f_i \nu K^i + \eta^{iQ_j} f_j \nu K^j. \end{aligned} \tag{6}$$

For the necessary first order condition for the controls (i.e. investments) we obtain

$$\mathcal{H}^i_{I_i} = -cI_i - b + \xi^{iK_i} \leq 0. \tag{7}$$

Applying the control constraint $I_i(a,t) \geq 0$ yields

$$I_i(a,t) = \begin{cases} \frac{1}{c(a)}\big(\xi^{iK_i}(a,t) - b(a)\big) & \text{if } \xi^{iK_i}(a,t) > b(a) \\ 0 & \text{otherwise} \end{cases}. \tag{8}$$

In the optimum the adjoint variables have to fulfill the following dynamics:

$$\begin{aligned} \Big(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\Big)\xi^{iK_i} &= (\rho_i + \delta)\xi^{iK_i} - \eta^{iQ_i} f_i \nu, \\ \Big(\frac{\partial}{\partial a} + \frac{\partial}{\partial t}\Big)\xi^{iK_j} &= (\rho_i + \delta)\xi^{iK_j} - \eta^{iQ_j} f_j \nu, \\ \eta^{iQ_i} &= \int_0^\omega \frac{1}{\omega}\Big(1 - 2\gamma_i Q_i - \beta Q_j\Big)\ da, \\ \eta^{iQ_j} &= \int_0^\omega \frac{1}{\omega}\Big(-\beta Q_i\Big)\ da, \end{aligned} \tag{9}$$

where $\xi^{iK_i}(a,t)$ and $\xi^{iK_j}(a,t)$ denote firm i's adjoint variable for capital i and j, respectively, and $\eta^{iQ_i}(t)$ and $\eta^{iQ_j}(t)$ are firm i's adjoint variables related to the output of firm i and j respectively. Note that, analogously to the corresponding state variables, $\xi^{iK_i}(a,t)$ and $\xi^{iK_j}(a,t)$ develop over age and time, whereas $\eta^{iQ_i}(t)$ and $\eta^{iQ_j}(t)$ are a function of just time. Furthermore, we have the following transversality conditions for the adjoint variables:

$$\begin{aligned}
\xi^{iK_i}(a,T) &= \frac{\partial l_i(a,K_i(a,T))}{\partial K_i}, \\
\xi^{iK_j}(a,T) &= 0, \\
\xi^{iK_i}(\omega,0) &= 0, \\
\xi^{iK_j}(\omega,0) &= 0.
\end{aligned} \tag{10}$$

We first consider the issue of time-consistency. In the open-loop case both firms fix their optimal strategy over the time horizon $[0,T]$ at the initial point of time $t=0$. If a firm has the opportunity to reconsider its investment strategy later at $t=\hat{t}>0$, but still chooses the same investment strategy, time-consistency is fulfilled. If time-consistency is not fulfilled, the firms would have the incentive to deviate from their initially chosen strategy. The following theorem proves time-consistency of the open-loop Nash equilibrium of our model.

Theorem 4.1 *The open-loop Nash equilibrium of the age-specific differential game (4) is time-consistent.*

The proof is very technical and thus shifted to Appendix A. As explained in footnote 6, this open-loop Nash equilibrium is time-consistent. An important but demanding question is what could be stated about the feedback equilibrium. It may be possible to perform an analysis analogous to what has been obtained for the conjectural equilibrium in Dockner et al. (2009, chapter 9). This is for sure an interesting topic for future research.

Employing the transversality conditions (10), the partial differential equations for the adjoint variables corresponding to the capital stocks can be solved. For firm i we obtain:

$$\begin{aligned}
\xi^{iK_i}(a,t) &= \int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\eta^{iQ_i}(t-a+s)\ ds \\
\xi^{iK_j}(a,t) &= \int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\eta^{iQ_j}(t-a+s)\ ds.
\end{aligned} \tag{11}$$

for $t-a+\omega \le T$ and

$$\begin{aligned}
\xi^{iK_i}(a,t) &= \int_a^{T-(t-a)} e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\eta^{iQ_i}(t-a+s)\ ds + \frac{\partial l_i(a,K_i(a,T))}{\partial K_i} \\
\xi^{iK_j}(a,t) &= \int_a^{T-(t-a)} e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\eta^{iQ_j}(t-a+s)\ ds.
\end{aligned} \tag{12}$$

for $t-a+\omega > T$. The first expression is used to prove the following proposition that shows that the investments corresponding to one generation of machines are positive at the beginning (when they are relatively new) and that they are zero before they reach the maximal age ω.

Proposition 4.2 *If $b(s)$, $f_i(t-s)$ and $\nu(s)$ are continuous and $b(\omega) > 0$, then for $t-a+\omega \leq T$ firm i's investments satisfy*

- $I_i(a, t+a) > 0$ *for* $a \in [0, a']$
- $I_i(a, t+a) = 0$ *for* $a \in [a'', \omega]$

for $0 \leq a' \leq a'' < \omega$.

Proof:

The proof exploits the explicit expression of the adjoint variable and the continuity of the functions $b(\cdot)$, $f_i(\cdot)$ and $\nu(\cdot)$. From (9) and (11) we obtain

$$\begin{aligned} \xi^{iK_i}(a,t) &= \int_a^{\omega} e^{-\rho_i(s-a)-\int_a^s \delta \ ds'} \frac{\partial R_i(t-a+s)}{\partial Q_i(t-a+s)} f_i(t-a)\nu(s) \ ds \\ &= \int_a^{\omega} e^{-\rho_i(s-a)-\int_a^s \delta \ ds'} \Big(\frac{R_i(t-a+s)}{Q_i(t-a+s)} - \gamma_i Q_i(t-a+s) \Big) f_j(t-a)\nu(s) \ ds. \end{aligned} \tag{13}$$

Obviously, it holds that $\xi^{iK_i}(\omega,t) = 0$. As the adjoint variables are continuous, we obtain $\xi^{iK_i}(\bar{a}, t-\omega+\bar{a}) < b(\bar{a})$ for $\bar{a} \in (a'', \omega]$ with $0 \leq a'' < \omega$ since we have assumed $b(\omega) > 0$, where a'' is the largest age for which it holds that $\xi^{iK_i}(a, t-\omega+a) < b(a)$. Thus investments are zero in this interval according to the necessary first order conditions.

For the second assertion we distinguish two cases. (i) If $\xi^{iK_i}(0,t) \leq b(0)$ investment in young machines is zero according to the necessary first order condition, i.e. $a' = 0$. (ii) If $\xi^{iK_i}(0,t) > b(0)$, we obtain $\xi^{iK_i}(\bar{a}, t+\bar{a}) > b(\bar{a})$ for $\bar{a} \in [0, a']$ by continuity, which implies positive investment. Similarly, as before a' is the smallest age for which it holds that $\xi^{iK_i}(a, t-\omega+a) < b(a)$.

Trivially $a' \leq a''$ since a' and a'' are the smallest and largest age for which it holds that $\xi^{iK_i}(a, t+a) > b(a)$. $a' < a''$ only holds, if $\xi^{iK_i}(a, t+a) > b(a)$ is not unique.

□

Note that we cannot show that $a' = a''$. Therefore, two or even multiple changes between positive and zero investments cannot be excluded. Two changes might occur in case the acquisition costs $b(s)$ shrink sharply or the learning-by-doing effect $\nu(s)$ increases sharply in between $[a', a'']$. Then it may be rational to wait a small time period until the conditions (for acquisition or production) for this specific machine generation are more profitable again.

Remark: For the opposite case $t-a+\omega > T$ above results depend on the salvage value function. If e.g. the salvage value is positive and higher than $b(\omega)$ the second assertion will not hold. It means that it will be rational to aquire old machines when their value at the end of the time horizon is higher than their acquisition costs. On the other hand, if the salvage value is negative and quite high, the first assertion will not hold. It means that expected benefits are smaller than the negative salvage value at the end of the time horizon.

In our model the existence of a steady state is not ensured. The following lemma provides a necessary (but not sufficient) condition such that a steady state can exist, while it also shows that the sign of the marginal revenue is positive.

Lemma 4.3 *Necessary for the existence of a steady state of the game is a constant productivity function $f_i(t)$ $(i = 1, 2)$. If further the optimal solution of the game reaches a stricktly positive steady state (i.e. $Q_i > 0$ for $i = 1, 2$), the marginal revenue of both firms with respect to their own output is positive.*

Proof:

A steady state of the game is defined as

$$\frac{dI_i(a,t)}{dt} = \frac{dK_i(a,t)}{dt} = \frac{d\xi^{iK_i}(a,t)}{dt} = \frac{d\xi^{iK_j}(a,t)}{dt} = 0. \tag{14}$$

Therefore, the marginal revenue of a firm with respect to its own output $\frac{\partial R_i(t)}{\partial Q_i(t)}$ does not change over time, as easily can be seen from (using the definition of $R_i(t)$, see (3))

$$\frac{d}{dt}\Big(\frac{\partial R_i(t)}{\partial Q_i(t)}\Big) = \frac{d}{dt}\Big(\frac{R_i(t)}{Q_i(t)} - \gamma_i Q_i(t)\Big) = 0. \tag{15}$$

Calculating $\frac{d\xi^{iK_i}(a,t)}{dt}$ by using (13)) and inserting the above result it can be concluded straightforwardly that a steady state can only occur if $f_i(t)$ (for $i = 1, 2$) does not depend on time, which proves the first result of the lemma.

Now we prove the second assertion. From $Q_i > 0$ it follows that $\xi^{iK_i}(a)$ exceeds $b(a)$ at least once (a very short interval because of continuity). By (13) this implies $R_i > \gamma_i Q_i^2$ as f_i and $\nu(a)$ are assumed to be positive. This implies a positive marginal revenue.

□

A time-independent $f_i(t)$ $(i = 1, 2)$ in fact implies that there is no technological progress. Therefore, the capital stock does not get more productive over time, which makes it plausible that a capital level being constant over time can be optimal.

The intuition of the second assertion of the lemma is straightforward. If marginal revenue would be negative, producing a smaller quantity would increase profit. If the marginal revenue would be zero, expression (13) implies a zero shadow price and thus zero investment. Thus in the steady state a higher production would increase the revenue, but decrease the profit, as it is too costly.

4.2 Economic Interpretation

In order to provide an economic interpretation of the necessary first order condition for investments we employ the explicit expression of the shadow price (13). Assuming $I_i(a,t) > 0$, it can be written in the following way for $t-a+\omega \leq T$ (case $t-a+\omega > T$ analogously):

$$\begin{aligned} 0 &= \int_a^{\omega} e^{-\rho_i(s-a)-\int_a^s \delta \; ds'} \frac{\partial R_i(t-a+s)}{\partial Q_i(t-a+s)} f_i(t-a)\nu(s) \; ds - cI_i(a,t) - b(a) \\ &= \int_a^{\omega} e^{-\rho_i(s-a)-\int_a^s \delta \; ds'} \Big(\frac{\partial R_i(t-a+s)}{\partial Q_i(t-a+s)} f_j(t-a)\nu(s) - c\bar{I}_i(a,t,s) - \bar{b}(a,s) \Big) \; ds, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \bar{I}_i(a,t,s) &= e^{\rho_i(s-a)+\int_a^s \delta \; ds'} \frac{I_i(a,t)}{\omega - a}, \\ \bar{b}(a,s) &= e^{\rho_i(s-a)+\int_a^s \delta \; ds'} \frac{b(a)}{\omega - a}. \end{aligned} \tag{17}$$

The first term between brackets, i.e. $\frac{\partial R_i(t-a+s)}{\partial Q_i(t-a+s)} f_i(t-s)\nu(s)$, is the instantaneous marginal revenue of firm i, which is proportional to the productivity and the learning-by-doing coefficient. The second term, i.e. $-c\bar{I}_i(a,t,s) - \bar{b}(a,s)$, denotes the marginal investment costs corresponding to vintage $t-a$ at time t, being transformed into running costs.[7] Hence, the expression between brackets stands for the instantaneous marginal profit associated with investment in capital goods of vintage $t-a$ at time t. The integral in (16) aggregates the present value of these instantaneous profits, which has to be zero. This is an intertemporal (by means of the life-cycle of a machine, i.e. along the 45 degree line in the Lexis diagram) version of the well known "marginal revenue = marginal costs"-condition.

The above formulation (16) and (17) is interesting also without the competition effect (see Feichtinger et al. (2006) or the ($\beta = 0$)-case in our model), as it enriches a very common optimality condition by an intertemporal facet. Later on we will again use it to explain the anticipation wave, which is another feature of the open-loop Nash equilibrium.

Dynamics of the investments

Next, we derive the change in the investments (given $I_i(a,t) > 0$) over time and/or age and give appropriate interpretations. This is especially important since (i) the age-specific model gives the investment path in three directions (age and time together, time with fixed age, age with fixed time) and (ii) the effect of the competing firm can be observed. For

[7] The marginal investment costs only arise at the time of investment t and not over the running time of the machine. However, with transformation (17) they can be interpreted as running costs.

the time path of investing in a specific vintage (age and time together) we obtain

$$\begin{aligned}\Big(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\Big)I_i(a,t) &= \underbrace{I_i(a,t)\Big(\rho_i+\delta-\frac{c'(a)}{c(a)}\Big)}_{i}+\underbrace{\frac{b(a)}{c(a)}\Big(\rho_i+\delta-\frac{b'(a)}{b(a)}\Big)}_{ii}- \\ &\quad \underbrace{-\frac{1}{c(a)}\frac{\partial R_i(t)}{\partial Q_i(t)}f_i(t-a)v(a)}_{iii}. \qquad (18)\end{aligned}$$

Note that when the right-hand side of (18) is positive (negative), the firm increases (decreases) investment in capital goods of a particular vintage when they get older. With this in mind we interpret the above labeled three terms separately:

i) Investments increase by $I_i(a,t)(\rho_i+\delta)$. This is because the older the capital goods are that the firm invests in, the less the remaining lifetime, thus the smaller the depreciation and the interest costs are. Since implementation becomes cheaper as time passes, i.e. $c'(a)\leq 0$, it holds that $-I_i(a,t)\frac{c'(a)}{c(a)}$ is non-negative. This effect accounts for a decrease of implementation costs relative to the value of them, and it positively attributes to the increase of investment in a particular vintage when capital goods of that vintage get older.

ii) $\frac{b(a)}{c(a)}(\rho_i+\delta)$ has the same interpretation as the corresponding term in i), but is weighted by the ratio of the cost parameters. The higher are the acquisition costs (corresponding to the linear term in the cost function) in relation to the implementation costs (corresponding to the quadratic term in the cost function), the stronger is the increase in the investments. $-\frac{b(a)}{c(a)}\frac{b'(a)}{b(a)}$ has the same interpretation as the corresponding term in i).

iii) The third factor is negative, because marginal revenue is positive. The older a capital good is, the shorter the time interval it can be used, implying that aggregate revenue will be smaller. This leads to a decrease of investments in a particular vintage over time.

Whereas effects i) and ii) are positive, and thus lead to increased investments when capital stock gets older, effect iii) is negative. Furthermore, the first two effects are independent of the competing firm and are thus also valid for a framework without competition. The third effect, however, depends on marginal revenue, which in turn depends negatively on the competitor's output.

Having discussed the change of investments over time for one particular vintage of capital goods, we now derive the change of investments in capital goods with age a (which is fixed) over time. Assuming again positive investments, we obtain the following expression:

$$\begin{aligned}\frac{dI_i(a,t)}{dt} &= \frac{1}{c(a)}\int_a^\omega e^{-\rho_i(s-a)-\int_a^s\delta\ ds'}\Big[f_i'(t-a)\nu(s)\frac{\partial R_i(t-a+s)}{\partial Q_i(t-a+s)}- \\ &\quad -f_i(t-a)\nu(s)\big(2\gamma_i Q_i'(t-a+s)+\beta Q_j'(t-a+s)\big)\Big]\ ds. \qquad (19)\end{aligned}$$

The first term between brackets accounts for the change in productivity due to technological progress. As time passes, capital goods of a given age across time get more productive, which implies higher investments. The second term between brackets accounts for the change in marginal revenue due to own and competitor's output. An increase of both outputs implies decreasing output price, which reduces investments.

Finally, the effect of age on investment at a given point in time t ($I_i(a,t)$) is replaced by

$$\begin{aligned}\frac{dI_i(a,t)}{da} &= \Big(\rho_i+\delta-\frac{c'(a)}{c(a)}\Big)I_i(a,t)+\frac{1}{c(a)}\Big((\rho_i+\delta)b(a)-b'(a)\Big)- \\ &-\frac{1}{c(a)}\frac{\partial R_i(t)}{\partial Q_i(t)}f_i(t-a)\nu(a)- \\ &-\frac{1}{c(a)}\int_a^\omega e^{-(\rho_i+\delta)(s-a)}\Big[f_i'(t-a)\nu(s)\frac{\partial R_i(t-a+s)}{\partial Q_i(t-a+s)}- \\ &-f_i(t-a)\nu(s)\big(2\gamma_i Q_i'(t-a+s)+\beta Q_j'(t-a+s)\big)\Big]\ ds \qquad (20)\end{aligned}$$

The first three terms also appear in (18) and the interpretation is similar: older capital goods are cheaper and have lower aggregate revenue. The last two terms are also contained in (19) with the same interpretation: older capital goods are less modern and thus less productive while also a correction for changes in output is needed.

Dependence on the price parameters

In the above three expressions we evaluated the dependence of investments on age, time and on both, with the latter such that $t-a$ is fixed, i.e. the same vintage over time is considered. We now investigate how investment depends on the demand parameters (again assuming that the investments are positive), i.e. γ_i, γ_j and β. Whereas the dependence on γ_i can also be obtained in the model without competition, the dependence on γ_j and β is due to including competition in the model.[8] These comparative dynamics are calculated by taking the total derivative of the necessary first order condition (7).

For the sake of a clearer notation we skip the time argument for the output values, i.e. $Q_i = Q_i(t-a+s)$ for $i=1,2$, in the following expressions. For the derivative with respect to the own output parameter γ_i we obtain

$$\frac{dI_i(a,t)}{d\gamma_i} = -\frac{1}{c(a)}\int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\Big(2Q_i+2\gamma_i\frac{dQ_i}{d\gamma_i}+\beta\frac{dQ_j}{d\gamma_i}\Big)\ ds. \qquad (21)$$

The sign of the above expression is ambiguous. The following proposition derives a condition under which $\frac{dI_i(a,t)}{d\gamma_i}$ is negative.

[8] Note that in our model all three parameters are assumed to be exogenous. Cellini and Lambertini (2002) studies a similar time-dependent differential game, where the firms can also invest into R&D in order to increase product heterogeneity (by increasing β) und reduce competition. The other price parameters γ_i $(i=1,2)$ are exogenous.

Proposition 4.4 *The derivative of firm i's investments with respect to γ_i (see (21)) is negative if the following condition is fulfilled*

$$1 > -\epsilon(Q_i, \gamma_i) - \frac{1}{2}\frac{\beta}{\gamma_i}\frac{Q_j}{Q_i}\epsilon(Q_j, \gamma_i), \tag{22}$$

where $\epsilon(x, y) := \frac{dx}{x}\frac{y}{dy}$ denotes the elasticity between arbitrary values x and y. Otherwise, the derivative is positive.

Proof:

A negative derivative implies that the term between brackets within the integrand of (21) is positive. Applying the definition of elasticity implies the result.

□

The proposition reveals two effects working in the opposite direction. An increase of γ_i directly implies a reduction in price and therefore also of marginal revenue. This direct effect, which represents the left-hand side of (22), reduces investment. This reduced investment leads to a lower output Q_i, which raises the price for both firms. This in turn leads to a higher investment. The latter indirect effect is covered by the right-hand side of (22), where the first term represents the effect of firm i and the second term that of the competing firm j. Note that the indirect effect is proportional to the elasticities of both outputs, which indeed measures the sensitivity of the quantities Q_i and Q_j to a change of γ_i. If the direct effect exceeds the indirect one, the derivative is negative. Furthermore, expression (22) allows to isolate the effect of competition on the derivative. The direct effect and the first term of the indirect one occurs also without competition. The second term is only due to the competing firm. Hence, although the parameter γ_i does not enter explicitly in the expression for p_j, due to the strategic effect of an increase of γ_i, that is, the output price of Q_j raises via the decrease of I_i, the competing firm j is still influenced by a change of γ_i.

The derivative with respect to the competition parameter β is derived in the same way. We obtain

$$\frac{dI_i(a,t)}{d\beta} = -\frac{1}{c(a)}\int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\left(2\gamma_i\frac{dQ_i}{d\beta} + \beta\frac{dQ_j}{d\beta} + Q_j\right)\ ds. \tag{23}$$

The following proposition derives the condition for $\frac{dI_i(a,t)}{d\beta}$ to be negative.

Proposition 4.5 *The derivative of firm i's investments with respect to β is negative if the following condition is fulfilled*

$$1 > -2\frac{\gamma_i}{\beta}\frac{Q_i}{Q_j}\epsilon(Q_i, \beta) - \epsilon(Q_j, \beta). \tag{24}$$

Otherwise the derivative is positive.

Proof:

Analogous to the proof of Proposition 4.2.

□

The interpretation of condition (24) of the above proposition is similar to that of condition (22). The left-hand side represents the direct effect and the right-hand side the indirect one. If the (absolute values of the) elasticities are sufficiently low, the direct effect exceeds the indirect one, and the derivative is negative. In this case an increase of β leads to lower investments.

Finally, we provide the derivative of investment with respect to γ_j, which enters in the price function of the competing firm:

$$\frac{dI_i(a,t)}{d\gamma_j} = -\frac{1}{c(a)} \int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\Big(2\gamma_i \frac{dQ_i}{d\gamma_j} + \beta \frac{dQ_j}{d\gamma_j}\Big)\ ds \qquad (25)$$

The following proposition provides the condition under which this derivative is positive.

Proposition 4.6 *The derivative of firm i's investments with respect to* γ_j *is positive if the following condition is fulfilled:*

$$0 > 2\frac{\gamma_i}{\beta}\frac{Q_i}{Q_j}\epsilon(Q_i,\gamma_j) + \epsilon(Q_j,\gamma_j). \qquad (26)$$

Otherwise the derivative is negative.

Proof:

Analogous to the proof of Proposition 4.2.

□

Parameter γ_j does not occur in the price of firm i, but only in that of firm j. Therefore, condition (26) has no direct effect but only an indirect one, which is represented by the right-hand side. This indirect effect is the mirror case of the indirect effect of γ_i, which we discussed above. The *direct* effect of firm j reduces investments of firm j. A decrease of investments of firm j leads to a lower output Q_j, which increases the price for firm i. This raises the incentive to invest for firm i and thus makes that investments of firm i increase when γ_j increases.

Anticipation wave

From the model without competition (see Feichtinger et al. (2006)) we know that a technological breakthrough that results in an upward jump of the technology level function $f_i(\cdot)$ ($i \in \{1,2\}$), generates a so called anticipation wave before the jump. An anticipation wave refers to a trajectory on which the firm consecutively produces more and less over time, compared to a scenario where the firm does not anticipate a future technological breakthrough. To discuss this phenomenon in more detail we proceed step by step. We compare two scenarios for technological progress: the first scenario is described by a given technology function $f_i(t)$ $(i \in \{1,2\})$, while in the second scenario the technology function is $\hat{f}_i(t)$ $(i \in \{1,2\})$, where

$$\hat{f}_i(t) = f_i(t) \ \text{ for } \ t \leq \hat{t}, \qquad \hat{f}_i(t) > f_i(t) \ \text{ for } \ t > \hat{t} \tag{27}$$

$(i \in \{1,2\})$. The technology function $\hat{f}_i(t)$ has a discontinuity (jump) at time $\hat{t}$, i.e.

$$\lim_{t \to \hat{t}-} \hat{f}_i(t) < \lim_{t \to \hat{t}+} \hat{f}_i(t) \qquad \text{for } \ i \in \{1,2\}, \tag{28}$$

which reflects a technological breakthrough right at time $\hat{t}$. We define an anticipation wave as follows.

Definition 4.7 *Let $I_i(\cdot)$ and $\hat{I}_i(\cdot)$ be the optimal investment bahavior corresponding to the two technology functions $f_i(t)$ and $\hat{f}_i(t)$, respectively. Furthermore, let t_0, t_1 and t_2 are three points in time such that $t_2 < t_1 < t_0$. An anticipation wave takes place if*

$$\begin{aligned} \hat{I}_i(a,t) < (>)\ I_i(a,t) \quad &\textit{for} \quad t \in [t_1, t_0) \\ \hat{I}_i(a,t) > (<)\ I_i(a,t) \quad &\textit{for} \quad t \in [t_2, t_1). \end{aligned}$$

Let us assume that the firms are in a situation where technology develops over time according to $\hat{f}_i(t)$. We consider

Anticipation scenario: The firm knows the real shape of the technology function, i.e. it is aware of a technological breakthrough beforehand. This implies that $\hat{f}_i(t)$ is known and the firm can behave optimally during the whole time horizon.

Non-Anticipation scenario: The firm is not aware of the technological breakthrough beforehand. In fact, $f_i(t)$ is the technology level to be expected for the whole planning period. Therefore, the firm determines its investment policy while having the wrong expectation. However, at time $\hat{t}$ the firm faces the technological breakthrough and adapts its investment strategy from there on.

Note that the above definition allows different scenarios for both firms. It is possible that firm i is in the anticipating and firm j $(j \neq i)$ in the non-anticipating scenario. A possible example would be that the anticipating firm has better information, because it has more knowledge about technological developments. Consequently, three scenarios are possible for the game with two firms: 1) both firms anticipate, 2) both firms do not anticipate, 3) one firm anticipates and the other one does not. Note that for $\beta = 0$ the third scenario the solution is the same as in scenario 1), whereas for the non-anticipating firm the solution of scenario 2) prevails.

Furthermore, it is possible that a productivity jump only occurs for one firm.[9] Even in that case all three above mentioned scenarios are possible. Therefore, it is possible that although firm j does not anticipate (it has no information about the productivity jump), it can observe the anticipation of firm i and reacts on it. In this way the anticipation of the technological breakthrough can also occur when the jump is not directly observed. In the numerical part we provide a special scenario to highlight this case.

We now give the economic intuition for the anticipation wave, which describes the case where the investments in new machines are lowered and increased substantially several times before the technological breakthrough. The deviation is greater the nearer the productivity jump is. In the numerical part we provide several figures showing this effect. For the interpretation we can distinguish between the following three effects[10]:

Productivity effect: The technological breakthrough results in an upward jump of $f_i(t)$. From (16) it follows that such a productivity jump implies that investment also jumps upward.[11]

Size effect: During a time interval right before the technological breakthrough the firm cuts down investment. This reduces production and increases output price. The result is that right after the technological breakthrough it is optimal for the firm to invest more in the new technology. In this way the firm can optimally benefit from the technology jump. Since during the *time interval before the jump* the output price has increased, profitability of investments taking place during the time right before this time interval, goes up. Therefore, firms increase investment there. This increase in investment leads to higher output and lower prices, which in turn reduces investment earlier in time. Proceeding with this reasoning while going backwards in time, provides the intuition for the anticipation wave.

Competition effect: In the case of a technological breakthrough also the output of the competing firm changes ex-ante. An increase (decrease) in the competing output leads to a decrease (increase) in the output price and a decrease (increase) in the investments.

Note that the productivity effect and the size effect also occur in the model without competition, i.e. for $\beta = 0$. The competition effect, however, only occurs in the case of competition, when $\beta > 0$.

[9] E.g. different environmental laws in two geographical regions.

[10] In the productivity effect we fix the marginal revenue for a moment and look at the implication of a change in the productivity function. In the competition effect we fix the productivity for a moment and look at the implication of a change in the marginal revenue (firstly by the change in the own output and then by the change in the output of the competitor) for a moment.

[11] This is also true for a continuous increase of the productivity.

5 Numerical Results

We start with the presentation of a benchmark scenario, for which we adopt the following parameters taken from Feichtinger et al. (2006):

- $\gamma_i = \beta = 0.0004$,
- $\omega = 20$,
- $\delta(a) = \frac{2a}{\omega^2} \ln \frac{1}{\kappa}$ with $\kappa = 0.2$,
- $\rho_i = \rho_j = 0.03$,
- $\nu(a) = \begin{cases} 2\sqrt{\frac{a}{v}} - \frac{a}{v} & \text{for } a \in [0, v) \\ 1 & \text{for } a \in [v, 1] \end{cases}$ with $v = 10$,
- $b(a) = \frac{\omega - a}{\omega} 3.92$,
- $c(a) = 1.2e^{-a/\omega}$.

Note that the competition parameter β is equal to the own price parameter γ_i ($i = 1, 2$), implying that this benchmark case corresponds to a homogeneous product market. Furthermore, it is assumed that the acquisition costs decrease with age in such a way that $b(\omega) = 0$, which does not satisfy the assumption under which Proposition 4.1 holds. Note that we have made no specification of the technology level function $f_i(t)$ ($i = 1, 2$), since the variation of it will be one of the central elements in this section.

Figure 3 plots the investment behavior for the benchmark model where both firms are in the anticipating scenario. It is assumed that the firms are identical (i.e. $\gamma_i = \gamma_j$ and $f_i(t) = f_j(t)$ for $i \neq j$) and start with the same amount of capital stock at $t = 0$, implying that the resulting investment path is optimal for both firms. The three investment paths in both panels correspond to different production efficiencies, i.e. stable $f_i(t) = 1$, linearly increasing $f_i(t) = 1 + \frac{t}{T}$ and exponentially increasing $f_i(t) = e^{1.02\,t}$ ($i = 1, 2$; $t \in [0, T]$). In the left panel the investment behavior over the "life-cycle" of one machine generation is plotted. The increase of the investments up to $a \approx 13$ is due to the increasing learning-by-doing function and due to the shrinking investment costs. Then the investments decrease but stay positive over the whole life-cycle. Note that this is no contradiction to Proposition 4.1 since assumption $b(\omega) > 0$ is violated. The shape over the life-cycle, which does not depend on the shape of the technology level function, shows that it is not optimal (in this benchmark case) to invest at the highest level in new machines. Instead, investments increase with age (up to a certain age) in order to exploit the learning-by-doing knowledge and the fact that investment costs decline with age. The right panel shows how the firms invest in the newest possible technology over time in the three different scenarios. In the case of constant production efficiencies the system is in steady state and investments are constant over time (see Lemma 4.1). In the case of linearly increasing $f_i(i)$ ($i = 1, 2$)

investments are shrinking slightly over time, as with better technology less capital is necessary for the production of the same output. If productivity is increasing exponentially, investments are sharply decreasing over time because of the same reason.

[Figure 3 about here]

The scenario we consider from now on is that there is a technological breakthrough at $t = 50$. We assume that the efficiency is constant before and after this event, i.e. $f_i(t) = 3$ for $t \in [0, 50)$ and $f_i(t) = 4$ for $t \in [50, T]$. In figure 4 we consider the case where the breakthrough is effective for both firms. Two scenarios are distinguished. In the first one both firms are aware of the productivity shock before it occurs and can anticipate it. In the second one only firm 1 anticipates the shock, while firm 2 does not. Investments of firm 1 are plotted in the left panels, that of firm 2 in the right ones. In the upper panels we took the investment path for the newest available machines and in the lower ones for machines in the middle of their "life-cycle"; ($a = 10$). For both machine types firm 1 anticipates the breakthrough and decreases investments immediately before the technological breakthrough followed by a jump afterwards. The reason for decreasing the investments is that the firm wants to benefit as much as possible from the raise in technology efficiency by investing in new machines just after this shock has occurred. The investment decline just before the shock makes that more new capital goods can be purchased after the shock without reducing the output price too much. Before and after the breakthrough an anticipation wave can be observed similarly to the model of Feichtinger et al. (2006). The economic interpretation for this wave can be explained as follows. Consider a positive anticipation phase occurring just before a negative anticipation phase. This one is caused by the fact that during this negative anticipation phase production is low leading to a high output price. This makes investing more profitable, which induces the positive anticipation phase. However, during a positive anticipation phase production is higher and thus price is lower, which in turn leads to another negative anticipation phase occurring just before the positive one.

Figure 4 clearly shows that anticipating the shock results in a larger market share after the technological breakthrough has occurred. In particular we see that, if firm 2 does not anticipate, part of the market share shifts from firm 2 to firm 1. We conclude that, if a firm contains the technological know-how that enables it to forecast the future technological developments, it can result in an increased market share and thus raise profits.

Two more very interesting effects can be observed. Firstly, for new machines it holds that the decrease in the investments before the breakthrough is less than in the case of older machines. For the increase after the breakthrough it is the other way around. This is due to the fact that the decrease of older machines in fact takes place at a moment in time that the technological breakthrough has already occurred for the new machines. Hence, this reduction of investment instantaneously raises the output price, which induces larger investments in those machines that already contain the technology from after the breakthrough. The result is that the shape of the anticipation wave is age dependent: for the new capital stock the reduction investment during the downward phase is smaller than for older machines, whereas the increase in investment during the upward phase is larger.

Secondly, even in case firm 2 does not anticipate the breakthrough (remember: firm 1 anticipates in both scenarios of these figures) it can observe the anticipation wave of firm 1 and react on that. The result is that firm 2 also anticipates, but with a little delay and not that intensive, which is due to the competition. However the investments of firm 2 do not jump immediately after $t = 50$, which gives firm 1 the opportunity to increase its market share at the expense of firm 2.

[Figure 4 about here]

To investigate the value of innovating and the value of information separately, we study an (admittedly) somewhat unrealistic scenario where the technological breakthrough (again at $t = 50$) occurs only for firm 2 (see figure 5), while at the same time firm 2 may not anticipate this breakthrough whereas firm 1 does anticipate it. In case firm 2 indeed does not anticipate, we have the interesting scenario that before the breakthrough firm 1 has more information than firm 2, whereas after the breakthrough firm 2 produces with a better technology than firm 1. In the left panel we plot the investments of firm 1 and in the right one those of firm 2 (again for new machines and machines in the middle of their life-cycle $a = 10$). Firm 1 (left panels) anticipates the breakthrough before and after $t = 50$, but without the dramatic decrease before and increase after $t = 50$. It only reacts on the changes in price due to the different (future) outputs of firm 2. As a result, if firm 2 does not anticipate the breakthrough, firm 1 starts to anticipate a bit later. Furthermore, the change in the investments is lower (in absolute values) than in the previous case (figure 4), where both firms anticipated. Remarkable is that firm 1's market share after the shock is considerably greater, if firm 2 does not anticipate the shock. The plots for firm 2 are quite similar to the previous case (figure 4) with the analogous interpretation.

Figures 6-8 present the dependence of optimal investments on the level of competition. Note that in the case of $\beta = 0$ we reproduce the outcome of the model without competition. The larger β, the more the products of both firms are alike, implying that competition increases. We plot four different investment paths in each panel related to different competition parameters, i.e. $\beta = 0$, $\beta = 0.0001$, $\beta = 0.0002$, and $\beta = 0.0003$. The case where firms produce identical products ($\beta = 0.0004$) has been depicted in the figures we discussed before.

In figure 6 the breakthrough occurs for both firms and both anticipate it. In figure 7 the situation is changed such that only firm 2 benefits (in terms of production efficiency) from the breakthrough. On the other, hand in figure 8 the breakthrough occurs for both firms, but only firm 1 is aware of it.

The following points can be observed. The investment paths decrease with increasing β. This implies that, as expected, investments are decreasing in the level of competition. In figure 6 we see that without competition firm 1's investments are constant over time. However, when β goes up, not only firm 1's investments are lower, but they also fluctuate more. The reason is that firm 1 gets more affected by firm 2's behavior, where firm 2's anticipative investment fluctuations result in fluctuations in output and thus in market price, which in turn causes fluctuations in the profitability of firm 1's investments.

In figure 8 firm 1 anticipates the breakthrough while firm 2 does not. As before, in the case of firm 2 a little anticipation wave can be observed. The interesting thing is, that this anticipation is larger in size the greater β is. Thus higher competition also leads to a more pronounced presence of the anticipation wave.

6 Conclusions

The contribution of this paper is twofold. First, it contributes to the literature of capital accumulation games. Second, it provides new results within the area of the economics of innovation. The first contribution within the class of capital accumulation games, an overview of which is given in Dockner et al. (2000), is Spence (1979), which was followed by, e.g., Fershtmann and Muller (1984,1986), Dockner and Takahashi (1990), and Reynolds (1987,1991). These papers have in common that the considered stocks of capital goods are homogenous. The present paper extends this research by adopting a vintage capital goods framework, which requires the adjustment of existing concepts within the theory of capital games. We derive and open-loop Nash equilibrium and show that it is time-consistent. An open-loop information structure prescribes that firms have to announce their strategy at the initial point of time and have to commit to this as time proceeds. A clear improvement would be to relax this commitment assumption so that a firm can immediately react to actions of the competitor. This is possible if we make the firms' investment rate a function of its own capital stock and of the capital stock of the competitor. It would be an interesting topic for future research to solve such a differential game and derive the resulting Markov perfect equilibria.

Our contribution to the economics of innovation is that, first, we have the obvious result that competition reduces investment of an individual firm. Second, in a situation where we have a future breakthrough time at which a more productive technology arrives, it is optimal for firms to change their investment behavior prior to the breakthrough time. This results in a so-called anticipation wave. As far as the shape of the anticipation wave is concerned, we showed that for newer capital goods the upward peaks are more pronounced, while for the older capital goods this holds for the downward peaks. Finally, we found that information pays off. In particular, if a firm is able to perfectly forecast future technological developments, this enables it to increase its long run market share.

A Proof of Theorem 4.1

According to the definition of the time-consistency we have to show that $\hat{I}_i(a,t) = I_i(a,t)$ $(i = 1,2)$ for $(a,t) \in [0,\omega] \times [\hat{t},T]$. Within this proof we assume that both firms have a zero salvage value, i.e. $l_i(a, K^i(a,T)) = 0$ for $a \in [0,\omega]$, $i = 1,2$. With a positive (or negative) salvage value no new arguments would be necessary, but the notation would be even more unclear.

The necessary first order condition implies that we have to show $\hat{\xi}^{iK_i}(a,t) = \xi^{iK_i}(a,t)$

for $i = 1, 2$. Using the definitions of the states, the corresponding expressions of the adjoint variables and the first order condition for the controls we obtain for the original game (starting at $t = 0$):

$$\begin{aligned}
\xi^{iK_i}(a,t) &= \int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\eta^{iQ_i}(t-a+s)\ ds \\
&= \int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\big(1-2\gamma_i Q_i(t-a+s)-\beta Q_j(t-a+s)\big)\ ds \\
&= \int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\Big(1-\int_0^\omega f_i(t-a+s-s')\nu(s')(2\gamma_i K_i(s',t-a+s)+ \\
&\quad +\beta K_j(s',t-a+s))\ ds'\Big)\ ds \\
&= \int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\Big(1-\int_0^\omega f_i(t-a+s-s')\nu(s')\times \\
&\quad \times\int_0^{s'} e^{-\int_{s''}^{s'} \delta\ ds'}\big(2\gamma_i I_i(s'',t-a+s-s'+s'')+ \\
&\quad +\beta I_j(s'',t-a+s-s'+s'')\big)\ ds''\ ds'\Big)\ ds \\
&= \int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\Big(1-\int_0^\omega f_i(t-a+s-s')\nu(s')\times \\
&\quad \times\int_0^{s'} e^{-\int_{s''}^{s'} \delta\ ds'}\big(2\frac{\gamma_i}{c(s'')}(\xi^{iK_i}(s'',t-a+s-s'+s'')-b(s'')+\mu_i(s'',t-a+s-s'+s''))+ \\
&\quad +\frac{\beta}{c(s'')}(\xi^{jK_j}(s'',t-a+s-s'+s'')-b(s'')+\mu_j(s'',t-a+s-s'+s''))\big)\ ds''\ ds'\Big)\ ds
\end{aligned} \tag{29}$$

By inserting recursively it gets clear that $\xi^{iK_i}(a,t)$ only depends on exogenous terms.

Now we turn to $\hat{\xi}^{iK_i}(a,t)$. In this case we have $K_i(a,\hat{t})$ for $a \in [0,\omega]$ $(i = 1, 2)$ as initial conditions. Analogous to the expression above we obtain the following expression for the game starting at time $\hat{t} > 0$:

$$\begin{aligned}
\hat{\xi}^{iK_i}(a,t) &= \int_a^\omega e^{-\rho_i(s-a)-\int_a^s \delta\ ds'} f_i(t-a)\nu(s)\Big[1-\int_0^{t-a+s-\hat{t}} f_i(t-a+s-s')\nu(s')\int_0^{s'} e^{-\int_{s'}^{s''} \delta\ ds'}\times \\
&\quad \times\big(2\frac{\gamma_i}{c(s'')}(\hat{\xi}^{iK_i}(s'',t-a+s-s'+s'')-b(s'')+\hat{\mu}_i(s'',t-a+s-s'+s''))+ \\
&\quad +\frac{\beta}{c(s'')}(\hat{\xi}^{jK_j}(s'',t-a+s-s'+s'')-b(s'')+\hat{\mu}_j(s'',t-a+s-s'+s''))\big)\ ds''\ ds'- \\
&\quad -\int_{t-a+s-\hat{t}}^\omega f_i(t-a+s-s')\nu(s')e^{-\int_{t-a+s-\hat{t}}^{s'} \delta\ ds''}\Big(2\gamma_i\hat{K}_i(s'-t+a-s+\hat{t},\hat{t})+ \\
&\quad +\beta\hat{K}_j(s'-t+a-s+\hat{t},\hat{t})\Big)\Big]\ ds
\end{aligned} \tag{30}$$

The only difference is that we cannot substitute for capital that has already been owned by the firms before $\hat{t}$. Thus we have to split the integral in the square bracket. The first one (from 0 to $t - a + s - \hat{t}$) is built up analogously to the corresponding part in (29). The second one (from $t - a + s - \hat{t}$ to ω) is completely exogenously given by the initial conditions of the new game started at $\hat{t}$.

By splitting the integral in the square brackets into two parts (29) can be transformed straightforwardly to an analogous expression than (30). By inserting recursively it is obvious that both adjoint variables depend only on exogenous terms, which implies equality. Since that means that the adjoint variables are equal in both cases, time-consistency follows immediately.

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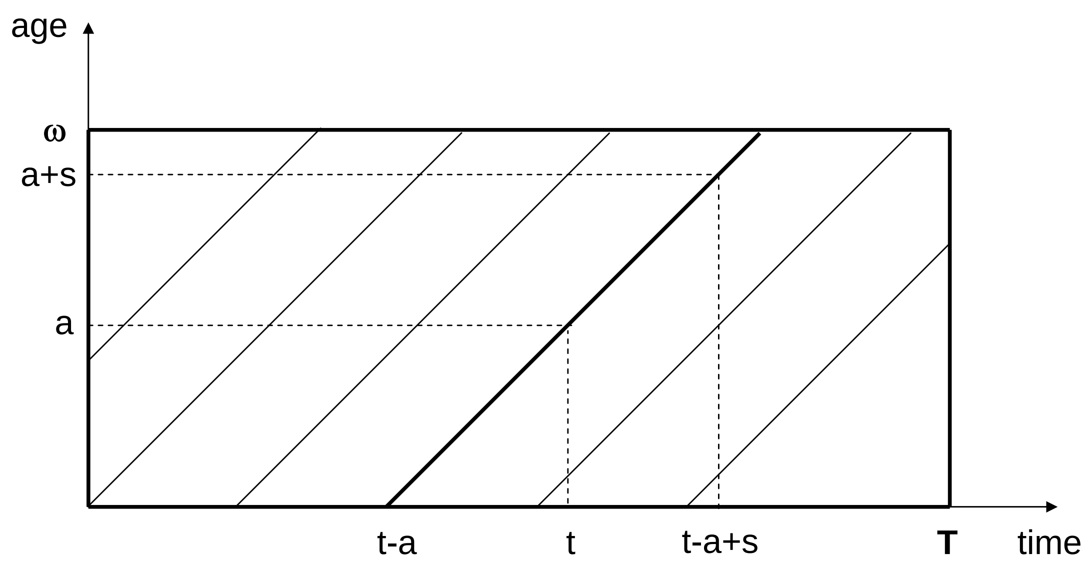

Figure 1: Lexis diagram

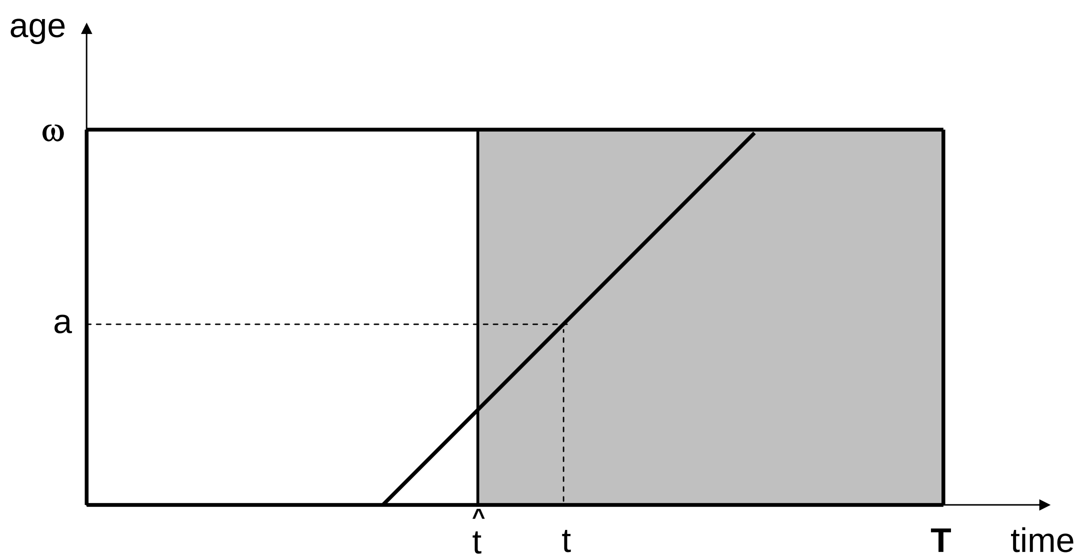

Figure 2: Time-consistency in the Lexis diagram

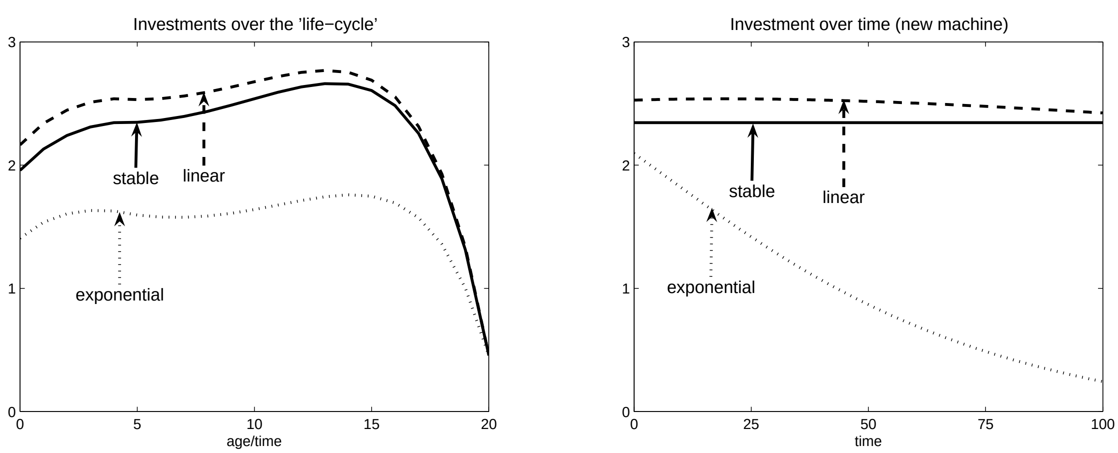

Figure 3: Benchmark case with different technological progresses

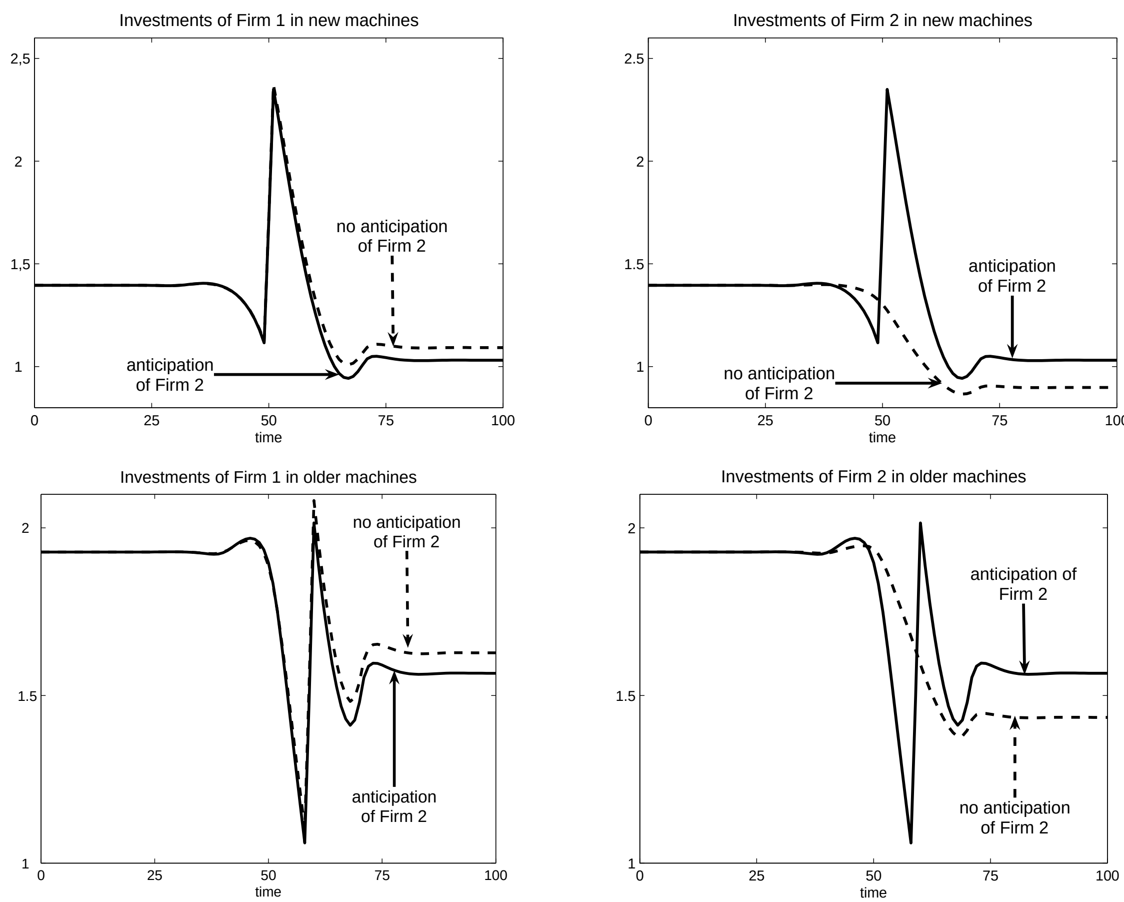

Figure 4: Technological breakthrough at $t = 50$ with and without anticipation of firm 2

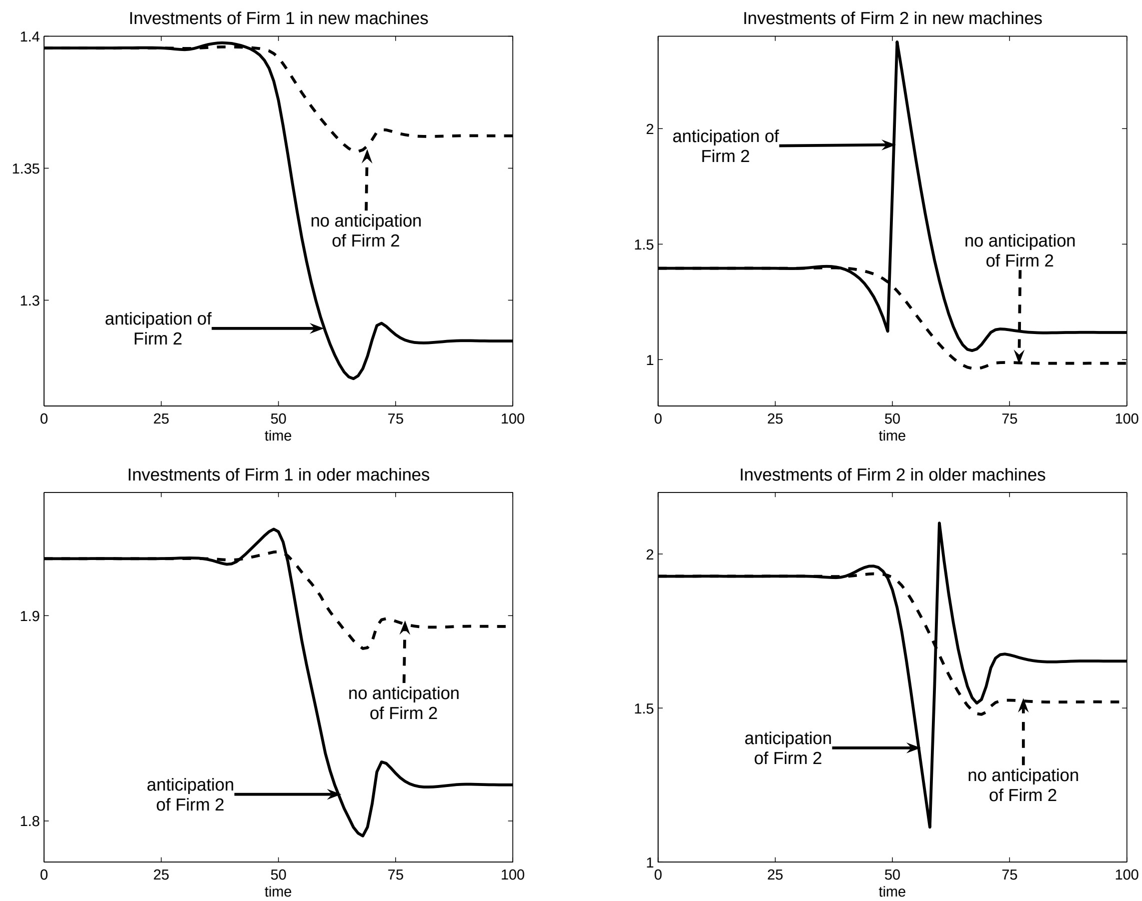

Figure 5: Technological breakthrough for firm 2 at $t = 50$ with and without anticipation of firm 2

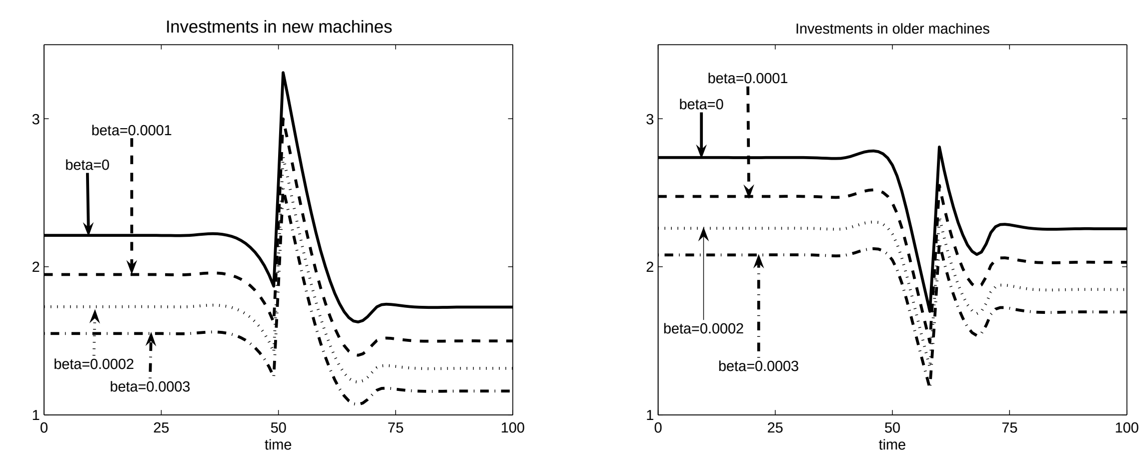

Figure 6: Investments depending on β: benchmark

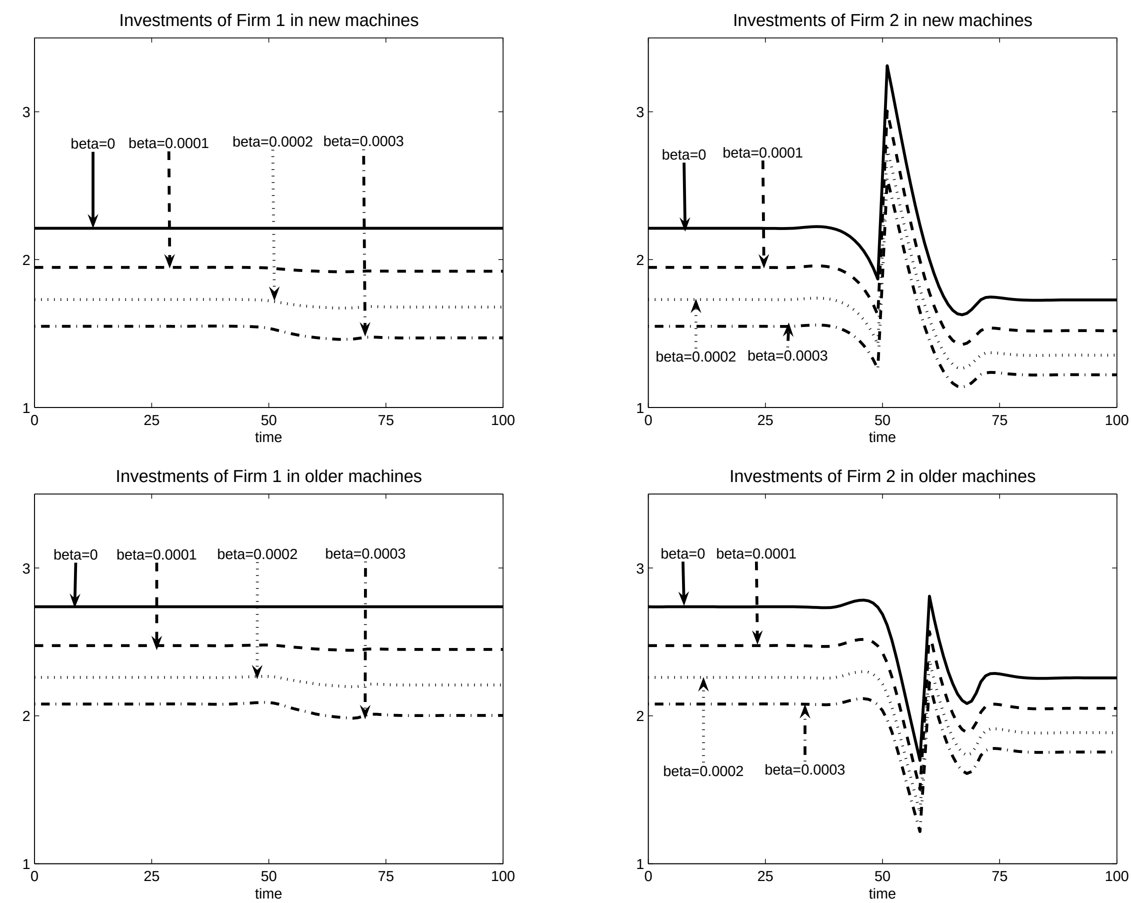

Figure 7: Investments depending on β: breakthrough only for firm 2, both anticipate

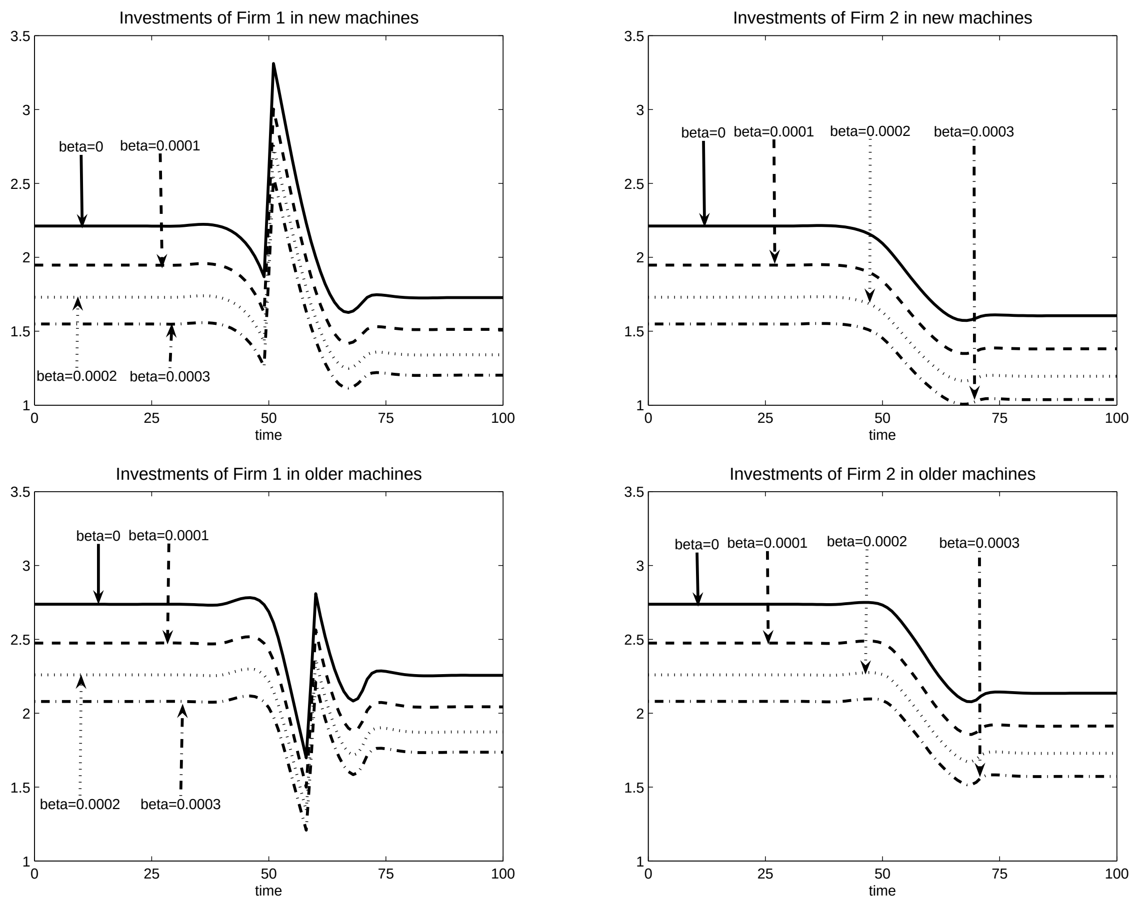

Figure 8: Investments depending on β: breakthrough for both firms, anticipation only by firm 1