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Optimality conditions and the Hamiltonian for a distributed optimal control problem on controlled domain *

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Abstract. The paper investigates an optimal control problem for a distributed system arising in the economics of endogenous growth. The problem involves a specific coupled family of controlled ODEs parameterized by a parameter (representing the heterogeneity) running over a domain that may dynamically depend on the control and on the state of the system. Existence of an optimal control is obtained and continuity of any optimal control with respect to the parameter of heterogeneity is proved. The latter allows to substantially strengthen previously obtained necessary optimality conditions and to obtain a Pontryagin's type maximum principle. The necessary optimality conditions obtained here have a Hamiltonian representation, and stationarity of the Hamiltonian along any optimal trajectory is proved in the case of time-independent data.

Keywords: optimal control, distributed control, controlled domain, Pontryagin-type maximum principle, endogenous economic growth

AMS Subject Classification: 49K20, 49K21, 49J20

1 Introduction

In modeling endogenous economic growth we encounter the following situation: at any given time t there exists a variety of technologies (or products, intermediate goods, etc.) represented by an interval $[0, Q(t)]$ of real numbers, so that every $q \in [0, Q(t)]$ labels some technology that exists at time t, and the larger is q, the newer is the technology. The technological frontier, $Q(t)$, has its own dynamics where investments in research and development appear as a control input (various ODE models of this dynamics are available in the literature, see e.g. [7]). In addition, there is a physical capital stock, $x(t, q)$, of technology $q \in [0, Q(t)]$. For every technology q the capital $x(t, q)$ obeys a linear ODE involving depreciation and investment (also a control variable) in this technology. That is, we have a family of ODEs parameterized by q. It is important to notice that this family of ODEs (viewed as a distributed control system) is meaningful only in the domain $\{(t, q) : \ t \geq 0, \ q \in [0, Q(t)]\}$ because investments in non-existing technologies are not possible. Since $Q(t)$ depends on the R&D control, we encounter a distributed control system on a controlled domain. Various optimization problems can

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be associated with this system, and we choose one that is general enough to cover many interesting economic models that are a subject of current investigations.

In order to make a link and a comparison with the existing economic literature we refer to [5, 6, 9], mentioning that although a variety of goods is present in these papers, true heterogeneity is not involved, since the products are viewed as identical, in contrast to the present paper. Models with truly heterogeneous goods are investigated in [8, 3], but in a rather different context, where the dynamics of the capital stock is ignored.

The investigation of optimization problems on a controlled domain is burdened by several circumstances: the possible non-differentiability of the function "control $\longrightarrow$ objective value" (shown in [4]); the non-concavity of this function—in a maximization problem!—shown in Section 3 below; technical complications due to the dependence of the spatial domain on the control.

Optimality conditions for a more general optimal control problem than the one considered in this paper are obtained in [4]. However, these conditions are shown in [4] to be efficient for analysis and numerical approaches only under an a priory continuity-type *regularity condition* for the optimal control. The verification of this regularity condition turns out to be uneasy even in simple special cases, since it cannot rely on the efficient optimality conditions requiring the regularity themselves.

One of the main aims of the present paper is to prove that the optimal control in the more specific problem considered here satisfies the regularity condition discussed above. As a consequence of this and the results in [4] we obtain optimality conditions of Pontryagin's type, which are good enough for analytic and numerical investigation (as described in [4]). In this sense the present paper is a substantial complement and enhancement of [4].

The obtained optimality conditions are reformulated in an appropriate Hamiltonian form, which turned out not to be a trivial task for the considered problem, since the distributed dynamics has to be appropriately extended outside the intrinsically relevant spatial domain. Under some additional conditions the Hamiltonian reformulation allows to prove stationarity of the Hamiltonian along the optimal solutions in case of stationary data.

An additional contribution of the paper is the proof of existence of an optimal solution of the considered maximization problem, which is not straightforward due to the non-convexity of the problem.

The plan of the paper is as follows. In Section 2 the problem and the assumptions on the data are stated in detail. In Section 3 existence of optimal control is proved and an example showing the non-convexity of the considered optimal control problem is given. The core of the present paper is Section 4 where continuity of any distributed optimal control with respect to the parameter of heterogeneity (i.e. its *regularity*) is proved. In section 5 we derive Pontryagin's type necessary optimality conditions for the optimal control problem we consider in this paper and prove constancy of the Hamiltonian along any optimal trajectory for the case of time-independent data.

2 Formulation of the problem and preliminaries

In this section we formulate a problem which is a particular case of the general one considered in [4]. Let $[0, T]$ be a fixed time-interval and let $[0, \bar{Q}]$ be an interval where the parameter of heterogeneity, q, will take values ($T > 0$ and $\bar{Q} > 0$ are given). Denote $D = [0, T] \times [0, \bar{Q}]$. State variables in the model below are the functions

$$x : D \mapsto \mathbf{R}^1, \quad Q : [0, T] \mapsto [0, \bar{Q}], \quad y : [0, T] \mapsto \mathbf{R}^1,$$

while $u : D \mapsto [\underline{u}, \overline{u}] \subset \mathbf{R}^1$ and $v : [0,T] \mapsto [\underline{v}, \overline{v}] \subset \mathbf{R}^1$ are control functions, $\underline{u} \geq 0$, $\overline{u} > 0$, $\underline{v} \geq 0$ and $\overline{v} > 0$ being given. For a given $Q : [0,T] \mapsto [0, \bar{Q}]$ we denote $D_Q := \{(t,q) \in D : t \in [0,T] \ \text{ and } \ q \in [0, Q(t)]\}$.

The optimal control problem we consider reads as follows:

$$\max_{(u(\cdot,\cdot),v(\cdot))} \int_0^T \left[\int_0^{Q(t)} \Big(L(t,q,Q(t),x(t,q),y(t)) - c_1(t,u(t,q)) \Big) \,\mathrm{d}q - c_2(t,v(t)) \right] \mathrm{d}t, \tag{1}$$

subject to the equations

$$\begin{aligned}
\dot{Q}(t) &= g_1(t,Q(t))y(t) + g_2(t,Q(t))v(t) \quad \text{for a.e. } t \in [0,T], \ \ Q(0) = Q^0, && (2)\\
y(t) &= \int_0^{Q(t)} d(t,q)u(t,q)\,\mathrm{d}q \quad \text{for a.e. } t \in [0,T], && (3)\\
\dot{x}(t,q) &= -\delta x(t,q) + u(t,q) \quad \text{for a.e. } (t,q) \in D_Q, && (4)\\
& x(0,q) = x^0(q) \ \text{ for a.e. } q \in [0,Q^0], && (5)\\
& x(t,Q(t)) = x^b(t) \ \text{ for } \ t \in (0,T], && (6)\\
u(t,q) &\in [\underline{u},\overline{u}] \ \text{ for a.e. } (t,q) \in D, \quad v(t) \in [\underline{v},\overline{v}] \ \text{ for a.e. } t \in [0,T]. && (7)
\end{aligned}$$

Everywhere, an upper "dot" means differentiation with respect to t, so that $\dot{x}(t,q) = \partial x(t,q)/\partial t$. The exact meaning of a solution of system (2)–(6) is given below.

From an economic point of view $Q(t)$ is the technological frontier at time t, which changes in accordance with (2) (where g_1 and g_2 are given functions), $x(t,q)$ is the amount of physical capital (or, alternatively, a quality measure) of technology q, $v(t)$ is a direct investment in development of new technologies, $u(t,q)$ is the investment in the technology $q \in [0,Q(t)]$, while $y(t)$ is the indirect effect of the capital investments on the development of new technologies. The objective functional represents the aggregated net revenue (L, c_1 and c_2 are given functions), where the dependence on t allows for discounting. The depreciation $\delta > 0$ and the function d are given, $Q^0 \in (0,\bar{Q})$ and $x^0(q)$, $q \in [0,Q^0]$ are given initial data, $x^b(t)$ is a boundary condition, which represents the amount of capital stock (or quality level) of technology q at the time this technology is developed, that is, when $Q(t) = q$.

The set of admissible controls is $\mathcal{U} \times \mathcal{V}$, where $\mathcal{U} = \{u \in L_\infty(D) : u(t,q) \in [\underline{u},\overline{u}] \ \text{ for a.e. } (t,q) \in D\}$ and $\mathcal{V} = \{v \in L_\infty([0,T]) : v(t) \in [\underline{v},\overline{v}] \ \text{ for a.e. } t \in [0,T]\}$.

We study problem (1)–(7) under the following *Standing Assumptions* (i) – (vii) (cf. [4]).

Standing Assumptions (i) – (v):

(i) The functions L, g_1, g_2, c_1 and c_2 are measurable in (t,q), locally essentially bounded, differentiable in (Q,x,y), with locally Lipschitz partial derivatives, uniformly with respect to $(t,q) \in D$. The function L is concave in (x,y), the function c_1 is strongly convex and twice continuously differentiable in u on an open set containing $[\underline{u},\overline{u}]$, uniformly in $t \in [0,T]$, the function c_2 is convex and differentiable in v on an open set containing $[\underline{v},\overline{v}]$, with locally Lipschitz derivative, uniformly in $t \in [0,T]$, the function d is measurable on D;

(ii) there exist $\underline{g}_i \geq 0$ and $\overline{g}_i > 0$, $i = 1,2$, such that $\underline{g}_i \leq g_i(t,Q) \leq \overline{g}_i$ for a.e. $t \in [0,T]$, for $Q \in [Q^0, \bar{Q}]$ and for $i = 1,2$;

(iii) there exist $\underline{d} > 0$ and $\overline{d} > 0$ such that $\underline{d} \leq d(t,q) \leq \overline{d}$ for a.e. $(t,q) \in D$;

(iv) $\underline{g}_1\ \underline{u} + \underline{g}_2\underline{v} > 0$;

(v) $x^b : [0,T] \mapsto \mathbf{R}^1$ is continuously differentiable, $x^0 : [0,Q^0] \mapsto \mathbf{R}^1$ is continuous, $x^0(Q^0) = x^b(0)$.

Since for any given $u \in \mathcal{U}$ and $v \in \mathcal{V}$ one can represent

$$\dot{Q}(t) = g_1(t, Q(t)) \int_0^{Q(t)} d(t,q)u(t,q)\,\mathrm{d}q + g_2(t, Q(t))v(t) \tag{8}$$

and the function in the right-hand side is locally Lipschitz in Q, equation (2) has locally an absolute continuous solution $Q = Q[u,v]$ and it is unique on its maximal interval of existence in $[0,T]$.

Standing Assumption (vi):
For every $(u,v) \in \mathcal{U} \times \mathcal{V}$ the solution $Q[u,v]$ exists on the whole interval $[0,T]$ and takes values in $[Q^0, \bar{Q}]$.

Given $u \in \mathcal{U}$ and $v \in \mathcal{V}$, we define for $q \in [0, \bar{Q}]$

$$\theta[u,v](q) := \begin{cases} 0 & \text{if } q \in [0, Q^0], \\ Q[u,v]^{-1}(q) & \text{if } q \in (Q^0, Q[u,v](T)), \\ T & \text{if } q \in [Q[u,v](T), \bar{Q}]. \end{cases}$$

The definition is correct, since $Q[u,v]$ is invertible according to Standing Assumptions (ii)–(iv), and its image is $[Q^0, Q[u,v](T)]$. We extend the definition of $x^0(\cdot)$ to $[0,\bar{Q}]$ as $x^0(q) = x^b(0)$ for $q \in (Q^0, \bar{Q}]$ and extend the dynamics of the distributed space variable $x(\cdot,\cdot)$ to the whole region D by replacing (4)–(6) with

$$\begin{aligned} \dot{x}(t,q) &= \dot{x}^b(t) && \text{for a.e. } q \in [0,\bar{Q}], \text{ and a.e. } t \in [0, \theta[u,v](q)), && (9) \\ \dot{x}(t,q) &= -\delta x(t,q) + u(t,q) && \text{for a.e. } q \in [0,\bar{Q}], \text{ and a.e. } t \in [\theta[u,v](q), T], && (10) \\ & x(0,q) = x^0(q) && \text{for a.e. } q \in [0,\bar{Q}]. && (11) \end{aligned}$$

By solution of (9)–(11) we mean a function $x(\cdot,\cdot)$, measurable on D, such that for almost every $q \in [0,\bar{Q}]$ it holds that (11) is satisfied, $x(\cdot,q)$ is absolutely continuous on $[0,T]$ and satisfies (9) and (10) almost everywhere in $t \in [0,T]$; $x(t,\cdot) \in L_\infty([0,\bar{Q}])$ for every $t \in [0,T]$. We thus view (9)–(11) as a family of ODEs (one for each $q \in [0,\bar{Q}]$), where the functions $y = y[u,v]$ and $Q = Q[u,v]$ are already defined in (2), (3) as described above.

For each $q \in [0,\bar{Q}]$ we define

$$x^*[u,v](q) := \begin{cases} x^0(q) & \text{if } q \in [0, Q^0], \\ x^b(\theta[u,v](q)) & \text{if } q \in (Q^0, Q[u,v](T)), \\ x^b(T) & \text{if } q \in [Q[u,v](T), \bar{Q}]. \end{cases}$$

Then for given controls (u,v), the solution of (4)–(6) on the domain $D_{Q[u,v]}$ is

$$x[u,v](t,q) = e^{-\delta(t-\theta[u,v](q))}x^*[u,v](q) + \int_{\theta[u,v](q)}^t e^{\delta(s-t)}u(s,q)\,\mathrm{d}s. \tag{12}$$

In accordance with (9)–(11), we extend $x[u,v]$ as

$$x[u,v](t,q) := x^b(t) \quad \text{for } (t,q) \in [0,T] \times [Q[u,v](t), \bar{Q}]. \tag{13}$$

An example satisfying the Standing Assumptions (including Standing Assumption (vii) before Proposition 2 in Section 4 below) is given in Section 6 in [4], when we add the control constraint $0 < \underline{u} \le u(t,q) \le \bar{u} < \infty$. The strictly positive lower bound on the investment $u(\cdot,\cdot)$ can be justified by the fact, that governments often require minimal investments when granting licenses to private companies.

3 Existence of optimal control

In this section we prove existence of a solution of problem (1)–(7). We present the proof in detail, since the standard Lebesgue-Tonelly approach has to be appropriately adapted to the considered maximization problem. The reason is that the problem turns out to be non-concave, as shown in the end of the section.

Proposition 1 *The optimal control problem (1)–(7) has at least one solution.*

Proof. Denoting the cost functional (1) by $J[u,v]$, let $J^* := \sup\{J[u,v] \, : \, (u,v) \in \mathcal{U} \times \mathcal{V}\}$. Let $\{(u_k, v_k)\}$ be a maximizing sequence for (1)–(7). Since all control functions are equibounded, there exist (after extracting subsequences) functions $\hat{u} \in L_2(D)$ and $\hat{v} \in L_2([0,T])$ such that $\{u_k\}$ converges weakly in $L_2(D)$ to $\hat{u}$ and $\{v_k\}$ converges weakly in $L_2([0,T])$ to $\hat{v}$. Since $\mathcal{U}$ and $\mathcal{V}$ are weakly closed in $L_2(D)$ and $L_2([0,T])$ respectively, we have that $(\hat{u},\hat{v}) \in \mathcal{U} \times \mathcal{V}$. Further we denote $Q_k := Q[u_k, v_k]$, $y_k := y[u_k, v_k]$ and $x_k := x[u_k, v_k]$ for every $k \in \mathbf{N}$. By Standing Assumption (vi) $\{Q_k\}$ is equibounded on $[0,T]$. From here and from Standing Assumptions (ii) and (iii) it follows that $\{\dot{Q}_k\}$ is equibounded a.e. on $[0,T]$, meaning that $\{Q_k\}$ is equicontinuous. According to the Arzelà–Ascoli theorem there is $\hat{Q} \in C([0,T])$ such that (after extracting a subsequence) $\{Q_k\}$ converges to $\hat{Q}$ uniformly on $[0,T]$. Further, because of (7) and of *Standing Assumptions* (iii) and (vi), $\{y_k\}$ is equibounded a.e. on $[0,T]$, so there is $\hat{y} \in L_2([0,T])$ such that (after extracting a subsequence) $\{y_k\}$ converges to $\hat{y}$ weakly in $L_2([0,T])$. Hence, for each $t \in [0,T]$

$$\int_0^t y_k(\tau)\,\mathrm{d}\tau \longrightarrow \int_0^t \hat{y}(\tau)\,\mathrm{d}\tau \quad \text{as} \;\; k \to \infty.$$

Also

$$\int_0^t y_k(\tau)\,\mathrm{d}\tau = \int_0^t \int_0^{Q_k(\tau)} d(\tau,q) u_k(\tau,q)\,\mathrm{d}q\,\mathrm{d}\tau \longrightarrow \int_0^t \int_0^{\hat{Q}(\tau)} d(\tau,q)\hat{u}(\tau,q)\,\mathrm{d}q\,\mathrm{d}\tau \quad \text{as} \;\; k \to \infty$$

because of the equiboundedness of all integrands on D, of the uniform on $[0,T]$ convergence of $\{Q_k\}$ to $\hat{Q}$ and of the weak in $L_2(D)$ convergence of $\{u_k\}$ to $\hat{u}$. Differentiating in t the equality

$$\int_0^t \hat{y}(\tau)\,\mathrm{d}\tau = \int_0^t \int_0^{\hat{Q}(\tau)} d(\tau,q)\hat{u}(\tau,q)\,\mathrm{d}q\,\mathrm{d}\tau,$$

we obtain that

$$\hat{y}(t) = \int_0^{\hat{Q}(t)} d(t,q)\hat{u}(t,q)\,\mathrm{d}q \quad \text{for a.e.} \;\; t \in [0,T],$$

i.e. $\hat{y} = y[\hat{u}, \hat{v}]$.

Further on, from (8) we obtain for every $k \in \mathbf{N}$ and for every $t \in [0, T]$

$$Q_k(t) - Q^0 = \int_0^t \dot{Q}_k(\tau)\,\mathrm{d}\tau = \int_0^t \left(g_1(\tau, Q_k(\tau)) \int_0^{Q_k(\tau)} d(\tau, q) u_k(\tau, q)\,\mathrm{d}q + g_2(\tau, Q_k(\tau)) v_k(\tau) \right) \mathrm{d}\tau.$$

From the uniform continuity of $g_i(\tau, \cdot)$ on $[Q^0, \bar{Q}]$ (uniformly in $\tau \in [0, T]$), $i = 1, 2$, the uniform on $[0, T]$ convergence of $\{Q_k\}$ to $\hat{Q}$ and from the weak convergence of $\{u_k\}$ and $\{v_k\}$, we obtain that

$$Q_k(t) \longrightarrow Q^0 + \int_0^t \left(g_1(\tau, \hat{Q}(\tau)) \int_0^{\hat{Q}(\tau)} d(\tau, q) \hat{u}(\tau, q)\,\mathrm{d}q + g_2(\tau, \hat{Q}(\tau)) \hat{v}(\tau) \right) \mathrm{d}\tau \quad \text{as} \;\; k \to \infty$$

for every $t \in [0, T]$. Taking into account that $Q_k(t)$ converges to $\hat{Q}(t)$ for all $t \in [0, T]$ and differentiating in t, we see that $\hat{Q}(t)$ satisfies (2), i.e. $\hat{Q} = Q[\hat{u}, \hat{v}]$.

Next, denoting $\theta_k := \theta[u_k, v_k]$ and $x_k^* := x^*[u_k, v_k]$, we first note that θ_k converges uniformly on $[0, \bar{Q}]$ to $\hat{\theta} := \theta[\hat{u}, \hat{v}]$ and this together with the continuity of x^b on $[0, T]$ yields that x_k^* converges uniformly on $[0, \bar{Q}]$ to $\hat{x}^*$. It is clear from (12) and (13) that the sequence $\{x_k\}$ is equibounded on D, hence, there is $\hat{x} \in L_2(D)$ which is the weak in $L_2(D)$ limit of (a subsequence of) $\{x_k\}$. So, for each $t \in [0, T]$ and each $q \in [0, \bar{Q}]$ we have

$$\int_0^t \int_0^q x_k(\tau, \sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \longrightarrow \int_0^t \int_0^q \hat{x}(\tau, \sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \quad \text{as} \quad k \to \infty. \tag{14}$$

Taking $(t, q) \in [0, T] \times [0, \hat{Q}(t))$, we first obtain from (12) that

$$\int_0^q x_k(t, \sigma)\,\mathrm{d}\sigma \longrightarrow \int_0^q e^{-\delta(t - \hat{\theta}(\sigma))} \hat{x}^*(\sigma)\,\mathrm{d}\sigma + \int_0^q \int_{\hat{\theta}(\sigma)}^t e^{\delta(s-t)} \hat{u}(s, \sigma)\,\mathrm{d}s\,\mathrm{d}\sigma$$

as $k \to \infty$, because of the uniform on $[0, \bar{Q}]$ convergence of $\{\theta_k^*\}$ and $\{x_k^*\}$ and the weak in $L_2(D)$ convergence of $\{u_k\}$. Next, the Lebesgue bounded convergence theorem yields

$$\int_0^t \int_0^q x_k(\tau, \sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \longrightarrow \int_0^t \int_0^q e^{-\delta(\tau - \hat{\theta}(\sigma))} \hat{x}^*(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_0^t \int_0^q \int_{\hat{\theta}(\sigma)}^\tau e^{\delta(s-\tau)} \hat{u}(s, \sigma)\,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}\tau \tag{15}$$

as $k \to \infty$. Taking into account (14) and (15) and differentiating first in t, then in q, we obtain

$$\hat{x}(t, q) = e^{-\delta(t - \hat{\theta}(q))} \hat{x}^*(q) + \int_{\hat{\theta}(q)}^t e^{\delta(s-t)} \hat{u}(s, q)\,\mathrm{d}s$$

for $(t, q) \in [0, T] \times [0, \hat{Q}(t))$, i.e. $\hat{x} = x[\hat{u}, \hat{v}]$.

Next we define the linear mappings $u \mapsto (\bar{x}[u], \bar{y}[u])$ in the following way:

$$\begin{aligned} \bar{x}[u](t, q) &:= e^{-\delta(t - \hat{\theta}(q))} \hat{x}^*(q) + \int_{\hat{\theta}(q)}^t e^{\delta(s-t)} u(s, q)\,\mathrm{d}s, \\ \bar{y}[u](t) &:= \int_0^{\hat{Q}(t)} d(t, q) u(t, q)\,\mathrm{d}q. \end{aligned}$$

One easily obtains that $x_k(t,q) = \bar{x}[u_k](t,q) + \gamma_{x,k}(t,q)$ and $y_k(t) = \bar{y}[u_k](t) + \gamma_{y,k}(t)$ with $\gamma_{x,k}(t,q) \to 0$ uniformly on D and $\gamma_{y,k}(t) \to 0$ uniformly on $[0,T]$ as $k \to \infty$. The functional

$$\bar{J}[u,v] := \int_0^T \left[\int_0^{\hat{Q}(t)} \Big(L(t,q,\hat{Q}(t),\bar{x}[u](t,q),\bar{y}[u](t)) - c_1(t,u(t,q)) \Big) \,\mathrm{d}q - c_2(t,v(t)) \right] \mathrm{d}t$$

is strongly in $L_2(D) \times L_2([0,T])$ continuous and concave, hence, it is weakly upper semicontinuous. Also, one easily obtains that $J[u_k,v_k] = \bar{J}[u_k,v_k] + \gamma_{J,k}$ with $\gamma_{J,k} \to 0$ as $k \to \infty$. So, we have

$$J^* = \lim_{k\to\infty} J[u_k,v_k] = \lim_{k\to\infty} \bar{J}[u_k,v_k] \le \bar{J}[\hat{u},\hat{v}] = J[\hat{u},\hat{v}],$$

i.e. $(\hat{u},\hat{v})$ is a solution of the optimal control problem (1)–(7). Q.E.D.

Remark 1 We note that problem (1)–(7) is *not* concave, so the obtained existence result is not an entirely trivial issue. The non-concavity is shown in the following example in which a particular case of (1)–(7) is considered. We show that the cost functional $J(\cdot)$ considered over the one-dimensional line segment in $\mathcal{U}$ consisting of all constant functions in $\mathcal{U}$ (the control v is absent) is not concave as a function of one variable.

Example. Consider the problem

$$\max_u J(u) := \int_0^T \int_0^{Q(t)} \left(x(t,q) - \alpha u^2(t,q) \right) \,\mathrm{d}q \,\mathrm{d}t,$$

subject to the equations

$$\begin{aligned}
\dot{Q}(t) &= \beta y(t) && \text{for a.e. } t \in [0,T], \quad Q(0) = 1, \\
y(t) &= \int_0^{Q(t)} u(t,q)\,\mathrm{d}q && \text{for a.e. } t \in [0,T], \\
\dot{x}(t,q) &= u(t,q) && \text{for a.e. } (t,q) \in D_Q, \\
& \quad x(0,q) = 0 && \text{for a.e. } q \in [0,Q^0], \\
& \quad x(t,Q(t)) = 0 && \text{for } \quad t \in (0,T], \\
& u(t,q) \in [\underline{u},\overline{u}] \;\; \text{for a.e. } (t,q) \in D. &&
\end{aligned}$$

Here $\alpha > 0$ and $\beta > 0$ are given, $0 < \underline{u} < \overline{u}$, and it is assumed that $\bar{Q} > e^{\beta\overline{u}T}$.

Now, take $u(t,q) \equiv u \in [\underline{u},\overline{u}]$. Then $Q[u](t) = e^{\beta u t}$ for $t \in [0,T]$ and $\theta[u](q) = \frac{1}{\beta u}\ln(q)$ for $q \in [1,e^{\beta u T}]$. Hence,

$$x[u](t,q) = \begin{cases} tu & \text{if} \quad (t,q) \in [0,T] \times [0,1] \\ \left(t - \frac{1}{\beta u}\ln(q) \right) u & \text{if} \quad (t,q) \in [\frac{1}{\beta u}\ln(q), T] \times [1, e^{\beta u T}] \end{cases}$$

and

$$J(u) = -\frac{\alpha u}{\beta} e^{\beta u T} + \frac{1}{\beta^2 u}\left(e^{\beta u T} - 1 \right) + \frac{1}{\beta}(\alpha u - T).$$

From here we obtain that

$$J''(u) = \left(-2\alpha T - \alpha\beta T^2 u - \frac{2T}{\beta u^2} + \frac{T^2}{u}\right) e^{\beta u T} + \frac{2}{\beta^2 u^3}\left(e^{\beta u T} - 1\right).$$

The last term above is positive. Denoting

$$\Phi(T,u) = -2\alpha T - \alpha\beta T^2 u - \frac{2T}{\beta u^2} + \frac{T^2}{u},$$

we have

$$\Phi(T,u) = \left[\left(-2\alpha - \frac{2}{\beta u^2}\right) + \left(-\alpha\beta u + \frac{1}{u}\right) T\right] T.$$

Take now $u_0 \in (\underline{u}, \overline{u})$ such that $-\alpha\beta u_0 + \frac{1}{u_0} > 0$ (if necessary, we decrease the lower bound $\underline{u} > 0$). Now, for $u \in [\underline{u}, u_0]$ we have $-\alpha\beta u + \frac{1}{u} > -\alpha\beta u_0 + \frac{1}{u_0}$ and $2\alpha + \frac{2}{\beta \underline{u}^2} > 2\alpha + \frac{2}{\beta u^2}$. Hence, if T is large enough, $\Phi(T,\cdot) > 0$ holds true on $[\underline{u}, u_0]$, i.e $J''(\cdot) > 0$ on $[\underline{u}, u_0]$, i.e. $J(\cdot)$ is strongly convex, hence non-concave, on the set of constant controls $u \in [\underline{u}, u_0]$.

4 Regularity of the optimal control

As it was shown in [4, Theorem 2], Pontryagin's type optimality conditions for an optimal control $\hat{u}$ are valid if this control satisfies the "regularity condition" that $\hat{u}(t,\cdot)$ is for almost every t continuous from the left at $q = Q(t)$. In this section, it shall be proved that any optimal control in problem (1)–(7) is "regular". We need the following additional assumption.

Standing Assumption (vii):
The functions L, L_x and d are continuous with respect to q uniformly in the rest of the variables.

Proposition 2 *Let Standing Assumptions (i)–(vii) be fulfilled. Then any optimal control $\hat{u}$ of problem* (1)–(7) *is continuous in q for almost every $t \in [0,T]$.*

Before proving the proposition, an auxiliary problem and two auxiliary results will be considered. Let $(\hat{x}, \hat{y}, \hat{Q}, \hat{u}, \hat{v})$ be a solution of problem (1)–(7), the existence of which was proved in the previous section. Consider the new optimal control problem

$$\max_u J(u) = \int_0^T \int_0^{\hat{Q}(t)} [L(t,q,\hat{Q}(t),x(t,q),\hat{y}(t)) - c_1(t,u(t,q))]\,\mathrm{d}q\,\mathrm{d}t \tag{16}$$

subject to

$$\begin{aligned}
\dot{x}(t,q) &= -\delta x(t,q) + u(t,q), \quad \text{for a.e. } (t,q) \in D_{\hat{Q}} && (17)\\
x(0,q) &= x^0(q) \ \text{ for a.e. } q \in [0,Q^0], && (18)\\
x(t,\hat{Q}(t)) &= x^b(t) \ \text{ for } \ t \in [0,T], && (19)\\
\int_0^{\hat{Q}(t)} d(t,q)u(t,q)\,\mathrm{d}q &= \hat{y}(t) \ \text{ for a.e. } \ t \in [0,T], && (20)\\
\underline{u} \ \le \ u(t,q) &\le \ \overline{u} \ \text{ for a.e. } \ q \in [0,\hat{Q}(t)]. && (21)
\end{aligned}$$

This is a reduction of the original problem, where y, Q and v are fixed at their optimal values, $\hat{y}, \hat{Q}, \hat{v}$. Obviously $\hat{u}$ is an optimal control for the reduced problem.

We introduce the following notational convention: dependencies on values of the control or the state variables, fixed at the considered optimal trajectory are suppressed. For example, $L(t, q, x(t, q)) := L(t, q, \hat{Q}(t), x(t, q), \hat{y}(t))$, $\; L_x(t, q) := L_x(t, q, \hat{Q}(t), \hat{x}(t, q), \hat{y}(t))$, $g_1(t) := g_1(t, \hat{Q}(t))$, etc.

Consider the following (decoupled) family of ODEs parameterized by $q \in [0, \hat{Q}(T)]$:

$$\dot{\hat{\lambda}}(t, q) = \delta \hat{\lambda}(t, q) - L_x(t, q), \quad \hat{\lambda}(T, q) = 0, \quad t \in [\theta(q), T]. \tag{22}$$

Clearly, a unique solution $\lambda(t, q)$, which is absolutely continuous in t and measurable and uniformly bounded in q exists on $D_{\hat{Q}}$.

Lemma 1 *Let the Standing Assumptions (i)–(vii) be fulfilled. Let $\hat{u}$ be an optimal control of problem* (16)–(21) *and let $\hat{\lambda}$ be the solution to* (22). *Then for almost every $t \in [0, T]$ the function $\hat{u}(t, \cdot)$ maximizes*

$$\int_0^{\hat{Q}(t)} [\hat{\lambda}(t, q) u(q) - c_1(t, u(q))] \, \mathrm{d}q \tag{23}$$

over the set of $u \in L_\infty([0, \hat{Q}(t)])$ satisfying

$$\int_0^{\hat{Q}(t)} d(t, q) u(q) \, \mathrm{d}q = \hat{y}(t), \quad \underline{u} \le u(q) \le \overline{u} \quad \textit{for a.e.} \quad q \in [0, \hat{Q}(t)]. \tag{24}$$

Proof. Let $u(t, q)$ be any measurable function on D satisfying the constraints (20), (21). Denote by $\hat{x}(t, q)$ and $x(t, q)$ the trajectories corresponding to $\hat{u}$ and u, respectively. Further, define $\Delta u(t, q) := u(t, q) - \hat{u}(t, q)$, $\Delta x(t, q) := x(t, q) - \hat{x}(t, q)$ and $\Delta J := J(u) - J(\hat{u})$.

$$\begin{aligned}
\Delta J &= \int_0^T \int_0^{\hat{Q}(t)} [L(t, q, x) - L(t, q, \hat{x}) - c_1(t, u(t, q)) + c_1(t, \hat{u}(t, q))] \, \mathrm{d}q \, \mathrm{d}t, \\
&= \int_0^T \int_0^{\hat{Q}(t)} [\langle L_x(t, q, \hat{x} + s(t, q)\Delta x(t, q)), \Delta x(t, q) \rangle - c_1(t, u(t, q)) + c_1(t, \hat{u}(t, q))] \, \mathrm{d}q \, \mathrm{d}t, \\
&= \int_0^T \int_0^{\hat{Q}(t)} [\langle L_x(t, q, \hat{x}), \Delta x(t, q) \rangle - c_1(t, u(t, q)) + c_1(t, \hat{u}(t, q))] \, \mathrm{d}q \, \mathrm{d}t + e(\Delta u),
\end{aligned} \tag{25}$$

where

$$e(\Delta u) := \int_0^T \int_0^{\hat{Q}(t)} \langle L_x(t, q, \hat{x}(t, q) + s(t, q)\Delta x(t, q)) - L_x(t, q), \Delta x(t, q) \rangle \, \mathrm{d}q \, \mathrm{d}t.$$

The difference $\Delta x(t, q)$ satisfies in $D_{\hat{Q}}$ the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta x(t, q) = -\delta \Delta x(t, q) + \Delta u(t, q)$$

with initial conditions $\Delta x(0,q) = \Delta x(t,\hat{Q}(t)) = 0$. Therefore, for some constant M_1, $\|\Delta x\|_\infty \leq M_1 \,\mathrm{meas}\,(\{t \in [0,T] : u(t,\cdot) \neq \hat{u}(t,\cdot)\})$.Since L_x is Lipschitz in x, it follows that

$$|e(\Delta u)| \leq M_2 \,\mathrm{meas}\,(\{t \in [0,T] : u(t,\cdot) \neq \hat{u}(t,\cdot)\})^2 . \tag{26}$$

Using (22) for $\hat{\lambda}$, the following holds true:

$$\begin{aligned}
\int_0^T \int_0^{\hat{Q}(t)} \langle L_x(t,q,\hat{x}), \Delta x(t,q)\rangle \,\mathrm{d}q\,\mathrm{d}t &= \int_0^T \int_0^{\hat{Q}(t)} \langle \delta\hat{\lambda}(t,q) - \dot{\hat{\lambda}}(t,q), \Delta x(t,q)\rangle \,\mathrm{d}q\,\mathrm{d}t \\
&= \int_0^T \int_0^{\hat{Q}(t)} \langle \delta\hat{\lambda}(t,q), \Delta x(t,q)\rangle \,\mathrm{d}q\,\mathrm{d}t - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{\hat{Q}(t)} \langle \hat{\lambda}(t,q), \Delta x(t,q)\rangle \,\mathrm{d}q\,\mathrm{d}t \\
&\quad + \int_0^T \langle \hat{\lambda}(t,\hat{Q}(t)), \Delta x(t,\hat{Q}(t))\rangle \dot{\hat{Q}}(t)\,\mathrm{d}t + \int_0^T \int_0^{\hat{Q}(t)} \langle \hat{\lambda}(t,q), -\delta\Delta x(t,q) + \Delta u(t,q)\rangle \,\mathrm{d}q\,\mathrm{d}t \\
&= \int_0^T \int_0^{\hat{Q}(t)} \langle \hat{\lambda}(t,q), \Delta u(t,q)\rangle \,\mathrm{d}q\,\mathrm{d}t - \int_0^{\hat{Q}(T)} \langle \hat{\lambda}(T,q), \Delta x(T,q)\rangle \,\mathrm{d}q \\
&\quad + \int_0^{Q^0} \langle \hat{\lambda}(0,q), \Delta x(0,q)\rangle \,\mathrm{d}q \\
&= \int_0^T \int_0^{\hat{Q}(t)} \langle \hat{\lambda}(t,q), \Delta u(t,q)\rangle \,\mathrm{d}q\,\mathrm{d}t.
\end{aligned}$$

Inserting the last expression in (25), and using the optimality of $\hat{u}$ which implies $\Delta J \leq 0$, we obtain

$$\int_0^T \int_0^{\hat{Q}(t)} [\hat{\lambda}(t,q)\hat{u}(t,q) - c_1(t,\hat{u}(t,q))] \,\mathrm{d}q\,\mathrm{d}t \geq \int_0^T \int_0^{\hat{Q}(t)} [\hat{\lambda}(t,q)u(t,q) - c_1(t,u(t,q))] \,\mathrm{d}q\,\mathrm{d}t + e(\Delta u). \tag{27}$$

Assume that the assertion of the lemma is not true, i.e. there exist an $\varepsilon > 0$ and a subset $A \subset [0,T]$ with $\mathrm{meas}(A) > 0$ such that for every $t \in A$ there exists a $u_t(\cdot) \in L_\infty([0,\hat{Q}(t)])$ such that

$$\int_0^{\hat{Q}(t)} [\hat{\lambda}(t,q)u_t(q) - c_1(t,u_t(q))] \,\mathrm{d}q \geq \int_0^{\hat{Q}(t)} [\hat{\lambda}(t,q)\hat{u}(t,q) - c_1(t,\hat{u}(t,q))] \,\mathrm{d}q + \varepsilon. \tag{28}$$

Next we want to show that we can take u_t to be measurable in (t,q). Consider the set valued mapping $\Gamma : A \rightrightarrows L_1([0,\bar{Q}])$, defined as

$$\Gamma(t) := \{u(\cdot) \in X : (G_1(t,u(\cdot)), G_2(t,u(\cdot))) \in [0,\infty) \times \{0\}\},$$

where $X := \left\{u(\cdot) \in L_1([0,\bar{Q}]) : \underline{u} \leq u(q) \leq \bar{u}\right\}$, and

$$\begin{aligned}
G_1(t,u(\cdot)) &= \int_0^{\hat{Q}(t)} [\hat{\lambda}(t,q)u(q) - c_1(t,u(q)) - \hat{\lambda}(t,q)\hat{u}(t,q) + c_1(t,\hat{u}(t,q))] \,\mathrm{d}q - \varepsilon, \\
G_2(t,u(\cdot)) &= \int_0^{\hat{Q}(t)} d(t,q)u(q)\,\mathrm{d}q - \hat{y}(t).
\end{aligned}$$

The so defined function fulfills the assumptions of Theorem 8.2.9 (p. 315 of [2]), because X is a complete separable metric space, $(G_1(t,u(\cdot)), G_2(t,u(\cdot)))$ is Caratheodory, $\Gamma(t) \neq \emptyset \quad \forall t \in A$.

Therefore, a measurable selection $w(t) \in \Gamma(t)$ exists. Since w is a measurable function from A to $L_1([0,\bar{Q}])$, there exists an equivalent function $u(t,q)$ measurable in (t,q) (Lusin's theorem and Lemma 2.1 on page 25 in [11]).

Now choose m big enough such that $\frac{M_2}{m} < \varepsilon$ and $\frac{1}{m} < \mathrm{meas}(A)$, then choose a subset A_m of A with $\mathrm{meas}(A_m) = \frac{1}{m}$ and define

$$u_m(t,q) = \begin{cases} u(t,q) & \text{if} \quad t \in A_m \\ \hat{u}(t,q) & \text{if} \quad t \in [0,T]\backslash A_m \end{cases}$$

The so defined control u_m is admissible because $\hat{u}$ and u are measurable and fulfill the conditions (20)–(21). It differs only on a set of measure m^{-1} from the optimal control and therefore, using (26), $|e(\Delta u_m)| \le M_2 m^{-2}$.

Using (28) and the definition of A_m and u_m, it follows that

$$\begin{aligned}
&\int_{A_m}\int_0^{\hat{Q}(t)} [\hat{\lambda}(t,q)\hat{u}(t,q) - c_1(t,\hat{u}(t,q))]\,\mathrm{d}q\,\mathrm{d}t + \frac{\varepsilon}{m} \\
&\le \int_{A_m}\int_0^{\hat{Q}(t)} [\hat{\lambda}(t,q)u_m(t,q) - c_1(t,u_m(t,q))]\,\mathrm{d}q\,\mathrm{d}t \\
&\le \int_{A_m}\int_0^{\hat{Q}(t)} [\hat{\lambda}(t,q)\hat{u}(t,q) - c_1(t,\hat{u}(t,q))]\,\mathrm{d}q\,\mathrm{d}t - e(\Delta u_m) \\
&< \int_{A_m}\int_0^{\hat{Q}(t)} [\hat{\lambda}(t,q)\hat{u}(t,q) - c_1(t,\hat{u}(t,q))]\,\mathrm{d}q\,\mathrm{d}t + \frac{\varepsilon}{m},
\end{aligned}$$

where the second inequality comes from (27) and the last one holds true because of the choice of m. The obtained contradiction completes the proof. Q.E.D.

In what follows c_1' denotes differentiation with respect to u. Also, $(c_1'(t))^{-1}(\eta)$ denotes $\left(\frac{\partial c_1}{\partial u}(t,\cdot)\right)^{-1}(\eta)$.

Lemma 2 *Let the Standing Assumptions (i)–(vii) be fulfilled and let $\hat{\lambda}$ be the solution to* (22)*. The function*

$$U(t,q,\lambda,\xi) := \begin{cases} \underline{u} & \text{if} \quad \lambda + \xi\, d(t,q) < c_1'(t,\underline{u}) \\ (c_1'(t))^{-1}(\lambda + \xi\, d(t,q)) & \text{if} \quad c_1'(t,\underline{u}) \le \lambda + \xi\, d(t,q) \le c_1'(t,\overline{u}) \\ \overline{u} & \text{if} \quad \lambda + \xi\, d(t,q) > c_1'(t,\overline{u}) \end{cases} \tag{29}$$

is continuous in q, Lipschitz in λ, and there exists a measurable on $[0,T]$ function $\hat{\xi}(t)$, such that the optimal control $\hat{u}(t,q)$ of problem (1)–(7) *fulfills $\hat{u}(t,q) = U(t,q,\hat{\lambda}(t,q),\hat{\xi}(t))$.*

Proof. According to Lemma 1, for almost every $t \in [0,T]$ the function $\hat{u}(t,\cdot)$ maximizes (23) subject to (24). According to the Theorem in Section 4.2 (p. 218) in [1], there exist $\xi_0(t) \ge 0$ and $\xi^*(t) \in \mathbf{R}$ with $\xi_0^2(t) + (\xi^*(t))^2 > 0$, such that for a.e. $q \in [0,\hat{Q}(t)]$, $\hat{u}(t,q)$ solves the problem

$$\max_{w\in[\underline{u},\overline{u}]} \left\{ \xi_0(t)[\hat{\lambda}(t,q)w - c_1(t,w)] + \xi^*(t)\left[d(t,q)w - \frac{\hat{y}(t)}{\hat{Q}(t)}\right]\right\}. \tag{30}$$

It can be shown that $\xi_0(t)$ can be taken equal to 1.

Fix $(t,q) \in D_{\hat{Q}}$. For $\xi_0(t) = 1$, problem (30) is equivalent to minimization of $c_1(t,w) - [\xi^*(t)d(t,q)+\hat{\lambda}(t,q)]w$ over $w \in [\underline{u}, \overline{u}]$. Since $c_1(t,\cdot)$ is strongly convex, that is, $c_1''(t,w) \geq \bar{\epsilon} > 0$ for all w, the function $c_1'(t,\cdot)$ is invertible. Since $(c_1'(t))^{-1}(c_1'(t,w)) = w$ we have $[(c_1'(t))^{-1}(c_1'(t,w))]' = 1$ and therefore

$$[(c_1'(t))^{-1}]'(c_1'(t,w)) = \frac{1}{c_1''(t,w)} \leq \frac{1}{\bar{\epsilon}}. \tag{31}$$

The maximizer is thus obtained by the sign of

$$\hat{\lambda}(t,q) - c_1'(t,w) + \xi^*(t)d(t,q)$$

for $w \in [\underline{u}, \overline{u}]$. The optimal control is therefore $(c_1'(t))^{-1}(\hat{\lambda}(t,q) + \xi^*(t)d(t,q))$ projected on $[\underline{u}, \overline{u}]$. That is, the optimal control can be written in feedback form (29), $\hat{u}(t,q) = U(t,q,\hat{\lambda}(t,q),\xi^*(t))$. It is clear from (29) and (31), that $U(t,q,\lambda,\xi)$ is Lipschitz with respect to λ with constant $(\bar{\epsilon})^{-1}$. The continuity in q follows from that of $d(t,\cdot)$

The Lagrange multiplier ξ^* from above may not be measurable. We will now prove that there exists a measurable multiplier $\hat{\xi}(\cdot)$ such that $U(t,q,\hat{\lambda}(t,q),\hat{\xi}(t)) = U(t,q,\hat{\lambda}(t,q),\xi^*(t))$ on $D_{\hat{Q}}$, hence $\hat{u}(t,q) = U(t,q,\hat{\lambda}(t,q),\hat{\xi}(t))$. Consider the set valued mapping

$$G(t) := \left\{ \xi \in \mathbf{R} : \int_0^{\hat{Q}(t)} d(t,q)U(t,q,\hat{\lambda}(t,q),\xi)\,\mathrm{d}q - \hat{y}(t) = 0 \right\},\ t \in [0,T]. \tag{32}$$

Since $\hat{y}(\cdot)$ is measurable, from Theorem 8.2.9 in [2], it follows that $G(\cdot)$ is measurable, therefore, a measurable selections exists which we denote by $\hat{\xi}(\cdot)$.

Note that the mapping $\xi \to U(t,q,\hat{\lambda}(t,q),\xi)$ is monotone increasing. Let $\hat{\xi}(t) \geq (\leq)\ \xi^*(t)$ hold for some $t \in [0,T]$. Due to the monotonicity of U this implies the corresponding inequality $U(t,q,\hat{\lambda}(t,q),\hat{\xi}(t)) \geq (\leq)\ U(t,q,\hat{\lambda}(t,q),\xi^*(t))$ for $q \in [0,\hat{Q}(t)]$. Since the equality in (32) is satisfied with both $\xi = \hat{\xi}(t)$ and $\xi = \xi^*(t)$, and since $d(t,q) > 0$, we conclude that $U(t,q,\hat{\lambda}(t,q),\xi^*(t)) = U(t,q,\hat{\lambda}(t,q),\hat{\xi}(t))$ for a.e. $q \in [0,\hat{Q}(t)]$. This completes the proof.

Q.E.D.

Proof of Proposition 2. Let $\hat{\lambda}(t,q)$ be the solution of (22), and let $U(\cdot,\cdot,\cdot,\cdot)$ and $\hat{\xi}(\cdot)$ be the functions defined in Lemma 2. We know from Lemma 2 that $\hat{u}(t,q) = U(t,q,\hat{\lambda}(t,q),\hat{\xi}(t))$.

Let us consider the following boundary value problem on $D_{\hat{Q}}$:

$$\dot{x}(t,q) = -\delta x(t,q) + U(t,q,\lambda(t,q),\hat{\xi}(t)), \qquad x(\hat{\theta}(q),q) = x^0(q) \tag{33}$$

$$\dot{\lambda}(t,q) = \delta\lambda(t,q) - L_x(t,q,x(t,q)), \qquad \lambda(T,q) = 0. \tag{34}$$

Obviously $(\hat{x},\hat{\lambda})$ is a solution of this system. Our goal below will be to prove that it is the only solution and that it depends continuously on q, hence $\hat{u}$ also depends continuously on q.

Consider the initial value problem (33)–(34) with $\lambda(\hat{\theta}(q),q) = p$ instead of the end point condition for λ. Due to the standing assumptions and Lemma 2, the right-hand side of the differential system in (33), (34) is Lipschitz continuous in (λ,x). Then for every q the initial value problem has a unique solution $(x(t,q;p),\lambda(t,q;p))$ on $[\hat\theta(q),T]$. Let us fix q and suppress it, as well as $\hat\xi(t)$, in the notations below.

To prove uniqueness of the solution of (33)–(34), assume that there are two solutions, (x_1,λ_1) and (x_2,λ_2). If $\lambda_1(\hat\theta)=\lambda_2(\hat\theta)$, then both solutions coincide with $(x(t;p),\lambda(t;p))$ for $p=\lambda_1(\hat\theta)$. Therefore, let us assume that $\lambda_2(\hat\theta)-\lambda_1(\hat\theta)>\varepsilon$ for some $\varepsilon>0$. Let τ be the maximal number in $[\hat\theta,T]$ such that $\lambda_2(t)-\lambda_1(t)\geq\varepsilon$ for all $t\in[\hat\theta,\tau]$. Using that the function $\lambda\mapsto U(t,\lambda)$ is non-decreasing due to its definition in (29) and the convexity of c, we obtain that for $t\in[\hat\theta,\tau]$

$$\dot x_2(t)-\dot x_1(t)=-\delta(x_2(t)-x_1(t))+U(t,\lambda_2(t))-U(t,\lambda_1(t))\geq-\delta(x_2(t)-x_1(t)).$$

Since $x_1(\hat\theta)=x_2(\hat\theta)$, the above inequality implies $x_2(t)-x_1(t)\geq 0$ for all $t\in[\hat\theta,\tau]$. Using the last inequality and the fact that the function $x\mapsto L_x(t,x)$ is non-increasing due to the concavity of L, we obtain

$$\begin{aligned}\dot\lambda_2(t)-\dot\lambda_1(t) &= \delta(\lambda_2(t)-\lambda_1(t))-(L_x(t,x_2(t))-L_x(t,x_1(t)))\\ &\geq \delta(\lambda_2(t)-\lambda_1(t))\geq 0,\quad t\in[\hat\theta,\tau],\end{aligned}$$

which implies $\lambda_2(\tau)-\lambda_1(\tau)>\varepsilon$. This means that $\tau=T$ and, in particular, $\lambda_2(T)-\lambda_1(T)\geq\varepsilon$. This contradicts the boundary condition in (34), and implies that the solution of (33), (34) is unique (namely, $(\hat x(\cdot,q),\hat\lambda(\cdot,q))$).

Next, we shall prove the continuity of $\hat\lambda$ with respect to q. Due to the boundedness of $\hat u$ it is easy to verify that there is a compact interval $P\subset\mathbf{R}$ containing all values $\hat\lambda(\hat\theta(q),q),\ q\in[0,\hat Q(T)]$. Let us prove the following property (P): for every $\varepsilon>0$ there exists $\delta>0$ such that for any $q_1,\ q_2\in[0,\hat Q(T)]$ and $p_1,\ p_2\in P$ satisfying $|q_1-q_2|<\delta$ and $\lambda(T,q_1;p_1)=\lambda(T,q_2;p_2)=0$, it holds that $|p_1-p_2|\leq\varepsilon$.

According to the continuous dependence of the solution of an ODE with Lipschitz continuous right-hand side with respect to initial data and parameter (see e.g. [10, Theorem 2], where the required continuity in t is not necessary.) the mapping $[0,\hat Q(T)]\times\mathbf{R}\ni(q,p)\to\lambda(\cdot,q;p)\in C([\hat\theta(q),T])$ is continuous, hence it is uniformly continuous on $[0,\hat Q(T)]\times P$.

Assume that property (P) does not hold. Then there exists $\varepsilon>0$ such that for every $\delta>0$ there exist $q_1,\ q_2\in[0,\hat Q(T)]$ and $p_1,\ p_2\in P$ such that $|q_1-q_2|<\delta$, $\lambda(T,q_1;p_1)=\lambda(T,q_2;p_2)=0$, and $p_2-p_1>\varepsilon$. Due to the (uniform) continuous dependence we may choose $\delta>0$ so small that

$$|\lambda(T,q_2;p_2)-\lambda(T,q_1;p_2)|\leq\varepsilon/2.$$

We have

$$\lambda(\hat\theta(q_1),q_1;p_2)-\lambda(\hat\theta(q_1),q_1;p_1)=p_2-p_1\geq\varepsilon,\quad x(\hat\theta(q_1),q_1;p_2)=x(\hat\theta(q_1),q_1;p_1).$$

Then we can prove in exactly the same way as a few paragraphs above that $\lambda(t,q_1;p_2)-\lambda(t,q_1;p_1)\geq\varepsilon$ for all $t\in[\hat\theta(q_1),T]$. Hence

$$\lambda(T,q_2;p_2)-\lambda(T,q_1;p_1)\geq\lambda(T,q_1;p_2)-\lambda(T,q_1;p_1)-\varepsilon/2\geq\varepsilon/2.$$

This contradicts the equality $\lambda(T,q_1;p_1)=\lambda(T,q_2;p_2)$ and proves property (P).

Applying property (P) for $p_i=\hat\lambda(\hat\theta(q_i),q_i)$, $i=1,\ 2$, we obtain that for every $\varepsilon>0$ there exists $\delta>0$ such that $|q_1-q_2|<\delta$ implies $|\hat\lambda(\hat\theta(q_1),q_1)-\hat\lambda(\hat\theta(q_2),q_2)|<\varepsilon$. Then using again the continuous dependence of the solution of ODEs we conclude that $q\mapsto\hat\lambda(\cdot,q)$ is continuous. From the equality $\hat u(t,q)=U(t,q,\hat\lambda(t,q),\hat\xi(t))$ and Lemma 2 we obtain the desired continuity of $\hat u(t,\cdot)$.
Q.E.D.

5 Pontryagin maximum principle and stationarity of the Hamiltonian along any optimal trajectory

In this section we derive necessary optimality conditions of Pontryagin's type for problem (1)–(7) and prove stationarity of the Hamiltonian along any optimal trajectory in the case of stationary data.

Define the adjoint variables $(\hat{\lambda}(t,q), \hat{\mu}(t), \hat{\nu}(t))$ for $(t,q) \in D_{\hat{Q}}$ as solutions to the following adjoint system (we use the notational convention from the beginning of Section 4)

$$\dot{\lambda}(t,q) = \delta\lambda(t,q) - L_x(t,q) \quad (t,q) \in D_{\hat{Q}}, \quad \lambda(T,q) = 0, \quad q \in [0, \hat{Q}(T)], \tag{35}$$

$$\begin{aligned} \dot{\mu}(t) = \mu(t) & \left[-\frac{\partial g_1}{\partial Q}(t)\, \hat{y}(t) - \frac{\partial g_2}{\partial Q}(t)\, \hat{v}(t) \right] - L(t, \hat{Q}(t)) + c_1(t, \hat{u}(t, \hat{Q}(t))) \\ & - \lambda(t, \hat{Q}(t))[-\delta x^b(t) - \dot{x}^b(t) + \hat{u}(t, \hat{Q}(t))] - \nu(t) d(t, \hat{Q}(t)) \hat{u}(t, \hat{Q}(t)) \\ & - \int_0^{\hat{Q}(t)} L_Q(t,q)\, \mathrm{d}q, \qquad t \in [0,T], \quad \mu(T) = 0, \end{aligned} \tag{36}$$

$$\nu(t) = \mu(t)\, g_1(t) + \int_0^{\hat{Q}(t)} L_y(t,q)\, \mathrm{d}q, \quad t \in [0,T]. \tag{37}$$

The distributed equation (35) is viewed as a family of ODEs parameterized by q, with a zero end-point condition at $t = T$.

Define $H : D \times \mathbf{R} \times [0, \bar{Q}] \times \mathbf{R} \times [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \mapsto \mathbf{R}$ as

$$\begin{aligned} H(t,q,x,Q,y,u,v,\lambda,\mu,\nu) = & [L(t,q,x,Q,y) - c_1(t,u) - c_2(t,v)] + \\ & + \lambda\,(-\delta x + u) + \mu\, [g_1(t,Q) y + g_2(t,Q) v] + \nu\, d(t,q) u. \end{aligned}$$

Theorem 1 *Let Standing Assumptions (i)–(vii) be fulfilled, let* $(\hat{u}, \hat{v}) \in L_\infty(D) \times L_\infty([0,T])$ *be an optimal control of the problem* (1)–(7) *and denote by* $\hat{z} := (\hat{x}, \hat{Q}, \hat{y})$ *the corresponding state trajectory. Then the adjoint system* (35)–(37) *has a unique solution* $\hat{\pi} := (\hat{\lambda}, \hat{\mu}, \hat{\nu})$ *and for a.e.* $(t,q) \in D_{\hat{Q}}$

$$H(t,q,\hat{z}(t,q),\hat{u}(t,q),\hat{v}(t),\hat{\pi}(t,q)) = \max_{(u,v) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]} H(t,q,\hat{z}(t,q),u,v,\hat{\pi}(t,q)). \tag{38}$$

Proof. Essentially, the result follows from the more general Theorem 2 in [4]. A key requirement in [4, Theorem 2] (and the main difficulty) is the regularity condition for the optimal u, which we proved in Section 4. A minor additional work is needed due to the non-distributed control $v(t)$ involved in the present paper in order to cover some existing economic models. Only distributed controls are involved in [4], but in all other respects the model there is more general, in particular, the control and the states may be multi-dimensional.

It is easy to bring our problem (1)–(7) to a form tractable by [4, Theorem 2]. Indeed, we introduce a second (artificial) distributed control $u_2(t,q) \in [\underline{v}, \overline{v}]$ and an additional integral state

$$y_2(t) = \int_0^{Q(t)} u_2(t,q)\, \mathrm{d}q. \tag{39}$$

It is clear that whatever is $Q(t) > 0$, every measurable v with $v(t) \in [\underline{v}, \overline{v}]$ can be represented as $v(t) = y_2(t)/Q(t)$ with $u_2(t,q) = v(t) \in [\underline{v}, \overline{v}]$ in (39), and u_2 satisfies the regularity condition. Moreover, every measurable $u_2(t,q) \in [\underline{v}, \overline{v}]$ defines in the same way a measurable $v(t) \in [\underline{v}, \overline{v}]$.

We modify problem (1)–(7) by replacing everywhere v with y_2/Q, and adding equation (39) as a second integral equation along with (3). Clearly, $\hat{u}$ and $\hat{u}_2(t,q) = \hat{v}(t)$ forms an optimal control pair for the modified problem. Applying Theorem 2 in [4] and returning to the original control v we obtain the claim of the theorem. The technicalities are straightforward. Q.E.D.

Our next aim will be to obtain a Hamiltonian representation of the optimality conditions and to prove stationarity of the Hamiltonian (for stationary problems) along any optimal solution. This turned out to be a non-straightforward task, and even the correct definition of a Hamiltonian is not evident, due to the endogenous change of the spatial domain in which the distributed equation (4) acts. In order to correctly define the Hamiltonian we replace equations (4)–(6) with the extended system (9)–(11), which has the advantage of defining a solution x on all the rectangular D (we remind that x^0 was extended to $[0, \bar{Q}]$ by setting $x^0(q) = x^b(0)$ for $q \in (Q^0, \bar{Q}]$). Similarly, we extend the adjoint function λ to the whole rectangular by defining $\dot{\lambda}(t,q) = 0$ on $D \backslash D_Q$. This is consistent with the redefinition of x, since L and the dynamics $\dot{x}(\cdot,\cdot)$ do not depend on x for (t,q) on $D \backslash D_Q$ (cf. the adjoint equation (11) on p. 293 in [4]).

We define the Hamiltonian as a functional depending on the following variables: $t \in [0,T]$, $x \in L_\infty([0,\bar{Q}])$, $Q \in [0,\bar{Q}]$, $y \in \mathbf{R}^1$, $u \in L_\infty([0,\bar{Q}])$, $v \in \mathbf{R}^1$, $\lambda \in L_\infty([0,\bar{Q}])$, $\mu \in \mathbf{R}^1$, $\nu \in \mathbf{R}^1$. The definition is as follows:

$$
\begin{aligned}
&\mathcal{H}(t,x,Q,y,u,v,\lambda,\mu,\nu) \\
&\qquad := \int_0^Q \left(L(t,q,Q,x(q),y) - c_1(t,u(q))\right) \mathrm{d}q - c_2(t,v) + \int_0^Q \lambda(q)\left(-\delta x(q) + u(q)\right) \mathrm{d}q \\
&\qquad\quad + \dot{x}^b(t) \int_Q^{\bar{Q}} \lambda(q)\, \mathrm{d}q + \mu\left(g_1(t,Q)y + g_2(t,Q)v\right) + \nu\left(-y + \int_0^Q d(t,q)u(q)\, \mathrm{d}q\right).
\end{aligned}
$$

Using this Hamiltonian functional, the primal control system (2), (3), (9)–(11) and the adjoint system (35)–(37) can be written in a more compact form. The primal system takes the form

$$
\begin{aligned}
\dot{x}(t,\cdot) &= \frac{\partial \mathcal{H}}{\partial \lambda}(t,x(t,\cdot),Q(t),y(t),u(t,\cdot),v(t),\lambda(t,\cdot),\mu(t),\nu(t)), & x(0,\cdot) &= x^0(\cdot), \\
\dot{Q}(t) &= \frac{\partial \mathcal{H}}{\partial \mu}(t,x(t,\cdot),Q(t),y(t),u(t,\cdot),v(t),\lambda(t,\cdot),\mu(t),\nu(t)), & Q(0) &= Q^0, \\
0 &= \frac{\partial \mathcal{H}}{\partial \nu}(t,x(t,\cdot),Q(t),y(t),u(t,\cdot),v(t),\lambda(t,\cdot),\mu(t),\nu(t)),
\end{aligned}
$$

while the adjoint system can be written as

$$
\begin{aligned}
\dot{\lambda}(t,\cdot) &= -\frac{\partial \mathcal{H}}{\partial x}(t,x(t,\cdot),Q(t),y(t),u(t,\cdot),v(t),\lambda(t,\cdot),\mu(t),\nu(t)), & \lambda(T,\cdot) &= 0, \\
\dot{\mu}(t) &= -\frac{\partial \mathcal{H}}{\partial Q}(t,x(t,\cdot),Q(t),y(t),u(t,\cdot),v(t),\lambda(t,\cdot),\mu(t),\nu(t)), & \mu(T) &= 0, \\
0 &= \frac{\partial \mathcal{H}}{\partial y}(t,x(t,\cdot),Q(t),y(t),u(t,\cdot),v(t),\lambda(t,\cdot),\mu(t),\nu(t)).
\end{aligned}
$$

We mention that in general, the derivatives $\frac{\partial \mathcal{H}}{\partial \lambda}$ and $\frac{\partial \mathcal{H}}{\partial x}$ are elements of the dual space of $L_\infty([0,\bar{Q}])$, but they turn out to be representable by L_∞-functions.

The condition of maximization of the Hamiltonian takes the following form: for each $t \in [0,T]$

$$\mathcal{H}(t, \hat{z}(t,\cdot), \hat{u}(t,\cdot), \hat{v}(t), \hat{\pi}(t,\cdot)) = \max_{u,v} \mathcal{H}(t, \hat{z}(t,\cdot), u(\cdot), v, \hat{\pi}(t,\cdot)), \tag{40}$$

where, as in Theorem 1, $\hat{z} := (\hat{x}, \hat{Q}, \hat{y})$ and $\hat{\pi} := (\hat{\lambda}, \hat{\mu}, \hat{\nu})$, and the maximization takes place on the set of all $u \in L_\infty([0, \hat{Q}(t)])$ with $u(q) \in [\underline{u}, \overline{u}]$ and all $v \in [\underline{v}, \overline{v}]$.

Having in mind the specific form of the Hamiltonian, above maximization condition splits in two:

$$\max_u \int_0^{\hat{Q}(t)} \left[-c_1(t, u(q)) + (\lambda(t,q) + \nu(t) d(t,q))\, u(q)\right] \mathrm{d}q, \quad u(q) \in [\underline{u}, \overline{u}]$$

$$\max_{v \in [\underline{v}, \overline{v}]} \{-c_2(t,v) + \mu(t) g_2(t)\, v\}.$$

Clearly, the first maximization problem is equivalent to point-wise maximization for every q separately, thus it is equivalent to (38).

Having at hand the correct Hamiltonian representation of the optimality conditions, we can prove that for stationary problems the Hamiltonian is constant along any optimal solution. Our proof, however, requires an additional assumption formulated in the next theorem.

Theorem 2 *Let Standing Assumptions (i)–(vii) hold and let L be linear in y. Let the functions L, c_1, c_2, g_1, g_2, d and $\dot{x}^b$ do not depend on t and let c_2 be strongly convex and twice continuously differentiable on an open set containing $[\underline{v}, \overline{v}]$. Let $(\hat{x}, \hat{Q}, \hat{y}, \hat{u}, \hat{v})$ be an optimal trajectory and let $\hat{\lambda}$, $\hat{\mu}$, and $\hat{\nu}$ be the respective solutions to the adjoint system* (35)–(37)*. Then the maximized Hamiltonian*

$$\mathcal{M}(\hat{x}(t,\cdot), \hat{Q}(t), \hat{y}(t), \hat{\lambda}(t,\cdot), \hat{\mu}(t), \hat{\nu}(t)) := \max_{u,v} \mathcal{H}(\hat{x}(t,\cdot), \hat{Q}(t), \hat{y}(t), u(\cdot), v, \hat{\lambda}(t,\cdot), \hat{\mu}(t), \hat{\nu}(t)),$$

where the maximization takes place over the set of all $u \in L_\infty([0, \hat{Q}(t)])$ with $u(q) \in [\underline{u}, \overline{u}]$ and all $v \in [\underline{v}, \overline{v}]$, is constant on $[0,T]$.

Proof. First of all we shall prove uniform boundedness of the trajectories and the adjoint variables. Denote for brevity $z := (x, Q, y)$ and $\pi := (\lambda, \mu, \nu)$. Due to the boundedness of x^0, $\dot{x}^b$, g_1, g_2, d, $[\underline{u}, \overline{u}]$, and $[\underline{v}, \overline{v}]$ (see the Standing Assumptions) there is a compact set $Z \in \mathbf{R}^5$ such that $(z(t,q), u(t,q), v(t)) \in Z$ for every admissible control (u, v) and corresponding trajectory $z = (x, Q, y)$. Then the right-hand sides of the adjoint equations (35)–(37) are uniformly bounded, hence there exists a compact set $\Pi \in \mathbf{R}^3$ such that $\pi(t,q) \in \Pi$ for the adjoint variables $\pi = (\lambda, \mu, \nu)$ corresponding to any admissible control and the corresponding trajectories. Then the set $\mathcal{A} := \{(z(t,\cdot), u(t,\cdot), v(t), \pi(t,\cdot)) \,:\, t \in [0,T],\ (u(\cdot,\cdot), v(\cdot)) \in \mathcal{U} \times \mathcal{V}\}$ is bounded in $L_\infty([0,\bar{Q}]) \times [0,\bar{Q}] \times \mathbf{R}^1 \times L_\infty([0,\bar{Q}]) \times \mathbf{R}^1 \times L_\infty([0,\bar{Q}]) \times \mathbf{R}^1 \times \mathbf{R}^1$.

Since L is linear in y, we can represent $L(q,x,Q,y) = L_1(q,x,Q)y + L_2(q,x,Q)$, where L_1 and L_2 have the same properties as L in the Standing Assumptions. From the corresponding differential equations and the boundedness of the right-hand sides, we have that $\hat{x}(\cdot, q)$, $\hat{Q}(\cdot)$, $\hat{\lambda}(\cdot, q)$ and $\hat{\mu}(\cdot)$ are Lipschitz continuous, uniformly in $q \in [0, \bar{Q}]$. Further, we have $L_y = L_1$ and, hence,

$$\hat{\nu}(t) = \hat{\mu}(t)\; g_1(\hat{Q}(t)) + \int_0^{\hat{Q}(t)} L_1(q, \hat{x}(t,q), \hat{Q}(t))\,\mathrm{d}q. \tag{41}$$

The first term in the right-hand side above is Lipschitz continuous in t because $\hat{\mu}$ and $\hat{Q}$ are such, and g_1 is continuously differentiable in Q, hence Lipschitz on $[0, \bar{Q}]$. The second term is also Lipschitz continuous due to the Lipschitz continuity of $\hat{Q}$, the boundedness of the integrand, the Lipschitz continuity of L_1 in (x, Q), and the Lipschitz continuity of $\hat{x}(\cdot, q)$ uniformly in q. Thus $\nu(\cdot)$ is absolutely continuous.

From (29), taking into account that now c_1 and d do not depend on t, we obtain that

$$\hat{u}(t,q) = \tilde{U}(q, \hat{\lambda}(t,q), \hat{\nu}(t)), \tag{42}$$

where

$$\tilde{U}(q,\lambda,\nu) := \begin{cases} \underline{u} & \text{if} \quad \lambda + \nu\, d(q) < c_1'(\underline{u}) \\ (c_1')^{-1}(\lambda + \nu\, d(q)) & \text{if} \quad c_1'(\underline{u}) \le \lambda + \nu\, d(q) \le c_1'(\overline{u}) \\ \overline{u} & \text{if} \quad \lambda + \nu\, d(q) > c_1'(\overline{u}). \end{cases}$$

Obviously, $\tilde{U}$ is Lipschitz in both λ and ν, uniformly in q. Due to the (uniform) Lipschitz continuity of $\hat{\lambda}(\cdot, q)$, there exists a constant C such that

$$|\hat{u}(t,q) - \hat{u}(\tau,q)| \le C\Big(t - \tau + |\hat{\nu}(t) - \hat{\nu}(\tau)|\Big)$$

for $0 \le \tau < t \le T$ and $q \in [0, \hat{Q}(\tau)]$. From here, for $0 \le \tau < t \le T$ we obtain

$$\begin{aligned} |\hat{y}(t) - \hat{y}(\tau)| &= \left| \int_0^{\hat{Q}(t)} d(q)\hat{u}(t,q)\,\mathrm{d}q - \int_0^{\hat{Q}(\tau)} d(q)\hat{u}(\tau,q)\,\mathrm{d}q \right| \\ &\le \left| \int_0^{\hat{Q}(\tau)} d(q)\left(\hat{u}(t,q) - \hat{u}(\tau,q)\right)\,\mathrm{d}q + \int_{\hat{Q}(\tau)}^{\hat{Q}(t)} d(q)\hat{u}(t,q)\,\mathrm{d}q \right| \\ &\le \overline{d}\,\bar{Q}C\Big(t - \tau + |\hat{\nu}(t) - \hat{\nu}(\tau)|\Big) + \overline{u}\,\overline{d}|\hat{Q}(t) - \hat{Q}(\tau)|. \end{aligned}$$

The absolute continuity of $\hat{\nu}(\cdot)$ and $\hat{Q}(\cdot)$ yield the absolute continuity of $\hat{y}(\cdot)$.

We next show that the function $t \longmapsto \mathcal{M}(\hat{z}(t,\cdot), \hat{\pi}(t,\cdot))$ is absolutely continuous on $[0, T]$. From (40) we have

$$\mathcal{M}(\hat{z}(t,\cdot), \hat{\pi}(t,\cdot)) = \mathcal{H}(\hat{z}(t,\cdot), \hat{u}(t,\cdot), \hat{v}(t), \hat{\pi}(t,\cdot)). \tag{43}$$

First we shall show that for any fixed $t' \in [0, T]$ the function $t \mapsto \mathcal{H}(\hat{z}(t,\cdot), \hat{u}(t',\cdot), \hat{v}(t'), \hat{\pi}(t,\cdot))$ is Lipschitz continuous, uniformly in t'. Due to the uniform boundedness and the Lipschitz continuity in t of the arguments of this function, it is enough to prove that it is Lipschitz continuous on $\mathcal{A}$.

Although this is a routine task, we show this property with respect to $x(\cdot)$:

$$\begin{aligned}
& \Big|\mathcal{H}(x_1(\cdot), Q, y, u(\cdot), v, \lambda(\cdot), \mu, \nu) - \mathcal{H}(x_2(\cdot), Q, y, u(\cdot), v, \lambda(\cdot), \mu, \nu)\Big| \\
= & \left|\int_0^Q \Big[L(q, x_1(q), Q, y) - L(q, x_2(q), Q, y)\Big]\,\mathrm{d}q - \delta \int_0^Q \lambda(q)\Big(x_1(q) - x_2(q)\Big)\,\mathrm{d}q\right| \\
\le & \int_0^Q M_L |x_1(q) - x_2(q)|\,\mathrm{d}q + \delta \int_0^Q |\lambda(q)|\,|x_1(q) - x_2(q)|\,\mathrm{d}q \\
\le & \ (M_L + \delta|\Pi|)\,\bar{Q}\,\|x_1(\cdot) - x_2(\cdot)\|_{L_\infty([0,\bar{Q}])},
\end{aligned}$$

where M_L is the Lipschitz constant of L on Z and $|\Pi| := \sup_{\pi\in\Pi} |\pi|$.

Thus there exists a constant K such that

$$|\mathcal{H}(\hat{z}(t,\cdot), \hat{u}(t,\cdot), \hat{v}(t), \hat{\pi}(t,\cdot)) - \mathcal{H}(\hat{z}(t',\cdot), \hat{u}(t,\cdot), \hat{v}(t), \hat{\pi}(t',\cdot))| \le K|t - t'|,$$

for every $t,\ t' \in [0,T]$. From here, using also (40), we obtain that

$$\begin{aligned}
-K|t-t'| &\le \mathcal{H}(\hat{z}(t,\cdot), \hat{u}(t',\cdot), \hat{v}(t'), \hat{\pi}(t,\cdot)) - \mathcal{H}(\hat{z}(t',\cdot), \hat{u}(t',\cdot), \hat{v}(t'), \hat{\pi}(t',\cdot)) \le \\
&\le \mathcal{H}(\hat{z}(t,\cdot), \hat{u}(t,\cdot), \hat{v}(t), \hat{\pi}(t,\cdot)) - \mathcal{H}(\hat{z}(t',\cdot), \hat{u}(t',\cdot), \hat{v}(t'), \hat{\pi}(t',\cdot)) \le \\
&\le \mathcal{H}(\hat{z}(t,\cdot), \hat{u}(t,\cdot), \hat{v}(t), \hat{\pi}(t,\cdot)) - \mathcal{H}(\hat{z}(t',\cdot), \hat{u}(t,\cdot), \hat{v}(t), \hat{\pi}(t',\cdot)) \le K|t-t'|,
\end{aligned}$$

hence, $|\mathcal{M}(\hat{z}(t,\cdot), \hat{\pi}(t,\cdot)) - \mathcal{M}(\hat{z}(t',\cdot), \hat{\pi}(t',\cdot))| \le K|t-t'|$, which yields the Lipschitz continuity of $t \longmapsto \mathcal{M}(\hat{z}(t,\cdot), \hat{\pi}(t,\cdot))$.

Now we shall prove that $\frac{d}{dt}\mathcal{M}(\hat{z}(t,\cdot), \pi(t,\cdot)) = 0$ almost everywhere on $[0,T]$. We remind that both $t \mapsto \hat{x}(t,\cdot) \in L_\infty([0,\bar{Q}])$ and $t \mapsto \hat{\lambda}(t,\cdot) \in L_\infty([0,\bar{Q}])$ are Lipschitz continuous, as well as $\hat{Q}$ and $\hat{\mu}$. Fix τ in the subset (of full measure) of $[0,T]$ on which (43), the state equations, and the adjoint equations (in their Hamiltonian form) are satisfied. For almost every $t \in [0,T]$ it holds that

$$\mathcal{M}(\hat{z}(t,\cdot), \hat{\pi}(t,\cdot)) \ge \mathcal{H}(\hat{z}(t,\cdot), \hat{u}(\tau,\cdot), \hat{\pi}(t,\cdot)),$$

and thus,

$$\mathcal{M}(\hat{z}(t,\cdot), \hat{\pi}(t,\cdot)) - \mathcal{M}(\hat{z}(\tau,\cdot), \hat{\pi}(\tau,\cdot)) \ge \mathcal{H}(\hat{z}(t,\cdot), \hat{u}(\tau,\cdot), \hat{\pi}(t,\cdot)) - \mathcal{H}(\hat{z}(\tau,\cdot), \hat{u}(\tau,\cdot), \hat{\pi}(\tau,\cdot)).$$

Assume that t converges to τ from above, divide by $(t-\tau) > 0$ and take the limit $t \longrightarrow \tau$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(\hat{z}(t,\cdot), \hat{\pi}(t,\cdot))\Big|_{t=\tau} &\ge \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\hat{z}(t,\cdot), \hat{u}(\tau,\cdot), \hat{\pi}(t,\cdot))\Big|_{t=\tau} \\
&= \Big(\langle \mathcal{H}_x, \dot{\hat{x}}\rangle + \langle \mathcal{H}_Q, \dot{\hat{Q}}\rangle + \langle \mathcal{H}_y, \dot{\hat{y}}\rangle + \langle \mathcal{H}_\lambda, \dot{\hat{\lambda}}\rangle + \langle \mathcal{H}_\mu, \dot{\hat{\mu}}\rangle + \langle \mathcal{H}_\nu, \dot{\hat{\nu}}\rangle\Big)_{t=\tau} \\
&= \langle -\dot{\hat{\lambda}}, \dot{\hat{x}}\rangle + \langle -\dot{\hat{\mu}}, \dot{\hat{Q}}\rangle + 0 + \langle \dot{\hat{x}}, \dot{\hat{\lambda}}\rangle + \langle \dot{\hat{Q}}, \dot{\hat{\mu}}\rangle + 0 = 0, \qquad (44)
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual product on the appropriate spaces and the point of evaluation is omitted for clarity. The two zeros in the left–hand side of the last line of (44) result from the facts that
$\mathcal{H}_y(\hat{z}(t,\cdot), \hat{u}(\tau,\cdot), \hat{\pi}(t,\cdot))\Big|_{t=\tau} = 0$ (this is the ajoint equation (37)) and
$\mathcal{H}_\nu(\hat{z}(t,\cdot), \hat{u}(\tau,\cdot), \hat{\pi}(t,\cdot))\Big|_{t=\tau} = 0$ (this is the state equation (3)).

Taking t to converge to τ from below, results in the opposite inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(\hat{z}(t,\cdot),\hat{\pi}(t,\cdot))\Big|_{t=\tau} \leq 0. \qquad (45)$$

Since (44) and (45) hold for a.e. $\tau \in [0,T]$, the time derivative of $\mathcal{M}$ is zero almost everywhere, thus, $\mathcal{M}$ is constant.

Q.E.D.

We conclude the paper with the following two remarks.

1. The Hamiltonian form of the maximum principle obtained above in this section is valid also for the general model considered in [4], provided that the regularity condition for the optimal control $\hat{u}$ is fulfilled. However, this regularity condition is difficult to verify a priori. Its verification for the more specific model considered in the present paper was one of our main goals.

2. The previous paper [4] contains numerical results for a model that is a particular case of the one considered here. The numerical approach proposed therein is based on the Pontryagin type optimality conditions, whose validity is proved under the abovementioned regularity condition. Thus the present paper provides a justification for the numerical (and analytic) results in [4]. A more profound economic analysis based on Theorem 1 is a subject of a current work by the first author.

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