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The Euler Method for Linear Control Systems Revisited*

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Abstract. Although optimal control problems for linear systems have been profoundly investigated in the past more than 50 years, the issue of numerical approximations and precise error analyses remains challenging due the bang-bang structure of the optimal controls. Based on a recent paper by M. Quincampoix and V.M. Veliov on metric regularity of the optimality conditions for control problems of linear systems the paper presents new error estimates for the Euler discretization scheme applied to such problems. It turns out that the accuracy of the Euler method depends on the "controllability index" associated with the optimal solution, and a sharp error estimate is given in terms of this index. The result extends and strengthens in several directions some recently published ones.

1 Introduction

In this paper we revisit the Euler discretization method applied to the following optimal control problem:

$$\min g(x(T)) \tag{1}$$

subject to

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t), \quad x(0) = x_0, \tag{2}$$

$$u(t) \in U. \tag{3}$$

Here $x \in \mathbf{R}^n$, $u \in U \subset \mathbf{R}^r$, the time interval $[0, T]$ is fixed, $g : \mathbf{R}^n \to \mathbf{R}$, and A and B are matrix functions with appropriate dimensions. The initial state x_0 is given. The control constraining set $U \subset \mathbf{R}^r$ is a convex compact polyhedron. The set of admissible controls, $\mathcal{U}$, consists of all measurable selections of U. The function $x = x[u] : [0, T] \to \mathbf{R}^n$ is a solution of (2) for a given $u \in \mathcal{U}$ if it is absolutely continuous and satisfies (2) for a.e. $t \in [0, T]$.

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Utilization of the Euler discretization scheme for this problem results in the following discrete-time optimal control problem:

$$\min g(x_N) \tag{4}$$

subject to

$$x_{i+1} = x_i + h\,(A(t_i)\,x_i + B(t_i)\,u_i)\,, \quad x_0 \text{ – given}, \tag{5}$$
$$u_i \in U, \quad i = 0, \ldots, N-1, \tag{6}$$

where N is a natural number, $h = T/N$, $t_i = ih$. The unknown variables here are $x_1, \ldots, x_N$ and $u_0, \ldots, u_{N-1}$.

The error analysis of the above discretization is burdened by the fact that the optimal control in problem (1)–(3) is typically discontinuous.

In the past few years a number of papers appeared that investigate the accuracy of discrete approximations of optimal control problems with a bang-bang structure of the optimal control. The first one seems to be [10], followed by [3, 1, 2]. The main result in the present paper shows that the accuracy of the approximation (measured in the relevant metric defined in the next section) provided by the Euler scheme is $O(h^{1/k})$, where k is the so-called *controllability index* of the optimal solution of problem (1)–(3). A comparison of this result with the abovementioned ones is given in the end of Section 3.

We mention also that the error analyses of discrete approximations to control problems for linear systems is facilitated by the recent papers [4–7]. However, our analyses is based on the "companion" paper [9], which extends in an appropriate way the concept of metric regularity of the optimality conditions for optimal control of linear systems.

The organization of the paper is as follows. In the next section we present some material from [9], which is needed for the proof of our main result. The order of accuracy of the Euler scheme is proved in Section 3. In the last section we give a numerical example that supports the theoretical result.

2 Assumptions and Preliminaries

In this section we present some necessary preliminary material from the "companion" paper [9]. We begin with some assumptions for problem (1)–(3).

Assumption (A1): The functions $A : [0,T] \to \mathbf{R}^{n\times n}$ and $B : [0,T] \to \mathbf{R}^{n\times r}$ are $\bar{k}$ times, respectively $\bar{k}+1$ times, continuously differentiable (for some natural number $\bar{k}$). Moreover, $g : \mathbf{R}^n \to \mathbf{R}$ is convex and differentiable with a locally Lipschitz derivative.

The reachable set $R = \{x[u](T) : u \in \mathcal{U}\}$ is a convex and compact subset of $\mathbf{R}^n$, hence problem (1)–(3) has at least one solution $(\hat{x}, \hat{u})$.

Define the sequence of matrices

$$B_0(t) = B(t), \quad B_{i+1}(t) = -A(t)B_i(t) + \dot{B}_i(t), \quad i = 0, \ldots, \bar{k}-1. \tag{7}$$

Moreover, denote by E the set of all (non-degenerate) edges of U, and by $\bar{E}$ – the set of all vectors $u_2 - u_1$, where $[u_1, u_2] \in E$.

Assumption (A2): $\mathrm{rank}[B_0(t)\, e, \ldots, B_{\bar{k}}(t)\, e] = n$ for every $e \in \bar{E}$ and every $t \in [0, T]$. Moreover, $\nabla g(x) \neq 0$ for every $x \in R$ (∇g denotes the gradient of g).

The rank condition in the above assumption is the well-known *general position hypotheses* [8]. The second part of the assumption makes the problem meaningful, since it rules out the possibility of infinitely many solutions.

The Pontryagin maximum principle claims that any optimal pair $(\hat{x}, \hat{u})$ together with a corresponding absolutely continuous function $\hat{p} : [0, T] \to \mathbf{R}^n$ satisfies the following (generalized) equations:

$$0 = \dot{x}(t) - A(t)\, x(t) - B(t)\, u(t), \quad x(0) = x_0, \tag{8}$$

$$0 = \dot{p}(t) + A^\top(t)\, p(t), \tag{9}$$

$$0 \in B^\top(t)\, p(t) + N_U(u(t)), \tag{10}$$

$$0 = p(T) - \nabla g(x(T)), \tag{11}$$

where $N_U(u)$ is the normal cone to U at u. Notice that (10) is equivalent to $u(t) \in \operatorname*{Argmin}_{w \in U} \langle B^\top(t)\, p(t), w \rangle$.

The following lemma is well-known.

Lemma 1. *Let the matrices A and B be measurable and essentially bounded, and let g be differentiable and convex. Then $(\hat{x}, \hat{u})$ is a solution of problem (1)–(3) if and only if the triple $(\hat{x}, \hat{p}, \hat{u})$ (with an absolutely continuous $\hat{p}$) is a solution of system (8)–(11). If (A1) and (A2) hold, then the solution $(\hat{x}, \hat{u})$ of (1)–(3) is unique, hence that of (8)–(11) is also unique. Moreover, $\hat{u}(t)$ is a vertex of U for a.e. $t \in [0, T]$.*

Let $(\hat{x}, \hat{u})$ be a solution of problem (1)–(3).

Definition 1 *Controllability index* of the solution $(\hat{x}, \hat{u})$ of problem (1)–(3) is the minimal number k such that for every $t \in [0, T]$ and for every $e \in \bar{E}$ at least one of the numbers $\langle B_i^\top(t)\, \hat{p}(t), e \rangle$, $i = 0, \ldots, k$, is not equal to zero. Here $\hat{p}$ is the solution of the equations (9), (11) with $x = \hat{x}$ and $u = \hat{u}$.

Clearly, if (A2) is fulfilled, then the number $k \leq \bar{k}$ exists.

The generalized equations (8)–(11) can be written in the form $0 \in F(x, p, u)$, where

$$F(x, p, u) := \begin{pmatrix} \dot{x} - A\, x - B\, u \\ \dot{p} + A^\top p \\ B^\top p + N_U(u) \\ p(T) - \nabla g(x(T)) \end{pmatrix}. \tag{12}$$

Thus the inclusion $0 \in F(x,p,u)$ is equivalent to our original problem (1)–(3). Namely, under (A1) and (A2) it has a unique solution $(\hat{x},\hat{p},\hat{u})$ and $(\hat{x},\hat{u})$ is the unique solution of problem (1)–(3).

The norms in $L^1(0,T)$ and $L^\infty(0,T)$ are denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. The notation $W^{1,s} = W^{1,s}([0,T];\mathbf{R}^n)$ (with $s=1$ or $s=\infty$) is used for the space of all absolutely continuous functions $x:[0,T]\to\mathbf{R}^n$ with the derivative $\dot{x}$ belonging to $L^s(0,T)$. The norm in this space is $\|x\|_{1,s} := \|x\|_\infty + \|\dot{x}\|_s$.

The set of admissible controls $\mathcal{U}$ is viewed as a subset of $L^\infty(0,T)$ equipped with the metric

$$d^\#(u_1,u_2) = \text{meas}\,\{t\in[0,T]:\, u_1(t)\neq u_2(t)\}.$$

This metric is shift-invariant and we shall shorten $d^\#(u_1,u_2) = d^\#(u_1-u_2,0) =: d^\#(u_1-u_2)$. Then the triple (x,p,u) is considered as an element of the (affine) space

$$\mathcal{X} = W^{1,1}_{x_0} \times W^{1,\infty} \times \mathcal{U},$$

where $W^{1,1}_{x_0} = \{x\in W^{1,1}:\, x(0)=x_0\}$.

The image space of F will be $\mathcal{Y} = L^1\times L^\infty\times L^\infty\times\mathbf{R}^n$ with the norm

$$\|y\| = \|(\xi,\pi,\rho,\nu)\| := \|\xi\|_1 + \|\pi\|_\infty + \|\rho\|_\infty + |\nu|.$$

We interpret the set $N_U(u)$ in (12) as $\{\rho\in L^\infty:\, \rho(t)\in N_U(u(t))\ \forall\, t\in[0,T]\}$ (strictly speaking, we should use the notation $N_{\mathcal{U}}(u)$ instead of the point-wise $N_U(u(t))$, but the overload of the latter does not lead to confusions).

The following is a simplified version of [9, Theorem 2].

Theorem 1 *Let assumptions (A1) and (A2) be fulfilled, let $(\hat{x},\hat{p},\hat{u})$ be a solution of the generalized equation $0\in F(x,p,u)$ (with F given in (12)) and let k be its controllability index. Then for every number $b>0$ there exists a number c such that for every $y=(\xi,\pi,\rho,\nu)\in\mathcal{Y}$ with $\|y\|\le b$ and for every solution $(x,p,u)\in\mathcal{X}$ of the inclusion $y\in F(x,p,u)$ it holds that*

$$\|x-\hat{x}\|_{1,1} + \|p-\hat{p}\|_{1,\infty} + \|u-\hat{u}\|_1 \;\le\; c\,\|y\|^{\frac{1}{k}}. \tag{13}$$

3 Euler discretization and its accuracy

Consider the discrete-time problem (4)–(6) as introduced in Section 1. The maximum principle for discrete-time optimal control problems claims that if $x^N=(x_0,\ldots,x_N)$, $u^N=(u_0,\ldots,u_{N-1})$ is a solution of problem (4)–(6) then

$$-B(t_i)^\top p_{i+1} \in N_U(u_i), \quad i=0,\ldots,N-1, \tag{14}$$

with $p^N=(p_0,\ldots,p_N)$ determined from the equations

$$p_N = \nabla g(x_N), \tag{15}$$

$$p_i = p_{i+1} + hA(t_i)^\top p_{i+1}, \quad i=N-1,\ldots,0. \tag{16}$$

We identify any sequence $u^N := (u_0, \dots, u_{N-1})$ with its piece-wise constant extension: $u^N(t) = u_i$ for $t \in [t_i, t_{i+1})$, $i = 0, \dots, N-1$. Moreover, we identify any sequence $x^N := (x_0, \dots, x_N)$ with its piecewise linear interpolation:

$$x^N(t) = x_i + \frac{t - t_i}{h}(x_{i+1} - x_i), \quad t \in [t_i, t_{i+1}), \quad i = 0, \dots, N-1.$$

Similarly for sequences $p^N := (p_0, \dots, p_N)$. Then we can view such sequences (x^N, p^N, u^N) as elements of the space $\mathcal{X}$.

The main result in this paper follows.

Theorem 2 *Let assumptions (A1), (A2) be fulfilled and let $(\hat{x}, \hat{u})$ be the unique solution of problem (1)–(3). Let k be the controllability index of this solution. Then there exists a number C such that for any natural number N and corresponding $h = T/N$, and for any solution (x^N, u^N) of the discretized problem (4)–(6) and for the corresponding adjoint functions $\hat{p}$ and p^N (given by equations (9), (11) and (15), (16), respectively) it holds that*

$$\|x^N - \hat{x}\|_{1,1} + \|p^N - \hat{p}\|_{1,\infty} + \|u^N - \hat{u}\|_1 \le C\, h^{1/k}.$$

Moreover, if all u_i, $i = 0, \dots, N-1$, are vertices of U, then

$$\|x^N - \hat{x}\|_{1,1} + \|p^N - \hat{p}\|_{1,\infty} + d^{\#}(u^N - \hat{u}) \le C\, h^{1/k}.$$

Proof. The proof is simple due to Theorem 1. Essentially, we only have to estimate the residual $y = (\xi, \pi, \rho, \nu)$ of (x^N, p^N, u^N) in the inclusion $0 \in F(x, p, u)$. That is, we have to ensure that there exists $y = (\xi, \pi, \rho, \nu) \in \mathcal{Y}$ such that $y \in F(x^N, p^N, u^N)$, and to estimate its norm.

First we mention that due to the boundedness of U there exists a number M (independent of N) such that any of the numbers $|u|$, $|\hat{x}(t)|$, $|\hat{p}(t)|$, $|x_i|$, $|p_i|$, where $u \in U$, $t \in [0, T]$, $i = 0, \dots, N$, is smaller than M. Also, let K be such that $|A(t)| \le K$ and $|B(t)| \le K$ for $t \in [0, T]$, where we use the operator norms of matrices. Moreover, we denote by L a Lipschitz constant of A and B.

We define $\xi(t) = \dot{x}^N(t) - A(t)x^N(t) - B(t)u^N(t)$. Clearly $\xi \in L^1$ and for $t \in [t_i, t_{i+1})$ we have

$$\begin{aligned}
|\xi(t)| &= \left| \frac{x_{i+1} - x_i}{h} - A(t)\left(x_i + \frac{t - t_i}{h}(x_{i+1} - x_i)\right) - B(t)u_i \right| \\
&= \left| A(t_i)x_i + B(t_i)u_i - A(t)\left(x_i + \frac{t - t_i}{h}(x_{i+1} - x_i)\right) - B(t)u_i \right| \\
&= \left| (A(t_i) - A(t))x_i + (B(t_i) - B(t))u_i - A(t)\left(\frac{t - t_i}{h}(x_{i+1} - x_i)\right) \right| \\
&\le 2hLM + |A(t)|\,|(x_{i+1} - x_i)| \le 2LMh + Kh|A(t_i)x_i + B(t_i)u_i| \\
&\le 2LMh + 2K^2Mh = 2(L + K^2)Mh.
\end{aligned}$$

Hence,

$$\|\xi\|_1 \le 2T(L + K^2)Mh.$$

Now we consider $\pi(t) := \dot{p}^N(t) + A(t)^\top p^N(t)$. Obviously $\pi \in L^\infty$ and using (16) we obtain that for $t \in [t_i, t_{i+1}]$

$$
\begin{aligned}
|\pi(t)| &= \left| \frac{p_{i+1} - p_i}{h} + A(t)^\top \left(p_i + \frac{t - t_i}{h}(p_{i+1} - p_i) \right) \right| \\
&= \left| -A(t_i)^\top p_{i+1} + A(t)^\top \left(p_i + \frac{t - t_i}{h}(p_{i+1} - p_i) \right) \right| \\
&\le \left| -A(t_i)^\top p_{i+1} + A(t)^\top p_i \right| + K|p_{i+1} - p_i| \\
&\le \left| -A(t_i)^\top p_{i+1} + A(t_i)^\top p_i \right| + \left| -A(t_i)^\top p_i + A(t)^\top p_i \right| + K|p_{i+1} - p_i| \\
&\le K|p_{i+1} - p_i| + LMh + K|p_{i+1} - p_i| \\
&\le (2K^2 + L)Mh.
\end{aligned}
$$

In order to estimate the residual in (10) we define the function $\rho(t) := B(t)^\top p^N(t) - B(t_i)^\top p^N(t_{i+1})$, $t \in [t_i, t_{i+1})$, which obviously belongs to L^∞. First of all, for $t \in [t_i, t_{i+1})$ we have from (14) that

$$
\rho(t) = \rho(t) + 0 \in \rho(t) + B(t_i)^\top p_{i+1} + N_U(u_i) = B(t)^\top p^N(t) + N_U(u^N(t)).
$$

We estimate for $t \in [t_i, t_{i+1}]$

$$
\begin{aligned}
|\rho(t)| &\le |(B(t)^\top - B(t_i)^\top) p^N(t)| + |B(t_i)|\,|p^N(t) - p^N_{i+1}| \\
&\le LMh + K|p^N_i - p^N_{i+1}| \le LMh + MK^2h = (L + K^2)Mh.
\end{aligned}
$$

The fourth residual is $\nu = 0$ since (11) is exactly satisfied by $p^N(T) = p_N$ and $x^N(T) = x_N$ due to (15).

Thus we have obtained so far that

$$
\|y\| \le 2T(L + K^2)Mh + (2K^2 + L)Mh + (L + K^2)Mh \le (2T + 3)(L + K^2)Mh.
$$

Now we apply Theorem 1 with $b = (2T + 3)(L + K^2)MT$ and with the corresponding number c from the formulation of this theorem. It claims that

$$
\begin{aligned}
\|x^N - \hat{x}\|_{1,1} + \|p^N - \hat{p}\|_{1,\infty} + \|u^N - \hat{u}\|_1 \;&\le\; c\,\|y\|^{\frac{1}{k}} \le c((2T + 3)(L + K^2)Mh)^{1/k} \\
&=: c_1\, h^{1/k}.
\end{aligned}
$$

This proves the first claim of the theorem with $C = c_1$.

To prove the second claim of the theorem we assume that u_i, $i = 0, \dots, N-1$, are vertices of U. We remind that according to Lemma 1 the values of $\hat{u}$ are also a.e. vertices of U. Then $|u^N(t) - \hat{u}(t)| \ge \eta$ whenever $u^N(t) \ne \hat{u}(t)$, where $\eta > 0$ is the minimal distance between different vertices of U. Then

$$
\eta\, d^{\#}(u^N - \hat{u}) \le \int_0^T |u^N(t) - \hat{u}(t)|\,\mathrm{d}t \le c_1 h^{1/k}.
$$

This proves the second claim of the theorem with $C := c_1/\eta$. Q.E.D.

In the rest of this section we compare the above result with those in [10, 3, 1, 2].

General Runge-Kutta schemes of (at least) second order global accuracy (third order local consistency) were applied in [10] instead of the Euler scheme. The accuracy of the approximation in the metric in $\mathcal{X}$ is proved to be $O(h^{1/k})$, where k is the controllability index of the optimal solution of problem (1)–(3). In the present paper we show that the same order $h^{1/k}$ is achieved by the Euler discretization (also the assumptions for g are relaxed).

In the recent paper [3] the authors consider linear problems with a linear function g and with controllability index $k = 1$. The result is similar to the one implied by Theorem 2 for the case $k = 1$. The control constraining set U is assumed to be a coordinate box in [3], which is a technical simplification. The linearity of g, however, is a substantial simplification, since the adjoint system (9), (11) is independent of the state x and can be treated by the Euler scheme separately from the rest of the equations in (8)–(11).

Papers [1, 2] make a substantial progress by considering a quadratic function g (also a quadratic in x integral term is present there). The $O(h)$ error estimate in this case (again with assumed $k = 1$) becomes nontrivial since the overall interconnected system (8)–(11) has to be investigated. However, its analysis is based on the structural stability of the switching structure of the optimal control (obtained in [4]). Such a stability is no longer valid if $k > 1$, which case is captured by Theorem 2. A different proof is needed in this case and in the present paper it is based on the results of [9].

4 A numerical test

The following test example is a slight modification of [3, Example 2.10]. The problem is

$$\begin{aligned}
&\min\ x_3(5)\\
&\dot{x}_1 = -x_2 + u, \quad x_1(0) = 1,\\
&\dot{x}_2 = u, \quad x_2(0) = 1,\\
&\dot{x}_3 = x_1 - 0.5\,u, \quad x_3(0) = 0,\\
&u \in [0, 1].
\end{aligned}$$

The only difference with [3, Example 2.10] is the coefficient -0.5 in the last equation, which equals 4 in the quoted paper. Here assumptions (A1) and (A2) are fulfilled with $\bar{k} = 2$. Therefore, the controllability index of the unique optimal pair (x, u) is $k \le 2$. In fact, for this example that $k = 2$. The numerical results presented on the table below show that the accuracy of the Euler approximation is $O(h^{1/2})$, indeed.

The optimal control in this problem is easily seen (by applying the maximum principle) to be $\hat{u}(t) \equiv 1$. However, the corresponding "switching function" $\sigma(t) = B(t)^\top p(t) = 5 - t - 0.5(5-t)^2 - 0.5$ has a double zero at $t = 4$. Thus $k = 2$ for this problem and the theoretical error estimate is $Err(h) := d^{\#}(u^N - \hat{u}) \le C\sqrt{h}$. In the table below we show the quantities $Err(h)$, $Err(2h)/Err(h)$, which is expected to be about $C\sqrt{2h}/C\sqrt{h} = \sqrt{2}$, and $Err(h)/\sqrt{h}$, which expected to be about the constant C.

h	$Err(h) := d^{\#}(u^N - \hat{u})$	$Err(2h)/Err(h)$	$C = Err(h)/\sqrt{h}$
0.01	0.28000000	1.4000	2.8000
0.01/2	0.20000000	1.4286	2.8284
$0.01/2^2$	0.14000000	1.4000	2.8000
$0.01/2^3$	0.10000000	1.4286	2.8284
$0.01/2^4$	0.07000000	1.4000	2.8000
$0.01/2^5$	0.05000000	1.4035	2.8284
$0.01/2^6$	0.03562500	1.4250	2.8500
$0.01/2^7$	0.02500000	1.4159	2.8284
$0.01/2^8$	0.01765625	1.4125	2.8250
$0.01/2^9$	0.01250000	$\sqrt{2} \approx 1.4142$	2.8284

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