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Aggregation and asymptotic analysis of an SI-epidemic model for heterogeneous populations*

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Abstract

The paper investigates a version of a simple epidemiologic model involving only susceptible and infected individuals, where the heterogeneity of the population with respect to susceptibility/infectiousness is taken into account. A comprehensive analysis of the asymptotic behaviour of the disease is given, based on an explicit aggregation of the model. The results are compared to those of a homogeneous version of the model to highlight the influence of the heterogeneity on the asymptotics. Moreover, the performed analysis reveals in which cases incomplete information about the heterogeneity of the population is sufficient in order to determine the long run outcome of the disease. Numerical simulation is used to emphasise that for a given level of prevalence the evolution of the disease under the influence of heterogeneity may in the long run qualitatively differ from the one "predicted" by the homogeneous model. Furthermore, it is shown that in a closed population the indicator for the survival of the population is in presence of heterogeneity distinct from the basic reproduction number.

Keywords: mathematical epidemiology, SI-model, heterogeneous epidemiologic models

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1 Introduction

In this paper we consider a heterogeneous version of a simple epidemiologic model of a population consisting of a non-infected (potentially susceptible) sub-population and an infected sub-population (SI-model). It is well recognized that modeling the population as homogeneous (with equal susceptibility/infectivity of all individuals) may give a rather distorted picture of the evolution of the disease, compared with the one that appears if the heterogeneity of the population is taken into account. The main goal is to quantitatively describe the differences (and similarities) in the asymptotic behaviour of the disease when modelling the population as heterogeneous, versus homogeneous.

In principle, it is well known that the heterogeneity plays an important role in epidemiologic models, therefore the issue of heterogeneity is introduced and investigated in this subject area in a large number of papers (see e.g. [8, 11, 15, 16, 25, 26, 27]). The main contribution of the present paper is that the asymptotic behaviour of the disease in a heterogeneous population (within an SI framework) is completely and explicitly described, and compared with the one resulting from the homogeneous version of the model. The proofs are rather technical, but not routine. Although the analysis is restricted to a very simple model, it can be useful as a benchmark case for more enhanced investigations of the influence of heterogeneity on the evolution of infectious diseases. Such are indicated in Section 6.

A crucial drawback of the heterogeneous models is that they require information about the distribution of the population along the space of heterogeneity, which is usually not available. As a by-product of our analysis, it becomes clear what information about the heterogeneity is actually essential for determining the ultimate epidemic state. The required information is (in generic cases) substantially less than the overall distribution.

The paper is organized as follows. In Section 2 we present the homogeneous and the heterogeneous versions of the considered SI-model. The aggregation of the heterogeneous model to an ODE model is done in Section 3. Section 4 is devoted to the asymptotic analysis of the disease, where a comparison between the results for the homogeneous and the heterogeneous model is also presented, as well as some illustrating numerical examples. The concept of reproduction number is adapted to the heterogeneous model and discussed in Section 5. Some concluding remarks and perspectives for further investigations of more complex epidemic models are given in Section 6. The more technical proofs are given in the appendix.

2 The SI-model

First we recall the standard SI-model for a population with a variable size, depending on fertility and mortality (natural, and such caused by the disease). Then, in the second subsection, we present a heterogeneous version of the same model.

2.1 The homogeneous model

The dynamics of the disease is given by the following equations, in which $S(t)$ and $I(t)$ denote the size of the susceptible and of the infected subpopulations at time $t \geq 0$:

$$\dot{S}(t) = -\sigma y(t) S(t) + \lambda S(t), \quad S(0) = S_0, \tag{1}$$

$$\dot{I}(t) = \sigma y(t) S(t) - \delta I(t), \quad I(0) = I_0, \tag{2}$$

where

$$y(t) = \frac{I(t)}{S(t) + I(t)} \tag{3}$$

is the prevalence. The meaning of the appearing parameters is as usual: λ (positive or negative) is the net inflow rate of susceptible individuals (i.e. the difference between birth and mortality rate), similarly δ is the net mortality rate of infected individuals, and σ is the infectiousness of the disease (strength of infection). Clearly, there is no recovery from the considered disease. In addition, it will be assumed that $\delta > 0$, that is, the infected individuals die at a higher rate than they reproduce. The initial data S_0 and I_0 are both positive.

SI-models are amongst the most basic epidemiological models and therefore an analysis of such models can be found in introductory texts on epidemiology, such as [3, 4, 9]. Here we will derive some of the results anew in order to make the considerations self-contained and to draw parallels between this homogeneous model and the heterogeneous model described below.

Closely related to SI-models are SIR-models where $R(t)$ stands for removed individuals and follows the dynamics

$$\dot{R}(t) = \delta I(t).$$

This population influences the dynamics of $S(t)$ and $I(t)$ because $y(t)$ now has to be defined as $y(t) = \frac{I(t)}{S(t)+I(t)+R(t)}$. However, if the removed no longer participate in infectious contacts, e.g. because the removal is due to quarantine, then the SI and SIR-models are equivalent.

2.2 The heterogeneous model

We introduce a heterogeneity into the above SI-model by differentiating the population according to some traits, such as genetic markers, natural resistance towards a disease, or social behaviour, that influence the spreading of the disease. We therefore assume that every individual has a certain h-state (heterogeneity-state) ω. We restrict ourselves to traits that are time invariant, i.e. an individual that has h-state ω stays in that state for all of its lifespan. Denote by Ω the space of all h-states. It will be assumed that Ω is a Borel measurable space with a finite measure μ. This allows Ω to be a continuous or discrete space, as well as a product space involving different traits. Previous works that use this or a similar notion of h-state include [8, 10, 11, 16, 25, 27].

We assume that the h-state ω influences the risk of an individual to become infected by a factor $p(\omega)$. More precisely, we denote by $\bar{S}(t,\omega)$ and $\bar{I}(t,\omega)$ the size of the susceptible and infected sub-populations of h-state ω and assume that the sub-population of each h-state develops similarly as in the homogeneous S-I-model:

$$\dot{\bar{S}}(t,\omega) = -\sigma z(t)\, p(\omega)\, \bar{S}(t,\omega) + \lambda \bar{S}(t,\omega), \quad \bar{S}(0,\omega) = S_0(\omega), \tag{4}$$

$$\dot{\bar{I}}(t,\omega) = \sigma z(t)\, p(\omega)\, \bar{S}(t,\omega) - \delta \bar{I}(t,\omega), \quad \bar{I}(0,\omega) = I_0(\omega). \tag{5}$$

Here the "dot" means differentiation with respect to t (for every fixed ω). The difference is that now the infectivity of the environment of the susceptible individuals is represented by the *weighted prevalence* $z(t)$ defined as

$$z(t) = \frac{J(t)}{K(t) + J(t)}, \tag{6}$$

where

$$K(t) = \int_\Omega p(\omega)\, \bar{S}(t,\omega)\, \mathrm{d}\omega, \tag{7}$$

$$J(t) = \int_\Omega p(\omega)\, \bar{I}(t,\omega)\, \mathrm{d}\omega. \tag{8}$$

Notice that not only the individual risk factor $p(\omega)$ is a carrier of heterogeneity in the above model, but also the weighted prevalence $z(t)$, which depends on the current (heterogeneous) distribution of the infected and susceptible individuals. A discussion about the model is given after the assumptions below.

Further, we will use the notations

$$S(t) = \int_\Omega \bar{S}(t,\omega)\, \mathrm{d}\omega, \quad I(t) = \int_\Omega \bar{I}(t,\omega)\, \mathrm{d}\omega. \tag{9}$$

for the aggregated states. It is easy to see that in the particular case of $\mu(\Omega) = 1$ and $p(\omega) \equiv 1$, the aggregated states $S(t)$ and $I(t)$ follow the dynamics of the homogeneous model (1)–(3). Thus the homogeneous model is a special case of the heterogeneous one. Another way to embed the homogeneous model into the heterogeneous one is to consider a set Ω which is a singleton with an unit atomic measure μ.

Below we formulate the assumptions needed for the subsequent analysis.

Assumptions (A). Ω is a complete Borel measurable space (that is, a Lebesgue space) with a nonnegative measure $\mu \geq 0$ with $\mu(\Omega) = 1$. The initial population is normalized: $S(0) + I(0) = 1$. The function $p : \Omega \to [0, \infty)$

is measurable, bounded, almost everywhere strictly positive, and also normalized: $\int_\Omega p(\omega)\,\mathrm{d}\omega = 1$. The initial data $S_0(\cdot)$ and $I_0(\cdot)$ are non-negative and measurable, both are strictly positive on a set of positive measure, and $I_0(\omega) = 0$ wherever $S_0(\omega) = 0$. The parameters $\sigma > 0$, $\delta > 0$, and λ are real numbers.

Everywhere measurability and integration in ω is meant with respect to the measure μ. The differential equations (4), (5) are considered as ODEs for every ω separately. From Theorem 1 in [28] it follows that given a bounded continuous function $z(t)$, the functions $\bar{S}(t,\cdot), \bar{I}(t,\cdot)$ resulting from the ODE family (4), (5) are measurable for a.e. $t \geq 0$. Moreover, wherever it appears in the sequel, changing the order of integration in t and ω is justified by Fubini's Theorem (Theorem 2.1 in Chapter V of [12]), while changing the order of differentiation in t and integration in ω is justified due to a variant of Lebesgue's Dominated Convergence Theorem, namely Theorem 5.7 in Chapter IV of [12]. Below we shall perform these manipulations without further references.

Remark 1. Notice, that due to (4), if $S_0(\omega) > 0$ for some ω, then $\bar{S}(t,\omega) > 0$ for all $t > 0$. The same also applies to $\bar{I}$. Then, according to (A), we have $S(t) > 0$, $I(t) > 0$ for all $t > 0$, hence $y(t) > 0$. Consequently, also $K(t) > 0$, $J(t) > 0$, $z(t) > 0$ for all $t \geq 0$.

Obviously, the normalization assumptions in (A) are made only for convenience and do not restrict the generality. The same is valid to all positivity assumptions in (A). The assumption that $I_0(\omega) = 0$ wherever $S_0(\omega) = 0$ is natural if the same model is assumed to be also valid before time $t = 0$.

On the other hand, the structure of the model (4)–(8) is somewhat restrictive. First, the definition of the weighted prevalence (resulting from a proportional mixing scenario, in principle) implicitly implies certain restrictions for the interpretation of the h-states. The model can be derived by considering a heterogeneous social contact network (see [26]), which is an often discussed way to model diseases [1, 5, 23, 24]. Other restrictions are due to the encapsulated assumptions of no-recovery and no-fertility (or fertility with infected off-springs only) of the infected population. Moreover, the off-springs of the susceptible individuals "statistically" inherit the h-state distribution of the current susceptible population. All this is the price to pay for the possibility to perform a detailed analytic investigation of the long run evolution of the disease.

While the homogeneous model only consists of a 2-dimensional ODE-system, the dynamics of the heterogeneous system form a generally infinite dimensional system. This makes its analysis more difficult. Another issue with this heterogeneous model is that the initial distributions $S_0(\omega)$ and $I_0(\omega)$ are either partially or completely unknown. Further, it is often difficult or impossible to measure the quantities $\bar{S}(t,\omega)$ or $\bar{I}(t,\omega)$. Therefore, it is desirable to represent the evolution of the aggregated states S and I by ODEs, if possible. It was shown in [27] even for a more general class of models that the evolution of S and I in the expansion phase of the disease is exactly described by a system like (1)–(3) where, however, the incidence rate $\sigma y(t)$ is replaced with an (implicitly defined) nonlinear function of the prevalence y. In the next section we obtain an explicit representation of the dynamics of S and I valid in the time horizon $[0,\infty)$. The approach is similar to that in [18, 19, 20, 25, 26, 27] but the result is more explicit due to the specific features of the model considered here.

3 Aggregation of the heterogeneous model

In this section we obtain an ODE system that describes the evolution of the aggregated states $S(t)$ and $I(t)$ in (9). For shortness we abbreviate $M(t) := K(t) + J(t)$.

Proposition 1. *Let F be the solution of the initial value problem*

$$\dot{F}(t) = 1 - \frac{1}{M(0)} \int_\Omega p(\omega)\, e^{F(t)(\lambda+\delta-\sigma p(\omega))}\, S_0(\omega)\, \mathrm{d}\omega, \quad F(0) = 0, \tag{10}$$

and let us define

$$\rho(t) = \dot{F}(t)\, \frac{\int_\Omega p(\omega) e^{-\sigma F(t)\, p(\omega)} S_0(\omega)\, \mathrm{d}\omega}{\int_\Omega e^{-\sigma F(t)\, p(\omega)} S_0(\omega)\, \mathrm{d}\omega}. \tag{11}$$

Then the aggregated states S and I of system (4)–(8) satisfy the equations

$$\dot{S}(t) = -\sigma\rho(t)\,S(t) + \lambda S(t), \quad S(0) = S_0, \tag{12}$$

$$\dot{I}(t) = \sigma\rho(t)\,S(t) - \delta I(t), \qquad I(0) = I_0. \tag{13}$$

Moreover, the weighted prevalence z in (6) is given by $z(t) = \dot{F}(t)$.

In [18] and [19] G. Karev gives a way to reduce equations of a similar form as equations (4)-(8) to a system of ODEs. However, he deals only with a single population and not, as is done in this paper, with two interacting ones. A. Novozhilov applies Karev's approach to epidemiological models in [25] and [26], but with different equations than discussed here. In [26] he encounters a special case of model (4)-(8) and acknowledges that Karev's approach is not applicable to it. The proof given here is inspired by the method of Karev, but takes into account specific features of equations (4)-(8) that allow a reduction to an ODE system.

Proof For the proof we first define $F(t) := \int_0^t z(\tau)\,\mathrm{d}\tau$ (which will be shown below to satisfy (10)). We begin with deriving some preliminary relations.

Integrating (4) we obtain

$$\begin{aligned}\dot{S}(t) &= \int_\Omega \dot{\bar{S}}(t,\omega)\,\mathrm{d}\omega = -\sigma z(t)\int_\Omega p(\omega)\bar{S}(t,\omega)\,\mathrm{d}\omega + \lambda\int_\Omega \bar{S}(t,\omega)\,\mathrm{d}\omega \\ &= -\sigma z(t)\,K(t) + \lambda S(t).\end{aligned} \tag{14}$$

Similarly we get

$$\dot{I}(t) = \sigma z(t)\,K(t) - \delta I(t). \tag{15}$$

Differentiating (7) yields

$$\begin{aligned}\dot{K}(t) &= \int_\Omega p(\omega)\dot{\bar{S}}(t,\omega)\,\mathrm{d}\omega \\ &= -\sigma z(t)\int_\Omega p(\omega)^2\bar{S}(t,\omega)\,\mathrm{d}\omega + \lambda\int_\Omega p(\omega)\,\bar{S}(t,\omega)\,\mathrm{d}\omega \\ &= \lambda K(t) - \sigma z(t)\int_\Omega p(\omega)^2\bar{S}(t,\omega)\,\mathrm{d}\omega.\end{aligned} \tag{16}$$

Similarly,

$$\dot{J}(t) = -\delta J(t) + \sigma z(t)\int_\Omega p(\omega)^2\bar{S}(t,\omega)\,\mathrm{d}\omega. \tag{17}$$

Then using the Cauchy formula for equation (16) gives

$$\begin{aligned} K(t) &= e^{\lambda t}\left(K(0) - \sigma\int_0^t z(s)\int_\Omega p(\omega)^2\, e^{-\sigma F(s)p(\omega)} S_0(\omega)\,\mathrm{d}\omega\,\mathrm{d}s\right) \\ &= e^{\lambda t}\left(K(0) + \int_\Omega p(\omega)\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} e^{-\sigma F(s)p(\omega)} S_0(\omega)\,\mathrm{d}s\,\mathrm{d}\omega\right) \\ &= e^{\lambda t}\int_\Omega p(\omega)\, e^{-\sigma F(t)p(\omega)} S_0(\omega)\,\mathrm{d}\omega. \end{aligned} \tag{18}$$

Since from (4)

$$\bar{S}(t,\omega) = S_0(\omega)\, e^{-\sigma\int_0^t z(\tau)\,\mathrm{d}\tau p(\omega) + \lambda t},$$

we obtain that

$$S(t) = e^{\lambda t}\int_\Omega e^{-\sigma F(t)p(\omega)} S_0(\omega)\,\mathrm{d}\omega. \tag{19}$$

For $M(t) := K(t) + J(t)$ we have from (16), (17) that

$$\dot{M}(t) = \lambda K(t) - \delta J(t).$$

From here one can represent

$$\begin{aligned} \dot{M}(t) &= \lambda K(t) - \delta J(t) = \lambda(1 - z(t))\, M(t) - \delta z(t)\, M(t) \\ &= M(t)\left(\lambda - \lambda z(t) - \delta z(t)\right). \end{aligned}$$

(This equality is easy to check starting from the last expression upwards.) Solving the above equation we obtain that

$$M(t) = M(0)\, e^{\int_0^t(\lambda - \lambda z(\tau) - \delta z(\tau))\,\mathrm{d}\tau} = M(0)\, e^{\lambda t - (\lambda+\delta)F(t)}. \tag{20}$$

Having relations (14)–(20) at hand, we may finalize the proof as follows.

Let us define

$$\rho(t) := z(t)\frac{K(t)}{S(t)} = \frac{J(t)}{M(t)}\frac{K(t)}{S(t)} = \left(1 - \frac{K(t)}{M(t)}\right)\frac{K(t)}{S(t)}.$$

From (14) and (15) it is evident that S and I satisfy (12), (13) with this function ρ. On the other hand, substituting the expressions (18)–(20) for K, S, and $K+J$, we obtain for ρ the representation (11). To complete the proof it remains to observe that

$$\dot{F}(t) = z(t) = \frac{J(t)}{M(t)} = 1 - \frac{K(t)}{M(t)},$$

and inserting the expressions (18), (20) results in (10). □

Equations (12), (13) for $S(t)$ and $I(t)$ have the same form as for the homogeneous system. The only difference is that in place of the incidence function $y(t)$ we use the function $\rho(t)$. Thus, the whole influence of the heterogeneity in this model is encapsulated in the *aggregated prevalence* $\rho(t)$ defined in (11) through the solution F of (10).

4 Asymptotics

4.1 The homogeneous system

Before we investigate the asymptotics of the heterogeneous system we analyze the homogeneous system (1), (3) in order to see later how the heterogeneity influences the asymptotic behavior.

Lemma 1. *If the number $\kappa := \sigma - \delta - \lambda$ is nonzero, then the solution of system (1), (3) is given by*

$$S(t) = S(0)\left(1 - y(0) + y(0)\, e^{\kappa t}\right)^{-\frac{\sigma}{\kappa}} e^{\lambda t}, \tag{21}$$
$$I(t) = I(0)\left(1 - y(0) + y(0)\, e^{\kappa t}\right)^{-\frac{\sigma}{\kappa}} e^{(\sigma-\delta)t}, \tag{22}$$
$$y(t) = \left(e^{-\kappa t}\left(y(0)^{-1} - 1\right) + 1\right)^{-1}. \tag{23}$$

The last equality is valid also for $\kappa = 0$.

The proof is routine but for completeness it is given in the appendix. Notice that $y(0)^{-1} - 1$ is well defined and positive due to Remark 1.

We split the analysis of the asymptotic behavior of system (1), (3) in 3 cases.

1. First, we consider the special case $\sigma = \lambda + \delta$. Then from Lemma 1, $y(t) \equiv y(0)$.

The solution $S(t)$ of (1) is

$$S(t) = S(0)\, e^{(\lambda - \sigma y(0))t}.$$

Thus, if $\lambda < \sigma y(0)$ both $S(t)$ and $I(t)$ converge to 0 when $t \to \infty$. If $\lambda > \sigma y(0)$ both tend to infinity, and if $\lambda = \sigma y(0)$ then $S(t) = S(0)$ and $I(t) = I(0)$ are constant. Notice that in the last case $I(0) = y(0) = \frac{\lambda}{\sigma}$. Thus

$$\frac{I(0)}{S(0)} = \frac{\frac{\lambda}{\sigma}}{1 - \frac{\lambda}{\sigma}} = \frac{\frac{\lambda}{\sigma}}{\frac{\sigma-\lambda}{\sigma}} = \frac{\lambda}{\delta}$$

and $I(0) = \frac{\lambda}{\delta} S(0)$.

2. Next, consider the case $\sigma > \delta + \lambda$. We have $\lambda - \sigma < -\delta < 0$. Then passing to the limit in (21) and (22) we obtain that both $S(t)$ and $I(t)$ converge to 0. According to (23) the prevalence $y(t)$ converges to 1.

3. Finally, let $\sigma < \delta + \lambda$. Using (21) and (22), we consider the following cases. If $\lambda < 0$ then $S(t) \to 0$ and $I(t) \to 0$ (since $\sigma - \delta < \lambda < 0$). If $\lambda = 0$ then $S(t) \to S^* := S(0)\,(1 - y(0))^{-\frac{\sigma}{\sigma-\delta}} = S(0)^{1-\frac{\sigma}{\sigma-\delta}} = S(0)^{\frac{\delta}{\delta-\sigma}}$, while $I(t) \to 0$. If $\lambda > 0$ then $S(t) \to \infty$, while the behaviour of $I(t)$ depends on $\sigma - \delta$. In this case, if $\sigma > \delta$ then $I(t) \to \infty$, if $\sigma < \delta$ then $I(t) \to 0$, and if $\sigma = \delta$ then $I(t) \to I^* := I(0)\,(1 - y(0))^{\frac{\sigma}{\lambda}} = I(0)\,(1 - I(0))^{\frac{\sigma}{\lambda}} = I(0)S(0)^{\frac{\sigma}{\lambda}}$. In all of these cases the prevalence $y(t)$ converges to 0 due to (23).

We summarize these results in Table 1, where we give the steady state to which $(S(t), I(t))$ and the prevalence $y(t)$ converge, depending on the parameter. Conventionally, we consider also ∞ as a steady state.

A few remarks follow. If $\lambda + \delta = \sigma$ the prevalence is constant, which is particularly interesting in the case where both $S(t)$ and $I(t)$ tend to infinity.

If $\lambda + \delta < \sigma$, then the prevalence tends to 1, which means that the susceptible individuals become infected "faster" than the infected ones die.

On the other hand, if $\lambda + \delta > \sigma$, then the prevalence goes to zero. This is again particularly interesting in the case where both $S(t)$ and $I(t)$ tend to infinity. In this case, although the total number of infected individuals is unbounded, they eventually make up only a negligible fraction of the total population.

Case	Subcases	Asymptotics of (S, I)	Prevalence
$\lambda + \delta = \sigma$	$\lambda < \sigma y(0)$	$(0, 0)$	$y(0)$
	$\lambda = \sigma y(0)$	$\big(S(0), \frac{\lambda}{\delta} S(0)\big)$	
	$\lambda > \sigma y(0)$	(∞, ∞)	
$\lambda + \delta < \sigma$	—	$(0, 0)$	1
$\lambda + \delta > \sigma$	$\lambda < 0$	$(0, 0)$	0
	$\lambda = 0$	$(S^*, 0)$	
	$\lambda > 0, \frac{\sigma}{\delta} < 1$	$(\infty, 0)$	
	$\lambda > 0, \frac{\sigma}{\delta} = 1$	(∞, I^*)	
	$\lambda > 0, \frac{\sigma}{\delta} > 1$	(∞, ∞)	

Table 1: Asymptotic behaviour of the homogeneous system.

4.2 The heterogeneous system

Now, we investigate the asymptotics of the aggregated states $S(t)$ and $I(t)$ of the heterogeneous system (4), (5), making use of Proposition 1. Also the asymptotics of the weighted prevalence $z(t)$ and the prevalence $y(t)$ will be obtained.

Notice that due to Remark 1 we have $z(t) > 0$, which implies that $F(t) = \int_0^t z(\tau)\,\mathrm{d}\tau$ is strictly increasing.

Define the following sets and numbers:

$$\Omega_+ := \{w \in \Omega : \lambda + \delta > \sigma p(\omega)\}, \qquad S^+(0) := \int_{\Omega_+} S_0(\omega)\,\mathrm{d}\omega,$$

$$\Omega_- := \{w \in \Omega : \lambda + \delta < \sigma p(\omega)\}, \qquad S^-(0) := \int_{\Omega_-} S_0(\omega)\,\mathrm{d}\omega,$$

$$\Omega_0 := \{w \in \Omega : \lambda + \delta = \sigma p(\omega)\}, \qquad S^0(0) := \int_{\Omega_0} S_0(\omega)\,\mathrm{d}\omega.$$

Let us abbreviate $\varphi(t, \omega) := e^{F(t)(\lambda+\delta-\sigma p(\omega))}$. According to Proposition 1 we have

$$\dot{F}(t) = 1 - \frac{1}{M(0)} \left[\int_{\Omega+} p(\omega)\,\varphi(t,\omega)\,S_0(\omega)\,\mathrm{d}\omega + \int_{\Omega-} p(\omega)\,\varphi(t,\omega)\,S_0(\omega)\,\mathrm{d}\omega + K^0(0) \right], \quad (24)$$

where

$$K^0(0) := \int_{\Omega_0} p(\omega)\, S_0(\omega)\, \mathrm{d}\omega.$$

Again from Proposition 1 we have

$$\rho(t) = z(t)\, \psi(F(t)), \quad \text{where} \;\; \psi(x) := \frac{\int_\Omega p(\omega) e^{-\sigma x\, p(\omega)} S_0(\omega)\, \mathrm{d}\omega}{\int_\Omega e^{-\sigma x\, p(\omega)} S_0(\omega)\, \mathrm{d}\omega}. \tag{25}$$

A remarkable property of the function ψ is that it is monotonically decreasing, which consequently also applies to $\psi(F(t))$. Indeed, if for any fixed $x \geq 0$ we denote $\Phi(\omega) := e^{-\sigma x\, p(\omega)} S_0(\omega)$, we can represent

$$\psi'(x) = -\sigma \frac{\int_\Omega (p(\omega))^2\, \Phi(\omega)\, \mathrm{d}\omega\, \int_\Omega \Phi(\omega)\, \mathrm{d}\omega - \left(\int_\Omega p(\omega)\, \Phi(\omega)\, \mathrm{d}\omega\right)^2}{\left(\int_\Omega \Phi(\omega)\, \mathrm{d}\omega\right)^2} \leq 0,$$

where we use the known inequality $\int p^2\, \Phi \int \Phi \geq \left(\int p\Phi\right)^2$ for the integral moments of Φ. As a consequence, $\psi^* := \lim_{t\to\infty} \psi(F(t))$ exists and

$$\lim_{t\to\infty} \rho(t) = \lim_{t\to\infty} z(t)\, \psi^*, \tag{26}$$

provided that $\lim_{t\to\infty} z(t)$ does exist.

Another preliminary assertion is that the prevalence $y(t) = \frac{I(t)}{S(t)+I(t)}$ satisfies the equation

$$\begin{aligned}
\dot{y}(t) &= \frac{\dot{I}(t)S(t) - I(t)\dot{S}(t)}{(S(t)+I(t))^2} = \frac{\sigma\rho(t)S(t)^2 - \delta I(t)S(t) - (\lambda - \sigma\rho(t))S(t)I(t)}{(S(t)+I(t))^2} \\
&= \sigma\, \rho(t)\, (1-y(t))^2 - (\lambda + \delta - \sigma\rho(t))\, y(t)\, (1-y(t)) \\
&= (1-y(t))\, (\sigma\rho(t) - (\lambda+\delta)\, y(t))\,.
\end{aligned} \tag{27}$$

We split the analysis of the asymptotic behavior of the heterogeneous system in four basic cases determined by the numbers $S^+(0)$, $S^-(0)$, and $S^0(0)$.

Case 1. $S^+(0) = S^-(0) = 0$.

In this case $S_0(\omega) = 0$ for almost every $\omega \in \Omega_+ \cup \Omega_-$, therefore $p(\omega) = (\lambda+\delta)/\sigma$ for almost every ω for which $S_0(\omega)$ is not zero. Thus, without restricting the generality we can assume that $p(\omega) = (\lambda+\delta)/\sigma$ is constant everywhere, as changing $p(\omega)$ for those ω where $S_0(\omega)$ (and thus $I_0(\omega)$) is zero does not influence the system. This turns the heterogeneous system into a homogeneous one, where σ is replaced by $\tilde{\sigma} := \sigma p(\omega) = \sigma(\lambda+\delta)/\sigma = \lambda+\delta$. The asymptotics in case 1 is presented in the first group of cases in Table 1.

Next we consider the following two cases:

Case 2. $S^+(0) = 0$, $S^-(0) > 0$, $S^0(0) > 0$,
Case 3. $S^+(0) = 0$, $S^-(0) > 0$, $S^0(0) = 0$.

In both cases we have from (24)

$$\dot{F}(t) = 1 - \frac{1}{M(0)} \left(\int_{\Omega_-} p(\omega)\, \varphi(t,\omega)\, S_0(\omega)\, \mathrm{d}\omega + K^0(0) \right). \tag{28}$$

If we assume that $F(\cdot)$ is bounded, then due to its monotonicity it would have a limit, $F_* \geq 0$. Then, having in mind the definition of φ, we obtain that

$$0 = \lim_{t\to\infty} \dot{F}(t) = 1 - \frac{1}{M(0)} \left(\int_{\Omega_-} p(\omega)\, e^{F_*(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\, \mathrm{d}\omega + K^0(0) \right).$$

Since the exponent does not exceed 1, we have

$$0 \geq 1 - \frac{1}{M(0)} \left(\int_{\Omega_-} p(\omega)\, S_0(\omega)\, \mathrm{d}\omega + K^0(0) \right) = 1 - \frac{K(0)}{M(0)} = \frac{J(0)}{M(0)},$$

which is a contradiction (see Remark 1). Thus $F(t) \to \infty$.

Using this last fact we obtain that $\varphi(t,\omega) \to 0$ for $\omega \in \Omega_-$, which implies that the integral in (28) converges to zero for $t \to \infty$. Therefore,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \dot{F}(t) = 1 - \frac{K^0(0)}{M(0)}. \tag{29}$$

Here we split the analysis of the two cases. We deal with Case 2 first.
In order to find the limit of $\rho(t)$ we use that $S_0(\omega) = 0$ for almost every $\omega \in \Omega_+$ and that $p(\omega) = (\lambda+\delta)/\sigma$ on Ω_0. Then

$$\begin{aligned} \psi^* &= \lim_{t\to\infty} \frac{\int_{\Omega_-} p(\omega) e^{-\sigma F(t)\, p(\omega)} S_0(\omega)\, \mathrm{d}\omega + \frac{\lambda+\delta}{\sigma} e^{-(\lambda+\delta)F(t)}\, S^0(0)}{\int_{\Omega_-} e^{-\sigma F(t)\, p(\omega)} S_0(\omega)\, \mathrm{d}\omega + e^{-(\lambda+\delta)F(t)}\, S^0(0)} \\ &= \lim_{x\to\infty} \frac{\int_{\Omega_-} p(\omega) e^{(\lambda+\delta-\sigma p(\omega))x}\, S_0(\omega)\, \mathrm{d}\omega + \frac{\lambda+\delta}{\sigma} S^0(0)}{\int_{\Omega_-} e^{(\lambda+\delta-\sigma p(\omega))x}\, S_0(\omega)\, \mathrm{d}\omega + S^0(0)}. \end{aligned}$$

Since $\lambda + \delta - \sigma p(\omega) < 0$ on Ω_-, the two integrals in the last expression converge to zero. We therefore have $\psi^* = (\lambda + \delta)/\sigma$, and due to (26) and (29)

$$\rho^* := \lim_{t\to\infty} \rho(t) = \left(1 - \frac{K^0(0)}{M(0)}\right) \frac{\lambda + \delta}{\sigma} > 0, \tag{30}$$

where the last inequality follows from the fact that $S^-(0) > 0$.

In order to investigate the asymptotics of S we use equation (12) to get

$$\begin{aligned} S(t) &= e^{\int_0^t (\lambda - \sigma\rho(\tau))\,\mathrm{d}\tau} S(0) = e^{t\left(\lambda - \frac{\sigma}{t}\int_0^t \rho(\tau)\,\mathrm{d}\tau\right)} S(0) \\ &= e^{t\left(\lambda - \sigma\rho^* + \frac{\sigma}{t}\int_0^t (\rho^* - \rho(\tau))\,\mathrm{d}\tau\right)} S(0). \end{aligned}$$

Since $\frac{\sigma}{t}\int_0^t (\rho^* - \rho(\tau))\,\mathrm{d}\tau$ converges to zero when $t \to \infty$, we obtain that

$$\lim_{t\to\infty} S(t) = \begin{cases} 0 & \text{if } \lambda < \sigma\rho^*, \\ \infty & \text{if } \lambda > \sigma\rho^*. \end{cases}$$

It is obvious from (13) that $I(t) \to 0$ for $\lambda < \sigma\rho^*$ and $I(t) \to \infty$ for $\lambda > \sigma\rho^*$. The "critical" case $\lambda = \sigma\rho^*$ requires an additional consideration. We state the result in the following lemma. The proof is given in the appendix.

Lemma 2. *If $\lambda = \sigma\rho^*$ in Case 2, then $S(t)$ converges to some $S^* > 0$ if $\Lambda := \int_{\Omega_-} \frac{S_0(\omega)}{\sigma p(\omega) - \lambda - \delta}\,\mathrm{d}\omega < \infty$. Otherwise $S(t) \to \infty$.*

If Λ is finite then we can use $\rho(t)S(t) \to \rho^* S^*$ and (13) to show that $I(t) \to \frac{\sigma\rho^*}{\delta} S^* = \frac{\lambda}{\delta} S^*$. If $\Lambda = \infty$ then $\rho(t)S(t) \to \infty$ and thus also $I(t) \to \infty$.

To analyse the prevalence we first consider the case $\lambda + \delta \le 0$. Here it is clear from (27) that $y(t) \to 1$ due to $\rho^* > 0$. If $\lambda + \delta > 0$ we can rewrite (27) as

$$\dot{y}(t) = (\lambda + \delta)(1 - y(t)) \left(\frac{\sigma\rho(t)}{\lambda + \delta} - y(t)\right).$$

If $\lambda + \delta < \sigma\rho^*$ then $\frac{\sigma\rho(t)}{\lambda+\delta} > 1$ for large enough t. Then obviously $y(t) \to 1$. If, on the other hand, $\lambda + \delta \ge \sigma\rho^*$, then $\dot{y}(t)$ is positive if $y(t)$ is smaller than $\frac{\sigma\rho(t)}{\lambda+\delta}$ and negative if it is bigger than that value. From this it is easy to see that $y(t) \to \frac{\sigma\rho^*}{\lambda+\delta}$. Note that these results can be summarised by the formula $y(t) \to \max\left\{\frac{\lambda+\delta}{\sigma\rho^*}, 1\right\}^{-1}$.

We now come to analyzing Case 3.
Here we easily obtain (using (29) and $K^0(0) = 0$) that $\sigma\rho^* = \sigma\psi^* \geq \sigma \inf_{\omega\in\Omega_-} p(\omega) \geq \lambda + \delta$. As above we use equation (12) to get

$$S(t) = e^{\int_0^t (\lambda - \sigma\rho(\tau))\,\mathrm{d}\tau} S(0) = e^{t\left(\lambda - \sigma\rho^* + \frac{\sigma}{t}\int_0^t (\rho^* - \rho(\tau))\,\mathrm{d}\tau\right)} S(0)$$

where $\frac{\sigma}{t}\int_0^t (\rho^* - \rho(\tau))\,\mathrm{d}\tau$ converges to zero. Because of $\lambda < \lambda + \delta \leq \sigma\rho^*$ we get $S(t) \to 0$ and consequently also $I(t) \to 0$.

When analysing the prevalence we again see that for $\lambda + \delta < 0$ we can use (27) to get $y(t) \to 1$. If $\lambda + \delta = 0$ an additional argument is needed, which is given in the appendix.

Lemma 3. *If $\lambda + \delta = 0$ in Case 3 then $y(t) \to 1$.*

For $\lambda + \delta > 0$ we can again use the same reasoning as in Case 2 above. Then, due to $\lambda + \delta \leq \sigma\rho^*$ we always have $y(t) \to 1$.

Case 4. $S^+(0) > 0$.

If we assume that $F(t) \to \infty$, then the second integral in (24) converges to zero, while the first integral converges to $+\infty$. Since $\dot{F}(t) = z(t) \geq 0$, this is a contradiction. Thus $F(t)$ is bounded, and since it is monotonically increasing and strictly positive it has a limit $F^* > 0$. In particular,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \dot{F}(t) = 0.$$

Then according to (26)

$$\lim_{t\to\infty} \rho(t) = \lim_{t\to\infty} z(t)\,\psi^* = 0.$$

Now we investigate the limit F^* of $F(t)$. Passing to the limit in (10) we obtain that F^* satisfies the equation

$$\frac{1}{M(0)} \int_\Omega p(\omega)\, e^{F^*(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\,\mathrm{d}\omega = 1. \tag{31}$$

The next lemma sates that this equation uniquely determines the value F^*.

Lemma 4. *Equation (31) has a unique positive solution.*

Proof. The existence of a solution was obtained above.

Consider the function

$$g(x) := \frac{1}{K(0)+J(0)} \int_\Omega p(\omega)\, e^{x(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\,\mathrm{d}\omega, \quad x \geq 0.$$

Since the function $x \mapsto e^{ax}$ is strictly convex for any $a \neq 0$ and is convex for $a = 0$, and since $p(\omega)\, S_0(\omega) \geq 0$ and the inequality is strict on a subset (of Ω_+) of positive measure, we have that g is strictly convex. Since $g(0) = K(0)/(K(0)+J(0)) < 1$, we conclude that F^* is the unique positive solution of (31). □

Now, we investigate the asymptotics. For $S(t)$ this is done easily enough. Using (12) and $\rho(t) \to 0$ we have $S(t) \to 0$ for $\lambda < 0$ and $S(t) \to \infty$ for $\lambda > 0$. If $\lambda = 0$ we get from (19)

$$S(t) = \int_\Omega e^{-\sigma F(t) p(\omega)} S_0(\omega)\,\mathrm{d}\omega \to \int_\Omega e^{-\sigma F^* p(\omega)} S_0(\omega)\,\mathrm{d}\omega =: S^\star. \tag{32}$$

So we have for $\lambda \leq 0$ that $S(t)$ converges to a finite value. Thus, $\rho(t)S(t) \to 0$ which implies $I(t) \to 0$. If $\lambda > 0$, deriving the asymptotics requires additional work. We state the results in the following lemma and refer to the appendix for the proof.

Lemma 5. *Let $\lambda > 0$ in Case 4. Define*

$$\Theta = \frac{M(0)}{\int_\Omega (p(\omega))^2 e^{F^*(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\,\mathrm{d}\omega}.$$

Then

$$\lim_{t\to\infty} I(t) = \begin{cases} 0 & \text{if } \frac{\sigma}{\delta} < \Theta, \\ I^\star & \text{if } \frac{\sigma}{\delta} = \Theta, \\ \infty & \text{if } \frac{\sigma}{\delta} > \Theta, \end{cases}$$

with $I^\star > 0$.

For the prevalence note that in the considered case Ω_+ has a positive measure. From $\lambda + \delta > \sigma p(\omega) > 0$ a.e. on Ω_+ we obtain that $\lambda + \delta > 0$. Since $\rho(t) \to 0$, we see from equation (27) that $y(t) \to 0$.

4.3 Summary and comparison

Table 2 gives for all cases and parameter configurations the asymptotic state (finite or infinite) to which the aggregated solution $(S(t), I(t))$ of the heterogeneous system (4), (5) and the corresponding prevalence $y(t)$ converge. We also use ρ^* defined by (30), S^* and Λ as given in Lemma 2, $S^\star$ as defined in (32), and $I^\star$ and Θ referred to in Lemma 5.

Case	Subcases	Asymptotics of (S, I)	Prevalence
$S^+(0) = S^-(0) = 0$: homogeneous case with $\lambda + \delta = \sigma$		see Table 1	see Table 1
$S^+(0) = 0,\ S^-(0) > 0,\ S^0(0) > 0$	$\lambda < \sigma\rho^*$	$(0, 0)$	$\frac{1}{\max\left\{\frac{\lambda+\delta}{\sigma\rho^*}, 1\right\}}$
	$\lambda = \sigma\rho^*$, $\Lambda < \infty$	$\left(S^*, \frac{\lambda}{\delta}S^*\right)$	
	$\lambda = \sigma\rho^*$, $\Lambda = \infty$	(∞, ∞)	
	$\lambda > \sigma\rho^*$	(∞, ∞)	
$S^+(0) = 0,\ S^-(0) > 0,\ S^0(0) = 0$	—	$(0, 0)$	1
$S^+(0) > 0$	$\lambda < 0$	$(0, 0)$	0
	$\lambda = 0$	$(S^\star, 0)$	
	$\lambda > 0, \frac{\sigma}{\delta} < \Theta$	$(\infty, 0)$	
	$\lambda > 0, \frac{\sigma}{\delta} = \Theta$	$(\infty, I^\star)$	
	$\lambda > 0, \frac{\sigma}{\delta} > \Theta$	(∞, ∞)	

Table 2: Asymptotic behaviour of the heterogeneous system

We see some obvious similarities between Table 1 and Table 2. Case 1 is of course itself a homogeneous system with $\lambda + \delta = \sigma$, but Case 2 is also very similar to the same homogeneous system. Once the initial incidence $y(0)$ is replaced by the final aggregated prevalence ρ^* in the differentiation of the sub-cases, the asymptotics are nearly the same. The difference only is that in the critical case $\lambda = \sigma\rho^*$ the heterogeneous system does not necessarily converge to a finite state. Also different is the asymptotic prevalence, which is constant in the homogeneous case, but may take different values in the heterogeneous model.

The cases $\lambda + \delta < \sigma$ and $\lambda + \delta > \sigma$ can be compared to the Cases 3 and 4 respectively. Note that while $S^+(0) = S^0(0) = 0$ implies that $\lambda + \delta < \sigma p(\omega)$ for a.e. ω for which $S_0(\omega) \neq 0$ and the comparison with the case $\lambda + \delta < \sigma$ of the homogeneous system is obvious, in Case 4 the inequality $\lambda + \delta < \sigma p(\omega)$

may still hold for a large portion of $\omega \in \Omega$ as $\mu(\Omega_+)$ can be arbitrarily small. Comparing the subcases of Case 4 with those of $\lambda + \delta > \sigma$ of the homogeneous system on the other hand shows a very close connection. The only difference here is that the threshold value for $\frac{\sigma}{\delta}$ that determines whether the infected population dies out or not is changed from 1 to Θ.

The main difference is however that once the parameters λ, σ, δ, and for the heterogeneous system $p(\omega)$, and the initial conditions $S(0)$ and $I(0)$ are fixed, the asymptotic behaviour of the homogeneous system is completely determined. In the heterogeneous system, however, the initial distribution $S_0(\omega)$ can still greatly influence the behaviour. Not only is $S_0(\omega)$ crucial in the definition of the values $S^+(0)$, $S^-(0)$, and $S^0(0)$ and thus in determining which case is at hand, but it (along with $I_0(\omega)$) also plays a role in the definition of ρ^* and Θ, so a difference in the initial distributions may result in a different asymptotic behaviour even though the different initial distributions stay within the same case.

One further aspect that is of interest is the question whether it makes a difference if the disease has a long history at time $t = 0$ (at which the data are given) when comparing the results of the homogeneous and heterogeneous systems. From (4) we get

$$\bar{S}(t,\omega) = S_0(\omega)\, e^{-\sigma \int_0^t z(\tau)\, \mathrm{d}\tau p(\omega) + \lambda t} = S_0(\omega)\, e^{-\sigma F(t) p(\omega) + \lambda t}.$$

Thus the proportion between the population sizes of two different h-states is given by

$$\frac{\bar{S}(t,\omega_1)}{\bar{S}(t,\omega_2)} = \frac{S_0(\omega_1)}{S_0(\omega_2)} e^{\sigma F(t)(p(\omega_2) - p(\omega_1))}.$$

We see that in the case $S^+(0) > 0$, where we have $F(t) \to F^*$, this proportion converges to some constant. This shows that the population retains a certain level of heterogeneity for all time.

In the cases where $S^+(0) = 0$ and $S^-(0) > 0$ however we have that $F(t) \to \infty$. Thus this proportion goes to zero whenever $p(\omega_1) > p(\omega_2)$. This implies that asymptotically only the h-state with the lowest value of $p(\omega)$ survives. Thus, in a sense, the population becomes more homogeneous.

This, however, does not mean that it can be more closely described by the original homogeneous system. It is more appropriate to say that the homogeneous system more closely resembles the heterogeneous one if σ is changed to σp^* where p^* is the infimum of $p(\omega)$ on Ω or, if this infimum is zero, to a sufficiently small number.

We emphasize that the above results show what information is needed to determine the ultimate state of the system. In particular we see that the final state can in some cases be determined without full knowledge of the initial distribution. For example, if $S^+(0) = S^0(0) = 0$ and $S^-(0) > 0$ then we know that $(S(t), I(t))$ will converge to $(0, 0)$ and the prevalence converges to 1. But this condition is just another way of saying that the set of $\omega \in \Omega$ where both $\lambda + \delta \geq \sigma p(\omega)$ and $S_0(\omega) > 0$ has measure zero. The only information about the function $S_0(\omega)$ that is needed here is its support. And even this information might not be needed in detail (e.g. in the simple case where $\lambda + \delta < \sigma p(\omega)$ for all $\omega \in \Omega$).

In some cases all information about $S_0(\omega)$ is needed. For example, in the case $S^+(0) > 0$, $\lambda > 0$, and $\sigma/\delta = \Theta$ the value Θ can only be calculated with full knowledge of $S_0(\omega)$, and $I^\star$ is only calculable with knowledge of $F(t)$ for all t, for which again $S_0(\omega)$ is needed in detail. However, to verify inequalities like $\sigma/\delta < \Theta$, incomplete information might in some cases suffice.

4.4 Numerical examples

The homogeneous SI-model is amongst the simplest epidemiological models. Consequently it is used only as a well understood starting point for the analysis of more complex models. For example, early attempts to understand the transmission of HIV started out by using simple SI-models (see e.g. [7, 22]). Some more recent models for HIV are still recognisable as SI-model, albeit more sophisticated ones using age-structured populations (e.g. [21]) or multiple stages of infection (e.g. [14]). Some diseases in animals have also been modelled using variants of SI-models (e.g. [2, 17]).

Since SI-models are mostly used as baseline models, we will not attempt

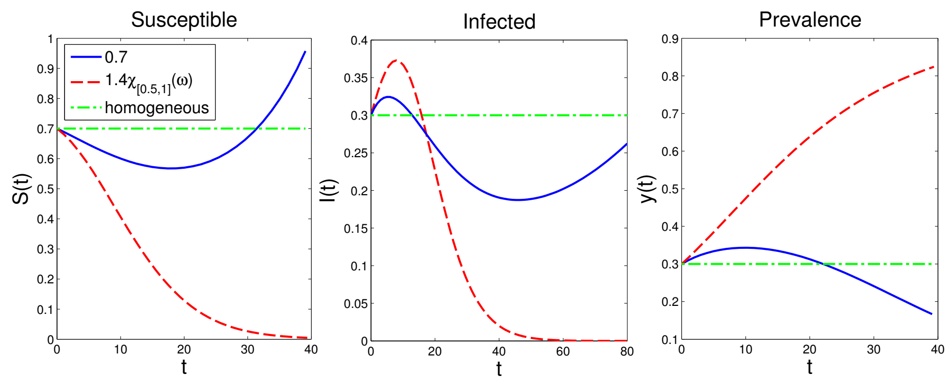

Figure 1: Comparison of the trajectories of the system for different choices of $S_0(\omega)$ and the homogeneous system. The solid lines show the results for a constant initial distribution $S_0(\omega) = 0.7$, the dashed lines for $S_0(\omega) = 1.4\chi_{[0.5,1]}(\omega)$. The dash-dotted line represents the solution for the homogeneous system.

to capture the exact dynamics of a specific disease. Rather we give some numerical examples to illustrate the effect that different initial distributions $(S_0(\omega), I_0(\omega))$ can have on the asymptotics, although we choose distributions that yield the same cumulative values, i.e. the initial conditions for the aggregated system (12)-(13) stay the same. Although the parameters used here are of a magnitude suitable for modelling HIV[1], the specific values have been chosen to highlight the differences in behaviour due to different choices of the initial conditions.

We first want to show how a different choice of $S_0(\omega)$ can influence the system. As Ω we take the interval $[0, 1]$ and for μ the Lebesgue measure. We set $I_0(\omega) = 0.6\chi_{[0.5,1]}$ where $\chi_{[0.5,1]}$ is the indicator function of the interval $[0.5, 1]$. Further we set $p(\omega) = \frac{1}{2} + \omega$, $\sigma = 0.3$, $\delta = 0.21$, and $\lambda = 0.09$. Note that $\lambda + \delta = \sigma$ and $\lambda = \sigma y(0)$, which means that if the population were homogeneous both $S(t)$ and $I(t)$ would be constant. It is easy to see that with our choice of parameters $\Omega_+ = [0, 0.5)$ and $\Omega_- = (0.5, 1]$. The

[1]The force of infection has been studied for example in [6], while parameters for population growth or mortality rates can be found in the internet data repository of the World Health Organisation at `www.who.int`

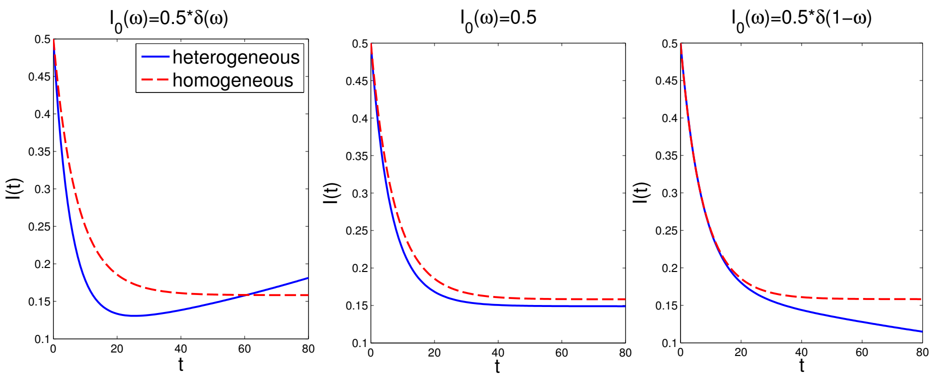

Figure 2: Comparison between the trajectories of $I(t)$ for different choices of $I_0(\omega)$ and the homogeneous system. The solution to the heterogeneous system is given by the solid line while the solution for the homogeneous system is represented by the dashed line.

set Ω_0 consists only of one point and, since $p(\omega)$ is strictly monotonically increasing, is a μ-null set.

Now we look at two different initial distributions $S_0(\omega)$. On the one hand, we consider $S_0(\omega) = 0.7$, on the other $S_0(\omega) = 1.4\chi_{[0.5,1]}(\omega)$. Obviously both cases yield $S(0) = 0.7$. But in the first case $S^+(0) > 0$ while in the second case $S^+(0) = 0$. The results can be seen in Figure 1. When $S^+(0) > 0$ the number $S(t)$ of susceptible individuals goes to infinity and since $\frac{\sigma}{\delta} > \Theta$ for our choice of parameters ($\Theta \approx 1.3091$, while $\frac{\sigma}{\delta} \approx 1.4286$), so does $I(t)$. The prevalence, however, still goes to zero. When, on the other hand, we take $S_0(\omega) = 1.4\chi_{[0.5,1]}(\omega)$, we see that both the populations of susceptible and infected individuals go to 0, while the prevalence increases towards 1. As mentioned above, both $S(t)$ and $I(t)$ are constant for the homogeneous system.

A second aspect we want to raise, is the influence of the choice of $I_0(\omega)$. Since $S(0) + I(0) = 1$, the value of $I(0)$ is fixed once $S_0(\omega)$ is chosen. However, the choice of $I_0(\omega)$ still influences $J(0)$. To illustrate the effects different values of $J(0)$ can have on the system we consider the following parameters: Ω, μ, and $p(\omega)$, are chosen as above, while we set $\sigma = 0.2$ and $\delta = 0.2$. We take $S_0(\omega) = 0.5$ to be constant. Thus we are in the case

$S^+(0) > 0$ and $S(t)$ always goes towards infinity as long as λ is positive. We now choose λ such that for $I_0(\omega) = 0.5$ we have $\frac{\sigma}{\delta} = \Theta$, which yields $\lambda \approx 0.1145$. Note that $\lambda + \delta > \sigma$, $\lambda > 0$, and $\sigma = \delta$. Thus, in a homogeneous population the population of susceptibles goes to infinity while the population of infected individuals converges. In our heterogeneous case $I(t)$ also converges. We then change $I_0(\omega)$ while keeping all other parameters fixed. We consider the two choices $I_0(\omega) = \frac{1}{2}\delta(\omega)$ and $I_0(\omega) = \frac{1}{2}\delta(1-\omega)$ where $\delta(x)$ is the Dirac delta distribution. Thus $I(0) = 0.5$ for all of these choices while $J(0) = 0.25$ for $I_0(\omega) = \frac{1}{2}\delta(\omega)$ and $J(0) = 0.75$ for $I_0(\omega) = \frac{1}{2}\delta(1-\omega)$ compared to $J(0) = I(0) = 0.5$ for constant $I_0(\omega)$. The results are shown in Figure 2. Since $S(t)$ goes to infinity in all of the cases, we restrict ourselves to giving the results for the infected population only. For $I_0(\omega) = 0.5$ we can see the convergence of $I(t)$. If $I_0(\omega) = \frac{1}{2}\delta(\omega)$ then $\frac{\sigma}{\delta} > \Theta$ and $I(t)$ goes to infinity, while for $I_0(\omega) = \frac{1}{2}\delta(1-\omega)$ we get $\frac{\sigma}{\delta} < \Theta$ and $I(t)$ goes towards 0.

5 Basic reproduction number

In this section we consider both the homogeneous and the heterogeneous model assuming that $\lambda = 0$, which means that the population has a fixed size if a disease is not present.

We use the definition of the basic reproduction number R_0 as given in [10] where it is defined as the expected number of secondary cases produced, in a completely susceptible population, by a typical infected individual during its entire period of infectiousness. The importance of this number lies in the threshold criterion which says that a disease can invade the population if $R_0 > 1$ and can not invade if $R_0 < 1$.

In the homogeneous model the basic reproduction number R_0^{hom} is given by $\frac{\sigma}{\delta}$ (see for example [9]). This is due to the fact that a single infected individual in an otherwise completely susceptible population has on average σ infectious contacts per unit of time and a life expectancy of $\frac{1}{\delta}$. In order to define a basic reproduction number for the heterogeneous system we use the following result obtained in [10].

Theorem 1. *Let $S(\omega)$ denote the density function of susceptibles describing the steady demographic state in the absence of the disease. Let $A(\tau, \zeta, \omega)$ be the expected infectivity of an individual which was infected τ units of time ago, while having h-state ω towards a susceptible which has h-state ζ. Assume that*

$$\int_0^\infty A(\tau, \zeta, \omega)\, \mathrm{d}\tau = a(\zeta)b(\omega).$$

Then the basic reproduction number R_0 for the heterogeneous system is given by

$$R_0 = \int_\Omega a(\omega)b(\omega)S(\omega)\, \mathrm{d}\omega.$$

In our heterogeneous system the initial condition $S_0(\omega)$ is, in the absence of any infected individuals, a steady state due to $\lambda = 0$. To derive an expression for the infectivity $A(\tau, \zeta, \omega)$ we note that σ denotes the strength of infection. The value $p(\omega)$ influences the number of contacts an individual has. Then the chance of an infectious contact between the infective ω individual and a specific ζ individual is given by $\sigma p(\omega) \frac{p(\zeta)}{\int_\Omega p(\xi)S_0(\xi)\, \mathrm{d}\xi}$. The infectivity of an individual is constant for its whole lifespan. In the absence of susceptible individuals the equation for the infected is given by $\dot{I}(t) = -\delta I(t)$, which suggests that the probability that an infected individual is still alive at time t is given by $e^{-\delta t}$. Thus $A(\tau, \zeta, \omega)$ is given by $\sigma p(\omega) \frac{p(\zeta)}{\int_\Omega p(\xi)S_0(\xi)\, \mathrm{d}\xi} e^{-\delta\tau}$. We have

$$\int_0^\infty A(\tau, \zeta, \omega)\, \mathrm{d}\tau = \int_0^\infty \sigma p(\omega) \frac{p(\zeta)}{\int_\Omega p(\xi)S_0(\xi)\, \mathrm{d}\xi} e^{-\delta\tau}\, \mathrm{d}\tau = \frac{\sigma}{\delta} \frac{p(\zeta)}{\int_\Omega p(\xi)S_0(\xi)\, \mathrm{d}\xi} p(\omega)$$

and thus

$$R_0 = \frac{\sigma}{\delta} \frac{\int_\Omega p(\omega)^2 S_0(\omega)\, \mathrm{d}\omega}{\int_\Omega p(\omega)S_0(\omega)\, \mathrm{d}\omega}.$$

Note that if $p(\omega) \equiv 1$, i.e. we are dealing with the homogeneous system, then $R_0 = \frac{\sigma}{\delta}$ as before.

Assume now that a fraction of the population is already infected. Since $\lambda = 0$, the population of infected individuals dies out in both models. Also, the population of susceptibles is strictly decreasing. An important question is whether the population of susceptible individuals also goes extinct or if some individuals remain uninfected.

For the homogeneous model this question can be answered immediately from Table 1.

Proposition 2. *If the basic reproduction number* $R_0^{hom} = \frac{\sigma}{\delta}$ *of the homogeneous model satisfies the inequality* $R_0^{hom} \geq 1$ *then the population of susceptible individuals dies out. If* $R_0^{hom} < 1$ *then a part,* $S^* = S(0)^{\frac{\delta}{\delta - \sigma}}$*, of the initial population stays uninfected.*

Note that S^* as given here is the same as in Table 1.

Similarly we can use Table 2 to get the following result for the heterogeneous model.

Proposition 3. *Define*

$$R^* = \frac{\sigma}{\delta} \operatorname*{essinf}_{\omega \in \Gamma} p(\omega) = \frac{\sigma}{\delta} \sup_{\substack{B \subseteq \Gamma \\ \mu(B)=0}} \inf_{\omega \in \Gamma \setminus B} p(\omega),$$

where $\Gamma = \{\omega \in \Omega : S_0(\omega) \neq 0\}$*. If* $R^* \geq 1$ *then the population of susceptible individuals dies out. If* $R^* < 1$ *then a part,* $S^\star = \int_\Omega e^{-\sigma F^* p(\omega)} S_0(\omega) \, \mathrm{d}\omega$*, of the initial population stays uninfected, where* F^* *is the unique positive solution of the equation*

$$\int_\Omega p(\omega) e^{F^*(\delta - \sigma p(\omega))} S_0(\omega) \, \mathrm{d}\omega = M(0).$$

Proof If $R^* \geq 1$ then $\frac{\sigma}{\delta} p(\omega) \geq 1$ almost everywhere on Γ. This means that $\mu(\Gamma \cap \Omega_+) = 0$ which implies $S^+(0) = 0$. If $R^* < 1$ then there is a set $A \subseteq \Gamma$ with positive measure on which $\frac{\sigma}{\delta} p(\omega) < 1$. Hence, $\mu(\Gamma \cap \Omega_+) > 0$ and $S^+(0) > 0$. Then the conclusions of the proposition in both cases follow from Table 2. The statement about F^* is proven in Lemma 4. □

Note that $R_0 = \frac{\sigma}{\delta}\frac{\int_\Omega p(\omega)^2 S_0(\omega)\,\mathrm{d}\omega}{\int_\Omega p(\omega) S_0(\omega)\,\mathrm{d}\omega} \geq \frac{\sigma}{\delta}\operatorname*{essinf}_{\omega\in\Gamma} p(\omega)\frac{\int_\Omega p(\omega) S_0(\omega)\,\mathrm{d}\omega}{\int_\Omega p(\omega) S_0(\omega)\,\mathrm{d}\omega} = R^*$. This also shows that $R_0 = R^*$ if and only if $p(\omega)$ is constant a.e. on Γ.

The above proposition exhibits an important difference between the homogeneous and heterogeneous model, which is that the indicator that determines whether the population dies out or not is in the homogeneous case given by the basic reproduction number R_0^{hom}, while in the heterogeneous case this indicator is R^* which in general is different from R_0.

In the homogeneous model a disease that can invade the population is characterised by $R_0^{hom} > 1$, hence it kills the whole population by Proposition 2. This is no longer necessarily the case in the heterogeneous model. Instead we encounter the following three possibilities:

- $1 \leq R^* \leq R_0$: the disease leads to an outbreak and $S(t) \to 0$,
- $R^* < 1 \leq R_0$: the disease leads to an outbreak and $S(t) \to S^\star$,
- $R^* \leq R_0 < 1$: the disease does not lead to an outbreak and $S(t) \to S^\star$.

6 Concluding remarks and perspectives

In this paper we show within a simple 2-dimensional distributed (SI) model how the asymptotic behaviour of a disease in a heterogeneous population depends on the distribution of the population among the space of heterogeneity (the h-distribution). The analysis is based on the fact that for this particular distributed model it is possible to obtain a 3-dimensional ODE model that exactly reproduces the evolution of the aggregated susceptible and infected individuals. The latter model involves only a few averaged characteristics of the h-distribution. We show that the asymptotic behaviour of the disease qualitatively depends on these characteristics and describe it comprehensively.

On the other hand, the information about the h-distribution of the population is usually scarce and uncertain. Moreover, the results in this paper are obtained under restrictive conditions in a model that is simplistic, anyway. As mentioned in the introduction, this analysis should be viewed as a baseline for more complex studies. In particular, we envisage two lines of further research indicated below. Both lines may involve distributed modes of dimension 3 or 4 with much richer structure than the one considered in the present paper.

1. The uncertainty in the h-distribution of the population gives rise to a tube of possible aggregated state-trajectories. The sections of this tube at any given time instant provides a set-membership estimation of the state of the disease, which is independent of the particular realizations of the uncertainties. To obtain numerically such set-estimations is tractable by involvement of known methods in the optimal control theory, and this is our next goal.

2. A second line of ongoing research involves distributed prevention control, which influences the transmission rate of the disease, but is costly. Using the technique of optimal sparse control one may address the following question: to which risk groups (in terms of the h-distribution of the population) should the prevention be allocated and how this allocation evolves with time. The results in [13] suggest that the answer critically depends on the current state of the disease. It turns out that the optimal allocation of prevention (with respect to reasonable intertemporal criteria) might vary from most risky to least risky groups. The necessary numerical analysis is tractable for much more complicated models as the one considered in this paper.

The results in this paper will conveniently serve as a benchmark case for testing the results of each of the investigations mentioned above, since the asymptotics are precisely known in this case.

Appendix

Proof of Lemma 1. Differentiating (3) we obtain

$$\begin{aligned}
y'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{I(t)}{S(t)+I(t)} = \frac{\dot{I}(t)S(t) - I(t)\dot{S}(t)}{\left(S(t)+I(t)\right)^2} \\
&= \frac{\sigma y(t)S(t)^2 - \delta I(t)S(t) + \sigma y(t)S(t)I(t) - \lambda S(t)I(t)}{\left(S(t)+I(t)\right)^2} \\
&= \sigma y(t)(1-y(t))^2 - \delta y(t)(1-y(t)) + \sigma y(t)^2(1-y(t)) - \lambda y(t)(1-y(t)) \\
&= \left(y(t) - y(t)^2\right)(\sigma - \delta - \lambda)\,.
\end{aligned}$$

This is a Bernoulli equation and its solution is given by (23).

From (1) we have

$$S(t) = S(0)e^{-\int_0^t \sigma y(s)\,\mathrm{d}s + \lambda t}.$$

Simple calculations give that

$$\int_0^t y(s)ds = t - \frac{\ln(y(t))}{\sigma-\delta-\lambda} + \frac{\ln(y(0))}{\sigma-\delta-\lambda},$$

thus

$$e^{\int_0^t -\sigma y(s)ds} = y(0)^{-\frac{\sigma}{\sigma-\delta-\lambda}}\, y(t)^{\frac{\sigma}{\sigma-\delta-\lambda}}\, e^{-\sigma t}.$$

Then using (23) we have

$$\begin{aligned}
S(t) &= S(0)\, y(0)^{-\frac{\sigma}{\sigma-\delta-\lambda}}\, y(t)^{\frac{\sigma}{\sigma-\delta-\lambda}}\, e^{(\lambda-\sigma)t} \\
&= S(0)\, y(0)^{-\frac{\sigma}{\sigma-\delta-\lambda}} \left(e^{-(\sigma-\delta-\lambda)t}\left(y(0)^{-1} - 1\right) + 1\right)^{-\frac{\sigma}{\sigma-\delta-\lambda}} e^{(\lambda-\sigma)t} \\
&= S(0)\left(1 - y(0) + y(0)\, e^{(\sigma-\delta-\lambda)t}\right)^{-\frac{\sigma}{\sigma-\delta-\lambda}} e^{\lambda t}.
\end{aligned}$$

Considering

$$\frac{I(t)}{S(t)} = \frac{\frac{I(t)}{S(t)+I(t)}}{\frac{S(t)}{S(t)+I(t)}} = \frac{y(t)}{1-y(t)} =$$

$$= \frac{1}{\left(e^{-(\sigma-\delta-\lambda)t}\left(y(0)^{-1}-1\right)+1\right)\left(1-\frac{1}{e^{-(\sigma-\delta-\lambda)t}(y(0)^{-1}-1)+1}\right)}$$

$$= \frac{1}{e^{-(\sigma-\delta-\lambda)t}\left(y(0)^{-1}-1\right)} = \frac{e^{(\sigma-\delta-\lambda)t}}{y(0)^{-1}-1} = e^{(\sigma-\delta-\lambda)t}\frac{y(0)}{1-y(0)}$$

$$= e^{(\sigma-\delta-\lambda)t}\frac{I(0)}{S(0)},$$

we get

$$I(t) = S(t)\frac{I(t)}{S(t)} = I(0)y(0)^{-\frac{\sigma}{\sigma-\delta-\lambda}}y(t)^{\frac{\sigma}{\sigma-\delta-\lambda}}e^{-\delta t}$$

$$= I(0)\left(1+y(0)e^{(\sigma-\delta-\lambda)t}-y(0)\right)^{-\frac{\sigma}{\sigma-\delta-\lambda}}e^{(\sigma-\delta)t}.$$

□

Proof of Lemma 2. On the assumptions of the lemma and in view of (12), $S(t)$ is given by

$$S(t) = S(0)\,e^{\int_0^t(\lambda-\sigma\rho(\tau))\,\mathrm{d}\tau} = S(0)\,e^{\sigma\int_0^t(\rho^*-\rho(\tau))\,\mathrm{d}\tau}.$$

We split the integral

$$\int_0^t(\rho^*-\rho(\tau))\,\mathrm{d}\tau = \int_0^t(z^*\psi^*-z(\tau)\psi(F(\tau)))\,\mathrm{d}\tau$$

$$= \int_0^t\left(z^*-z(\tau)\right)\psi^*\,\mathrm{d}\tau + \int_0^t z(\tau)\left(\psi^*-\psi(F(\tau))\right)\,\mathrm{d}\tau$$

in two parts and denote them by $\mathrm{Int}_1(t)$ and $\mathrm{Int}_2(t)$ respectively.

Using (28), (29) and the definition of φ we have

$$\mathrm{Int}_1(t) = \frac{\psi^*}{M(0)}\int_0^t\int_{\Omega_-}p(\omega)\,e^{F(\tau)(\lambda+\delta-\sigma p(\omega))}S_0(\omega)\,\mathrm{d}\omega\,\mathrm{d}\tau$$

From this we can see that $\mathrm{Int}_1(t)$ is monotonically increasing. It remains to see whether it is bounded or not. Observe that for $\omega \in \Omega_-$ the function $\varphi(\cdot, \omega)$ is strictly decreasing. Then (28) implies that $\dot{F}(t)$ is strictly increasing. Hence, $F(t) > t\dot{F}(0) = tz(0)$ for $t > 0$ and we get

$$\begin{aligned}
\mathrm{Int}_1(t) &= \frac{\psi^*}{M(0)} \int_0^t \int_{\Omega_-} p(\omega)\, e^{F(\tau)(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\, \mathrm{d}\omega\, \mathrm{d}\tau \\
&< \frac{\psi^*}{M(0)} \int_0^t \int_{\Omega_-} p(\omega)\, e^{z(0)\tau(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\, \mathrm{d}\omega\, \mathrm{d}\tau \\
&= \frac{\psi^*}{M(0)z(0)} \int_{\Omega_-} p(\omega) \frac{1-e^{-tz(0)(\sigma p(\omega)-\lambda-\delta)}}{\sigma p(\omega)-\lambda-\delta} S_0(\omega)\, \mathrm{d}\omega \\
&\leq \frac{\psi^* \sup_{\omega\in\Omega_-} p(\omega)}{M(0)z(0)} \int_{\Omega_-} \frac{1-e^{-tz(0)(\sigma p(\omega)-\lambda-\delta)}}{\sigma p(\omega)-\lambda-\delta} S_0(\omega)\, \mathrm{d}\omega.
\end{aligned}$$

Since $\dot{F}(t) = z(t) < 1$, which implies that $F(t) < t$, we have that

$$\begin{aligned}
\mathrm{Int}_1(t) \geq \mathrm{Int}_1(tz(0)) &= \frac{\psi^*}{M(0)} \int_0^{tz(0)} \int_{\Omega_-} p(\omega)\, e^{F(\tau)(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\, \mathrm{d}\omega\, \mathrm{d}\tau \\
&> \frac{\psi^*}{M(0)} \int_0^{tz(0)} \int_{\Omega_-} p(\omega)\, e^{\tau(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\, \mathrm{d}\omega\, \mathrm{d}\tau \\
&= \frac{\psi^*}{M(0)} \int_{\Omega_-} p(\omega) \frac{1-e^{-tz(0)(\sigma p(\omega)-\lambda-\delta)}}{\sigma p(\omega)-\lambda-\delta} S_0(\omega)\, \mathrm{d}\omega \\
&\geq \frac{\psi^* \inf_{\omega\in\Omega_-} p(\omega)}{M(0)} \int_{\Omega_-} \frac{1-e^{-tz(0)(\sigma p(\omega)-\lambda-\delta)}}{\sigma p(\omega)-\lambda-\delta} S_0(\omega)\, \mathrm{d}\omega.
\end{aligned}$$

Notice that $\inf\limits_{\omega\in\Omega_-} p(\omega) > (\lambda+\delta)/\sigma = \rho^* \left(1 - \frac{K^0(0)}{M(0)}\right)^{-1} > 0$. Then the above two inequalities for $\mathrm{Int}_1(t)$ show that $\mathrm{Int}_1(t)$ converges if and only if the integral $\int_{\Omega_-} \frac{1-e^{-tz(0)(\sigma p(\omega)-\lambda-\delta)}}{\sigma p(\omega)-\lambda-\delta} S_0(\omega)\, \mathrm{d}\omega$ is bounded in t. Since the exponent under the integral converges to zero when $t \to \infty$, we obviously have

$$\lim_{t\to\infty} \mathrm{Int}_1(t) < \infty \iff \int_{\Omega_-} \frac{S_0(\omega)}{\sigma p(\omega)-\lambda-\delta}\, \mathrm{d}\omega < \infty.$$

Now, we investigate $\mathrm{Int}_2(t)$. Since we know that $\psi(\cdot)$ is decreasing and $F(\cdot)$ is increasing, we have that $\psi^* - \psi(F(t)) \leq 0$. Thus $\mathrm{Int}_2(t)$ is monotonically decreasing. In order to prove that it converges we estimate it using that $z(t) \leq 1$, $\psi^* = \frac{\lambda+\delta}{\sigma}$, and the definition of $\psi(\cdot)$ in (25):

$$\begin{aligned}
|\mathrm{Int}_2(t)| &\leq \int_0^t \left| \frac{\lambda+\delta}{\sigma} - \frac{\int_\Omega p(\omega) e^{-\sigma F(\tau)\, p(\omega)} S_0(\omega)\, \mathrm{d}\omega}{\int_\Omega e^{-\sigma F(\tau)\, p(\omega)} S_0(\omega)\, \mathrm{d}\omega} \right| \mathrm{d}\tau \\
&= \int_0^t \left| \frac{\lambda+\delta}{\sigma} - \frac{\int_{\Omega_-} p(\omega) e^{(\lambda+\delta-\sigma p(\omega))F(\tau)}\, S_0(\omega)\, \mathrm{d}\omega + \frac{\lambda+\delta}{\sigma} S^0(0)}{\int_{\Omega_-} e^{(\lambda+\delta-\sigma p(\omega))F(\tau)}\, S_0(\omega)\, \mathrm{d}\omega + S^0(0)} \right| \mathrm{d}\tau \\
&= \int_0^t \left| \frac{\int_{\Omega_-} \left(\frac{\lambda+\delta}{\sigma} - p(\omega)\right) e^{(\lambda+\delta-\sigma p(\omega))F(\tau)}\, S_0(\omega)\, \mathrm{d}\omega}{\int_{\Omega_-} e^{(\lambda+\delta-\sigma p(\omega))F(\tau)}\, S_0(\omega)\, \mathrm{d}\omega + S^0(0)} \right| \mathrm{d}\tau.
\end{aligned}$$

Using again that $F(t) > \dot{F}(0)t$ we obtain that

$$\begin{aligned}
|\mathrm{Int}_2(t)| &\leq \frac{1}{S^0(0)} \int_0^t \int_{\Omega_-} \left| \frac{\lambda+\delta}{\sigma} - p(\omega) \right| e^{(\lambda+\delta-\sigma p(\omega))z(0)\tau}\, S_0(\omega)\, \mathrm{d}\omega\, \mathrm{d}\tau \\
&= \frac{1}{\sigma z(0) S^0(0)} \int_{\Omega_-} (\sigma p(\omega) - \lambda - \delta) \frac{1 - e^{-tz(0)(\sigma p(\omega)-\lambda-\delta)}}{\sigma p(\omega) - \lambda - \delta} S_0(\omega)\, \mathrm{d}\omega \\
&\leq \frac{\int_{\Omega_-} S_0(\omega)\, \mathrm{d}\omega}{\sigma z(0) S^0(0)} = \frac{S^-(0)}{\sigma z(0) S^0(0)}.
\end{aligned}$$

Thus $\mathrm{Int}_2(t)$ converges. Consequently $\mathrm{Int}_1(t)+\mathrm{Int}_2(t)$ converges if $\mathrm{Int}_1(t)$ converges and we have

$$S(t) = S(0) e^{\sigma \int_0^t (\rho^* - \rho(\tau))\, \mathrm{d}\tau} \to S^* < \infty, \tag{33}$$

(with some strictly positive number S^*), provided that $\int_{\Omega_-} \frac{S_0(\omega)}{\sigma p(\omega) - \lambda - \delta}\, \mathrm{d}\omega < \infty$. Otherwise $S(t) \to \infty$. □

Proof of Lemma 3 Since $\lambda + \delta = 0$ we have from (27)

$$\dot{y}(t) = (1 - y(t)) \sigma \rho(t).$$

The solution to this ODE is

$$y(t) = e^{-\sigma \int_0^t \rho(\tau)\, \mathrm{d}\tau} (y(0) - 1) + 1.$$

We see that

$$\lim_{t\to\infty} y(t) = 1 \Longleftrightarrow \int\limits_0^\infty \rho(t)\,\mathrm{d}t = \infty.$$

We remind that due to $S^+(0) = S^0(0) = 0$ and the fact that $\varphi(t,\omega)$ is decreasing on Ω_-, we have from (28) that $\dot{F}(t)$ is increasing. Then

$$\rho(t) = \dot{F}(t)\psi(F(t)) \geq \dot{F}(0)\psi(F(t)) = z(0)\psi(F(t)).$$

On the other hand,

$$\rho(t) = \dot{F}(t)\psi(F(t)) = z(t)\psi(F(t)) \leq \psi(F(t)).$$

Then

$$\lim_{t\to\infty} y(t) = 1 \Longleftrightarrow \int\limits_0^\infty \psi(F(t))\,\mathrm{d}t = \infty.$$

Since $\psi(x)$ is decreasing and $F(t) \leq t$, we have $\psi(F(t)) \geq \psi(t)$. Thus

$$\begin{aligned}
\int\limits_0^\infty \psi(F(t))\,\mathrm{d}t \geq \int\limits_0^\infty \psi(t)\,\mathrm{d}t &= \int\limits_0^\infty \frac{\int_\Omega p(\omega)e^{-\sigma t\,p(\omega)}S_0(\omega)\,\mathrm{d}\omega}{\int_\Omega e^{-\sigma t\,p(\omega)}S_0(\omega)\,\mathrm{d}\omega}\,\mathrm{d}t \\
&= \int\limits_0^\infty \frac{-\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{-\sigma t\,p(\omega)}S_0(\omega)\,\mathrm{d}\omega}{\int_\Omega e^{-\sigma t\,p(\omega)}S_0(\omega)\,\mathrm{d}\omega}\,\mathrm{d}t \\
&= -\frac{1}{\sigma}\int\limits_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\int_\Omega e^{-\sigma t\,p(\omega)}S_0(\omega)\,\mathrm{d}\omega\right)\,\mathrm{d}t \\
&= -\frac{1}{\sigma}\ln(0) + \frac{1}{\sigma}\ln(S(0)) = \infty.
\end{aligned}$$

□

Proof of Lemma 5. In order to obtain the asymptotics of $I(t)$ we shall use equation (13). We have to determine the asymptotic behaviour of the term $\rho(t)S(t)$, where we have $\rho(t) \to 0$ and $S(t) \to \infty$. Using (11) and (19) we get

$$\rho(t)S(t) = \dot{F}(t)e^{\lambda t}\int_\Omega p(\omega)e^{-F(t)\sigma p(\omega)}S_0(\omega)\,\mathrm{d}\omega.$$

Since $\int_\Omega p(\omega)e^{-F(t)\sigma p(\omega)}S_0(\omega)\,\mathrm{d}\omega$ converges, we only need to consider the asymptotic behaviour of the term $\dot{F}(t)e^{\lambda t}$. Differentiating $\dot{F}(t)$ as given in (10) and using (10) again yields

$$\ddot{F}(t)=(\lambda+\delta)(\dot{F}(t))^2-\left((\lambda+\delta)-\frac{\sigma}{M(0)}\int_\Omega(p(\omega))^2e^{F(t)(\lambda+\delta-\sigma p(\omega))}S_0(\omega)\,\mathrm{d}\omega\right)\dot{F}(t).$$

We now write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{F}(t)e^{\lambda t}\right)=\ddot{F}(t)e^{\lambda t}+\dot{F}(t)\lambda e^{\lambda t}=\dot{F}(t)e^{\lambda t}\left(\frac{\ddot{F}(t)}{\dot{F}(t)}+\lambda\right).$$

Integrating this equation for $\dot{F}(t)e^{\lambda t}$ and using the above expression for $\ddot{F}(t)$ we obtain that

$$\begin{aligned}\dot{F}(t)e^{\lambda t}&=\dot{F}(0)e^{\int_0^t\left[(\lambda+\delta)\dot{F}(\tau)-\left((\lambda+\delta)-\frac{\sigma}{M(0)}\int_\Omega(p(\omega))^2e^{F(\tau)(\lambda+\delta-\sigma p(\omega))}S_0(\omega)\,\mathrm{d}\omega\right)+\lambda\right]\mathrm{d}\tau}\\&=\dot{F}(0)e^{(\lambda+\delta)F(t)+\int_0^t\left[\frac{\sigma}{M(0)}\int_\Omega(p(\omega))^2e^{F(\tau)(\lambda+\delta-\sigma p(\omega))}S_0(\omega)\,\mathrm{d}\omega-\delta\right]\mathrm{d}\tau}.\end{aligned}$$

Since $F(t)$ converges we need to consider only the integral in the exponent. The first term in the integrand converges to $\sigma\Theta^{-1}$. Thus, if $\delta>\sigma\Theta^{-1}$ then $\dot{F}(t)e^{\lambda t}$ converges to zero and consequently $I(t)\to 0$. Analogously, $I(t)\to\infty$ if $\delta<\sigma\Theta^{-1}$. This leaves the case $\delta=\sigma\Theta^{-1}$. Here we have

$$\begin{aligned}&\int_0^\infty\frac{\sigma}{M(0)}\left[\int_\Omega(p(\omega))^2e^{F(t)(\lambda+\delta-\sigma p(\omega))}S_0(\omega)\,\mathrm{d}\omega-\delta\right]\mathrm{d}t\\&=\int_0^\infty\left[\frac{\sigma}{M(0)}\int_\Omega(p(\omega))^2e^{F(t)(\lambda+\delta-\sigma p(\omega))}S_0(\omega)\,\mathrm{d}\omega\right.\\&\qquad\left.-\frac{\sigma}{M(0)}\int_\Omega(p(\omega))^2e^{F^*(\lambda+\delta-\sigma p(\omega))}S_0(\omega)\,\mathrm{d}\omega\right]\mathrm{d}t\\&=\frac{\sigma}{M(0)}\int_0^\infty\int_\Omega(p(\omega))^2e^{F^*(\lambda+\delta-\sigma p(\omega))}\left(e^{(\lambda+\delta-\sigma p(\omega))(F(t)-F^*)}-1\right)S_0(\omega)\,\mathrm{d}\omega\,\mathrm{d}t\\&=\frac{\sigma}{M(0)}\int_\Omega(p(\omega))^2e^{F^*(\lambda+\delta-\sigma p(\omega))}\int_0^\infty\left[e^{(\lambda+\delta-\sigma p(\omega))(F(t)-F^*)}-1\right]\mathrm{d}t\,S_0(\omega)\,\mathrm{d}\omega.\end{aligned}$$

Using de l'Hospital's rule we get

$$\begin{aligned}\lim_{t\to\infty}\frac{\ln(\dot F(t))}{-t} &= \lim_{t\to\infty} -\frac{\ddot F(t)}{\dot F(t)}\\ &= \lim_{t\to\infty} -(\lambda+\delta)\dot F(t)\\ &\quad + \left((\lambda+\delta) - \frac{\sigma}{M(0)}\int_\Omega (p(\omega))^2 e^{F(t)(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\,\mathrm{d}\omega\right)\\ &= (\lambda+\delta) - \frac{\sigma}{M(0)}\int_\Omega (p(\omega))^2 e^{F^*(\lambda+\delta-\sigma p(\omega))} S_0(\omega)\,\mathrm{d}\omega\\ &= \lambda+\delta-\sigma\Theta^{-1} = \lambda > 0\end{aligned}$$

This shows that $\ln(\dot F(t))$ declines asymptotically linear. Thus $\dot F(t) = e^{\ln(\dot F(t))}$ goes to zero faster than e^{-ct} for some $c>0$. Because of the equation $\lim_{t\to\infty}\frac{\dot F(t)}{F(t)-F^*} = \lim_{t\to\infty}\frac{\ddot F(t)}{\dot F(t)} = -\lambda$ we can therefore conclude that $F(t)-F^*$ goes to zero faster than de^{-ct} for some constants d and c. From this it follows that the integral $\int_0^\infty \left[e^{(\lambda+\delta-\sigma p(\omega))(F(t)-F^*)}-1\right]\mathrm{d}t$ converges and due to the fact that $p(\omega)$ is bounded the convergence is uniform in ω. Consequently $\rho(t)S(t)$ also converges, which implies $I(t)\to I^\star<\infty$. □

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