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Research Report 2015-03

February, 2015

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Multiperiod Maximum Loss Is Time Unit Invariant - a Short Note*

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Abstract

Time unit invariance is introduced as an additional requirement for multiperiod risk measures: for a constant portfolio under an iid risk factor process, the multiperiod risk should equal the one period risk of the aggregated loss, for an appropriate choice of parameters and independent of the portfolio. Multiperiod maximum loss over a sequence of Kullback-Leibler balls is time unit invariant, whereas multiperiod Value at Risk and multiperiod Expected Shortfall are not.

Keywords: multiperiod risk measures, relative entropy, model risk, ambiguity aversion
JEL classification: C61, C44, D81
AMS classification: 62C20, 90B50, 91B30, 94A17

1 Introduction

Time consistency (recursiveness) is an important property, often required for multiperiod risk measures on top of the requirements of one period risk measures. From a sequence $\mathrm{R}_{\alpha_1}, \ldots, \mathrm{R}_{\alpha_T}$ of one period conditional risk measures (not necessarily coherent) with parameters α_t (which would be e.g. confidence levels when R is Value at Risk (VaR) or Expected Shortfall) one can construct in a canonical way a time consistent risk measure for processes with finitely many time steps: One defines the multiperiod risk $\mathrm{MR}_{\alpha_1,\ldots,\alpha_T}$ of a process $(L_1, \ldots, L_T)$ recursively by

$$\begin{aligned} \mathrm{MR}_{\alpha_T}(L_T|\mathcal{F}_{T-1}) &:= \mathrm{R}_{\alpha_T}(L_T|\mathcal{F}_{T-1}), \qquad (1) \\ \mathrm{MR}_{\alpha_t,\ldots,\alpha_T}(L_t, \ldots, L_T|\mathcal{F}_{t-1}) &:= \mathrm{R}_{\alpha_t}\left(L_t + \mathrm{MR}_{\alpha_{t+1},\ldots,\alpha_T}(L_{t+1}, \ldots, L_T|\mathcal{F}_{t-1})\right). \end{aligned}$$

*This work was funded by the FWF, Austrian Science Fund, Project No. L532-N18.
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Under weak technical assumptions this construction guarantees that $\text{MR}_{\alpha_1,\dots,\alpha_T}$ is time consistent [4].

In this paper we introduce *time unit invariance* as another requirement for process risk measures. The choice of time units should be irrelevant. Consider an institution holding a portfolio unchanged over some time period (which may e. g. be a day). It should make no difference into how many intervals this period is split, be it into 1440 minutes, or 24 hours, or 1 day: the risk of cash flows aggregated over one day should be equal to multiperiod risk over 24 single hours or 1440 minutes—if the risk factor changes during the day are independent of each other and of other information emerging during the day. We do not think risk factor changes are necessarily independent of each other and portfolios are actually held constant over a day. These assumptions describe a counterfactual situation. But if this were the case, a multiperiod risk of the loss process should be equal to one-period risk of aggregated losses, whatever the portfolio happens to be. A risk measure for processes satisfying this natural requirement is called time unit invariant. More formally, we call $\text{MR}_{\alpha_1,\dots,\alpha_T}$ time unit invariant if there exists a parameter value $\bar{\alpha}$, such that *for any iid sequence* of losses $L_1, \dots, L_T$

$$\text{MR}_{\alpha_1,\dots,\alpha_T}(L_1, \dots, L_T|\mathcal{F}_0) = R_{\bar{\alpha}}\left(\sum_{t=1}^{T} L_t|\mathcal{F}_0\right), \tag{2}$$

holds, such that the parameter $\bar{\alpha}$ may depend on the number of timesteps T but not on L. The intuition behind requiring the same $\bar{\alpha}$ for all loss processes is that the passage of time does not depend on the portfolio someone may hold. In this note we introduce multiperiod maximum loss (MML) and show that it is time unit invariant, whereas multiperiod VaR (MVaR) is not.

2 Multiperiod Maximum Loss

In the single period case consider a measurable space $(\Omega, \mathcal{F})$, a random vector $\mathbf{r}(\cdot) : \Omega \to \mathbb{R}^k$, representing risk factors, and a measurable real-valued loss function $L(\cdot)$. Conditions on L will be specified below. We may shortly write $L = L(\mathbf{r})$.

The true probability measure $\mathcal{P}$ is not known, but it is assumed that an estimated measure, P, is available. Furthermore, in order to account for model uncertainty it is assumed that the true probability measure lies within the "ball" of all probability measures Q whose I-divergence (also called relative entropy or Kullback-Leibler distance), $D(Q||P) := \int \log \frac{dQ}{dP}(\mathbf{r}) dQ(\mathbf{r})$, from P is not larger than some fixed threshold $k > 0$. Maximum Loss of the loss function L then is defined as the expected loss in the worst of the plausible distributions,

$$\begin{aligned}\mathrm{MaxLoss}_k(L) &:= \sup_{Q:D(Q||P)\le k} \mathbb{E}_Q(L) \\ &= \sup\left\{\mathbb{E}_P\left[LZ\right]:\ Z\ge 0, \mathbb{E}_P\left[Z\right]=1, \mathbb{E}_P\left[Z\log\left(Z\right)\right]\le k\right\}.\end{aligned} \tag{3}$$

MaxLoss is a coherent risk measure [1, 5] and a decision maker, trying to minimize it, is ambiguity averse [6]. I-divergence based sets of alternative models are often used in optimal control under model uncertainty [7].

Loss L is not assumed to be essentially bounded. Instead, given k in (3) we require L to satisfy conditions (i)-(iii) below. MaxLoss is different from the entropic risk measure [5], which describes divergence preferences [8], and whose dual representation also uses I-divergence, but as a penalty term. Still the two can be evaluated with the same techniques [3].

The loss maximisation problem occuring in the definition (3) of MaxLoss has a regular solution when L and k meet three conditions [2]:

(i) If $\operatorname{ess\,sup}(L)$ is finite, then k should be smaller than $k_{\max} := -\log(P(\{\mathbf{r} : L(\mathbf{r}) = \operatorname{ess\,sup}(L)\}))$,

(ii) $\theta_{\max}(L) := \sup\{\theta : \Lambda(\theta) < +\infty\}$ should be positive,

(iii) If $\theta_{\max}, \Lambda(\theta_{\max}),\ \Lambda'(\theta_{\max})$ are finite, then k should be smaller than $k_{\max}(L) := \theta_{\max}\Lambda'(\theta_{\max}) - \Lambda(\theta_{\max})$.

Here the function Λ is defined as

$$\Lambda(\theta, L) := \log\left(\int e^{\theta L(\mathbf{r})} dP(\mathbf{r})\right), \tag{4}$$

where θ is a positive real number. For fixed L, Λ as a function of θ is convex and lower semicontinuous on $\mathbb{R}$. Its essential domain of definition $D_\Lambda := \{\theta : \Lambda(\theta) < \infty\}$ is a finite or infinite interval, excluding the trivial case $D_\Lambda = \{0\}$. In the interval D_Λ, the function $\Lambda(\theta)$ is continuous and has derivative $\Lambda'(\theta) = \int L(\mathbf{r})\exp(\theta L(\mathbf{r}) - \Lambda(\theta))dP(\mathbf{r})$. At an endpoint of D_Λ that belongs to D_Λ, this derivative is understood as one-sided and is not necessarily finite. Moreover, $\Lambda'(\theta)$ is strictly increasing in D_Λ (unless $L(\mathbf{r})$ is constant ¶-almost surely). If $\sup D_\Lambda = \infty$ then $\Lambda'(\theta) \to \operatorname{ess\,sup}(L)$ as $\theta \to \infty$.

Under assumptions (i), (ii), (iii) the equation

$$\theta\Lambda'(\theta, L) - \Lambda(\theta, L) = k \tag{5}$$

has a unique positive solution $\overline{\theta}$. The maximum loss then is achieved and is given by

$$\mathrm{MaxLoss}_k(L) = \Lambda'(\overline{\theta}, L). \tag{6}$$

In the sequel (i)-(iii) are standing assumptions.

The pathological cases where some of the assumptions (i)–(iii) are violated can be solved with different methods [2].

Extending (3), conditional versions of the one-period maximum loss can easily be defined. Given some σ-field $\mathcal{F}' \subset \mathcal{F}$, define the conditional maximum loss ML for $L \in L_{exp}(\Omega, \mathcal{F}, P)$ by

$$\mathrm{ML}_k(L|\mathcal{F}') := \sup\left\{\mathbb{E}[LZ|\mathcal{F}'] :\ Z \geq 0, \mathbb{E}[Z|\mathcal{F}'] = 1, \mathbb{E}[Z\log(Z)|\mathcal{F}'] \leq k\right\}. \tag{7}$$

The sup here denotes the supremum of functions, with respect to almost sure ordering. The Z can be interpreted as densities of feasible probability Q with respect to P.

In a multiperiod setup we work with filtered spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, Q)$ and consider adapted loss processes $L_t = L_t(\mathbf{r}_t(\omega))$. If we set $Q = P$, the probability measure P again is interpreted as an estimated probability measure. If the portfolio is constant in the sense that no transactions occur, we have $L_t = L(\mathbf{r}_t(\omega))$. $\mathcal{F}_t$ represents the information available at time t, while $\mathbf{r}(\omega) = \{\mathbf{r}_1(\omega), \ldots, \mathbf{r}_T(\omega)\}$ represent paths of risk factor values. If the risk factor changes $\mathbf{r}_t$, and hence also losses $L(\mathbf{r}_t)$, are independent of information $\mathcal{F}_0, \ldots, \mathcal{F}_{t-1}$ for all t. This is called the independence assumption in the following. We omit the reference to $\mathcal{F}_t$.

Define now multiperiod maximum loss (MML) for a sequence of positive radii $(k_1, \ldots, k_T)$ by the recursive procedure (1) with ML_{k_t} playing the role of R_{α_t}. $\mathrm{MML}_{k_1,\ldots,k_T}$ inherits the properties of convexity, homogeneity, and law-invariance from conditional one-period ML. Because of its recursive construction it is time consistent, but it does not describe preference for temporal resolution of uncertainty [10].

3 Main results

Let us first look at the situation where a Kullback-Leibler radius $K > 0$ for the long period $[0, T]$ is given. Maximum cumulated loss over the whole period then is denoted $\mathrm{ML}_K\left(\sum_{t=1}^T L_t\right)$.

Lemma 1 (Disaggregation)**.** *Under the independence assumption, assuming (i) to (iii) for $\Lambda(\theta, \sum_{t=1}^T L_t)$ and a given $K > 0$, the equation $\theta \cdot \Lambda'\left(\theta, \sum_{t=1}^T L_t\right) - \Lambda\left(\theta, \sum_{t=1}^T L_t\right) = K$ has a unique solution $\overline{\theta}$. Choose*

$$k_t := \overline{\theta} \cdot \Lambda'(\overline{\theta}, L_t) - \Lambda(\overline{\theta}, L_t) \tag{8}$$

for $t = 1, \ldots, T$. With this choice we have

$$\mathrm{MML}_{k_1,\ldots,k_T}(L_1, \ldots, L_T) = \mathrm{ML}_K\left(\sum_{t=1}^T L_t\right), \tag{9}$$

and

$$K = \sum_{t=1}^{T} k_t. \tag{10}$$

Proof. We prove this by induction in the number T of time steps. For $T = 1$ the result is trivial. In what follows, choose $\overline{\theta}$ and the related k_t according to the assumptions of the Lemma and equation (8). Summation leads to $\sum_{t=2}^{T} k_t = \overline{\theta} \cdot \sum_{t=2}^{T} \Lambda'(\overline{\theta}, L_t) - \sum_{t=2}^{T} \Lambda(\overline{\theta}, L_t)$. Λ is the logarithm of a moment-generating function applied to a sum of independent random variables, hence

$$\sum_{t=2}^{T} k_t = \overline{\theta} \cdot \Lambda'(\overline{\theta}, \sum_{t=2}^{T} L_t) - \Lambda(\overline{\theta}, \sum_{t=2}^{T} L_t). \tag{11}$$

Assume now that the Lemma holds up to $T-1$ (induction hypothesis). Then – by renumbering – it holds also for losses $L_2, \ldots, L_T$. Now, (11) shows that $\overline{\theta}$ and the k_t are suitable for applying the Lemma to losses $L_2, \ldots, L_T$ if one uses ML with radius $\sum_{t=2}^{T} k_t$. Hence we have

$$\begin{aligned} \mathrm{MML}_{k_1,\ldots,k_T}(L_1, \ldots, L_T) &= \mathrm{ML}_{k_1}\left(L_1 + \mathrm{MML}_{k_2,\ldots,k_T}(L_2, \ldots, L_T)\right) && (12) \\ &= \mathrm{ML}_{k_1}\left(L_1\right) + \mathrm{ML}_{\sum_{t=2}^{T} k_t}(\sum_{t=2}^{T} L_t). && (13) \end{aligned}$$

By assumption and by (11) the relevant parameter is $\overline{\theta}$ for both ML in (13), therefore

$$\begin{aligned} \mathrm{MML}_{k_1,\ldots,k_T}(L_1, \ldots, L_T) &= \Lambda'\left(\overline{\theta}, L_1\right) + \Lambda'\left(\overline{\theta}, \sum_{t=2}^{T} L_t\right) && (14) \\ &= \Lambda'\left(\overline{\theta}, \sum_{t=1}^{T} L_t\right) = \mathrm{ML}_K\left(\sum_{t=1}^{T} L_t\right), && (15) \end{aligned}$$

which shows (9). Finally, to establish (10) sum up equation (8) over all t. $\square$

Now let us assume that positive real radii $k_1, \ldots, k_T$ are given.

Lemma 2 (Aggregation)**.** *Under the independence assumptions and assuming (i)-(iii) for each given k_t, there exists a unique $\overline{\theta}$ such that*

$$\sum_{t=1}^{T} \Lambda'(\overline{\theta}, L_t) = \sum_{t=1}^{T} \Lambda'(\theta_t, L_t), \tag{16}$$

where each θ_t is the unique solution of $\theta_t \cdot \Lambda'(\theta_t, L_t) - \Lambda(\theta_t, L_t) = k_t$. Define K by

$$K := \overline{\theta} \cdot \Lambda'\left(\overline{\theta}, \sum_{t=1}^{T} L_t\right) - \Lambda\left(\overline{\theta}, \sum_{t=1}^{T} L_t\right). \tag{17}$$

With this choice of K we have

$$\mathrm{ML}_K\left(\sum_{t=1}^{T} L_t\right) = \mathrm{MML}_{k_1,\dots,k_T}(L_1,\dots,L_T) = \sum_{t=1}^{T} \mathrm{ML}_{k_t}(L_t). \tag{18}$$

Proof. Since all $\Lambda'(\cdot, L_t)$ are strictly increasing and continuous, $\Lambda'(\cdot, \sum_{t=1}^{T} L_t)$ is also strictly increasing and continuous, taking values in the interval $[\Lambda'(\min(\theta_t), \sum_{t=1}^{T} L_t), \Lambda'(\max(\theta_t), \sum_{t=1}^{T} L_t)]$. Therefore, for any point in this interval there is a unique $\overline{\theta}$ in $[\min(\theta_t), \max(\theta_t)]$ such that $\Lambda'(\overline{\theta}, \sum_{t=1}^{T} L_t)$ takes this value. Observing that under the independence assumption the right hand side of (16) equals MML and the left hand side is the ML of the summed loss variables with K given by (17), this establishes (18) and thus the lemma. □

If in addition to the independence assumption the losses L_t are i.i.d., we can derive the simple relation announced in the introduction.

Proposition 1 (Aggregation for time independent loss function and iid risk factors)**.** *Assume i.i.d. risk factors and a time independent loss function L. Then under assumptions (i)-(iii):*

$$\mathrm{ML}_{kT}\left(\sum_{t=1}^{T} L_t\right) = \mathrm{MML}_{k,\dots,k}\left(L_1,\dots,L_T\right) = T \cdot \mathrm{ML}_k(L). \tag{19}$$

Proof. Proof. Under the assumptions the losses L_t are i.i.d. with the same distribution as some random variable L. By Lemma 2, there is unique $(\overline{\theta}, K)$ such that (18) holds. Putting $\overline{\theta}$ into (8) leads to $k_t = k$ and Lemma 1 gives $K = kT$. Equation (19) follows easily. □

As required for time unit invariance (2), the one period parameter k leads to the multi-period parameter kT, independently of the actual loss function or of the risk factor distribution.

4 Counterexamples

In the following, we analyze recursive concatenations of two important risk measures: value at risk (which is not a risk measure in the strict sense, but widely used in industry) and expected shortfall (see e.g. [9]). It shows that in both cases time unit invariance is violated.

4.1 Value at Risk

For a constant linear portfolio with loss function $L_t(r_t) = -lr_t$ whose loss equals a constant $-l$ times the value of a normally distributed risk factor $r_t \sim N(0, \sigma^2)$. Value at Risk at level α equals $\Phi^{-1}(\alpha) \cdot \sigma \cdot l$, where Φ^{-1} is the inverse of the standard normal distribution function. For a sequence of confidence levels $\alpha_1, \alpha_2, \dots, \alpha_T$, a time consistent multiperiod version MVaR of VaR may be defined by a procedure similar to (1) with VaR_α in the role of R_α. For two time steps we get

$$\mathrm{MVaR}_{\alpha_1,\alpha_2}(-lr_1, -lr_2) = (\Phi^{-1}(\alpha_1) + \Phi^{-1}(\alpha_2)) \cdot \sigma \cdot l.$$

Aggregated VaR at level $\bar{\alpha}$ over the period $[0, 2]$ equals

$$\mathrm{VaR}_{\bar{\alpha}}(-lr_1 - lr_2) = \sqrt{2}\Phi^{-1}(\bar{\alpha}) \cdot \sigma \cdot l.$$

$\mathrm{MVaR}_{\alpha,\alpha}$ equals one period VaR at level $\bar{\alpha}$ if

$$\bar{\alpha} = \Phi((\Phi^{-1}(\alpha_1) + \Phi^{-1}(\alpha_2))/\sqrt{2}). \tag{20}$$

This condition guarantees equality for all constant linear portfolios.

Consider now quadratic portfolios with loss function $-l^2 r^2$. VaR_α equals $F_1^{-1}(\alpha) l^2 \sigma^2$, where F_1^{-1} is the inverse of the distribution function of the χ^2-distribution with one degree of freedom. Multiperiod VaR for the parameter sequence (α_1, α_2) is

$$\mathrm{MVaR}_{\alpha_1,\alpha_2}(-l^2 r_1^2, -l^2 r_2^2) = (F_1^{-1}(\alpha_1) + F_1^{-1}(\alpha_2))\sigma^2 l^2.$$

Aggregated VaR at level $\bar{\alpha}$ over the period $[0, 2]$ is

$$\mathrm{VaR}_{\bar{\alpha}}(-l^2 r_1^2 - l^2 r_2^2) = F_2^{-1}(\bar{\alpha}) l^2 \sigma^2,$$

where F_2^{-1} is the inverse of the distribution function of the χ^2-distribution with two degrees of freedom. Multiperiod $\mathrm{MVaR}_{\alpha_1,\alpha_2}(-l^2 r_1^2, -l^2 r_2^2)$ equals $\mathrm{VaR}_{\bar{\alpha}}(-l^2 r_1^2 - l^2 r_2^2)$ if

$$\bar{\alpha} = F_2(F_1^{-1}(\alpha_1) + F_1^{-1}(\alpha_2)). \tag{21}$$

This guarantees equality for all constant quadratic portfolios. But (21) is different from (20), hence time unit invariance is not fulfilled for MVaR.

4.2 expected Shortfall

Assume that losses L_i are i.i.d. with a standard normal distribution. (Conditional) expected shortfall of L_2 at level q therefore is

$$ES_1[L_2] = ES[L_2] = \frac{\phi\left(\Phi^{-1}(q)\right)}{1-q}.$$

The multiperiod expected shortfall over two periods then is given by

$$ESM(q) = ES[L_1 + ES_1[L_2]] = ES[L_1] + \frac{\phi\left(\Phi^{-1}(q)\right)}{1-q}. = 2 \cdot \frac{\phi\left(\Phi^{-1}(q)\right)}{1-q}.$$

On the other hand the expected shortfall over two periods at level q' is given by the expected shortfall of the random variable $L_1 + L_2$, which has a normal distribution with mean zero and a variance of two. This leads to

$$ESS(q') = \sqrt{2} \cdot \frac{\phi\left(\Phi^{-1}(q')\right)}{1-q'}.$$

To find a level q' such that ESM and ESS are equal, one has to solve

$$\frac{\phi\left(\Phi^{-1}(q')\right)}{1-q'} = \sqrt{2} \cdot \frac{\phi\left(\Phi^{-1}(q)\right)}{1-q}. \tag{22}$$

Consider now the quadratic portfolio: $M_i = L_i^2$ are i.i.d. χ^2-distributed with one degree of freedom. Here we have

$$ES_1\left[M_2\right] = ES\left[M_2\right] = 1 + \sqrt{2}\frac{\sqrt{F_1^{-1}(q)\exp\left(-\frac{1}{2}F_1^{-1}(q)\right)}}{1-q},$$

and

$$ESM(q) = ES\left[M_1 + ES_1\left[M_2\right]\right] = 2 + 2\sqrt{2}\frac{\sqrt{F_1^{-1}(q)\exp\left(-\frac{1}{2}F_1^{-1}(q)\right)}}{1-q},$$

where $F_1^{-1}(q)$ denotes the q-quantile of the χ^2 distribution with one degree of freedom.

The expected shortfall over two periods is calculated for $M_1 + M_2$, which is χ^2-distributed with two degrees of freedom:

$$ESS(q') = \frac{\exp\left(-\frac{1}{2}F_2^{-1}(q')\right) \cdot \left(F_2^{-1}(q) + 2\right)}{1-q},$$

where $F_2^{-1}(q)$ denotes the q-quantile of the χ^2 distribution with two degrees of freedom.Again we have to solve ESM(q)=ESS(q'). Because of

$$F_2^{-1}(q') = -2\ln\left(1-q'\right),$$

this leads to

$$q' = 1 - \exp\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\frac{\sqrt{F_1^{-1}(q)\exp\left(-\frac{1}{2}F_1^{-1}(q)\right)}}{1-q}\right),$$

which is not in general a solution to (22).

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