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VALUATION AND PRICING OF ELECTRICITY DELIVERY CONTRACTS - THE PRODUCER'S VIEW

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Summary. This paper analyzes valuation and pricing of physical electricity delivery contracts. Values and prices should be consistent to production and fuel storage capacities. Using stochastic optimization problems in discrete time but general state space, the duals of production problems are used to derive no-arbitrage conditions for fuel and electricity prices as well as superhedging values and prices of OTC electricity delivery contracts. In particular we take the perspective of an electricity producer, serving contractual deliveries but avoiding unacceptable losses at the end of the planning horizon. The resulting no-arbitrage conditions, stochastic discount factors and superhedging prices account for typical frictions like limitation of storage and production capacity and for the fact that it is possible to produce electricity from fuel, but not to produce fuel from electricity. Similarities, but also substantial differences to purely financial results can be demonstrated in this way. Finally, using acceptability measures we analyze capital requirements and acceptability prices for delivery contracts, where the producer accepts some risk.

1. Introduction

The notions of arbitrage and market completeness are cornerstones of modern finance, in particular for the valuation of financial contracts. Essentially, a financial market is arbitrage free if and only if there exists a (local) martingale measure and an arbitrage free market is complete if and only if there exists a unique martingale measure. See e.g. [7, 8] for more detailed statements and proofs. At complete markets every contingent claim is attainable by hedging portfolios, and financial derivatives can be priced by calculating the expected discounted value of the derivatives payoff w.r.t. the unique martingale measure.

At incomplete markets, not every contingent claim can be attained. Still, under no-arbitrage one can calculate the minimal and the maximal discounted expectation under all expectations w.r.t. any equivalent martingale measure, which gives the lower and upper arbitrage boundary. This often leads to a large interval of possible prices. If the analyzed security is traded at an exchange, one can try to estimate a market price of risk, which leads to a price within the arbitrage bounds. For contracts that are traded OTC this is not possible because data are not available. Several methods for finding reasonable prices have been developed so far, e.g. good deal bounds [6], local risk minimization [16] or convex hedging [21]. Such approaches search for martingale measures with restricted or minimal risk, where the notion of risk is different for each approach.

In the present paper we move on from the sphere of purely financial markets and analyze OTC electricity delivery contracts from the viewpoint of an electricity producer. Electricity markets are typically incomplete markets with unique frictions, not existent at financial markets: In particular electricity is produced from fuels but can not be converted back to fuels. Electricity also can not be stored in large quantities. Furthermore, all kinds of physical storage and production restrictions are relevant for the production process. Therefore, and because produced and used electric power is balanced immediately in an electrical network, differences between demanded and produced power may lead to damaged equipment or even breakdown of the net. Some parts of electricity markets, in particular futures markets, are organized like financial markets, but even in this case the delivery profiles of traded futures can not fully replicate any OTC-traded delivery profile.

We start with an analysis of arbitrage and ask the question: Given the above frictions, how can we characterize arbitrage in a simple market model with electricity production? In particular we search for an analogon to equivalent martingale measures, when looking at prices for fuel and electricity instead of prices for financial securities.

We then proceed to valuation and pricing of delivery contracts. When valuating a specified contract (with a stochastic process of deliveries and a fixed delivery price) the producer may look at the smallest up-front payment (or cash-reserve) such that all contractual obligations can be

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satisfied at the given delivery price, and the asset value (consisting of cash and the value of fuel) at the end of the planning horizon is for sure not negative. This approach resembles what in a financial context is called superhedging and therefore we speak of the *superhedging value*. In similar manner, we analyze the smallest delivery price such that, starting with an asset value of zero, all contractual obligations can be fulfilled with a nonnegative end value. *Superhedging prices* are not market prices (the same is true for superhedging values and market values) but mark an important boundary: if delivery is agreed at smaller prices, the producer definitely has to take some risk. The costs of completely removing the risk related to a smaller price can be calculated as the related superhedging value.

Superhedging values and superhedging prices are dependent on the production equipment but independent of any preferences. However they may be very strict boundaries for feasible upfront payments or delivery prices, because any risk is eliminated. In practical situations, producers usually take some risk in order to achieve potentially higher profit. We therefore also analyze the smallest start capital leading to an acceptable distribution of the end value as well as the smallest delivery price that leads to an acceptable end distribution when starting with zero capital. In the first case we speak of the *capital requirement*. The delivery price in the second case will be called the *acceptability price*.

In the present paper we restrict the analysis to concave acceptability functionals [20], which are (up to sign) closely related to coherent risk measures [2]. Furthermore, we consider stochastic processes and optimization in discrete time, which is the usual approach taken by energy producers in practice. Similar methods have been applied to purely financial portfolios e.g. in [14] and [10], where it was shown that classical results on no-arbitrage pricing can be replicated in a discrete time, discrete state space stochastic optimization framework. In particular dual formulations can be used to characterize no-arbitrage and to derive the classical risk-free pricing results and arbitrage bounds. While those papers builds on discretized state spaces, the present work uses general state spaces. T. Pennanen in [17, 18] analyzes superhedging in a very general framework but again focuses on financial markets. In addition, [19] derives basic facts of capital requirements and acceptability pricing (in particular indifference pricing, see remark 3.3 below) in the context of convex analysis, again in a purely financial context. The present paper builds on the views expressed in those papers and uses convex analysis and duality theory but focuses on the case of electricity markets. It aims at deriving concrete no-arbitrage conditions and pricing principles in the complex situation with storage and production restrictions (like random outages of generators), asymmetric production possibilities between fuel and electricity and nonstorability of electricity.

The paper is organized as follows: Section 2 uses a basic optimization problem to derive and analyze no-arbitrage conditions for a model with spot prices for fuel and electricity, when electricity can be produced with given efficiency. In the main part, section 3, the optimization problem of a minimum up-front payment for a delivery contract is used to derive valuation formulas in terms of stochastic discount factors and equivalent measures. As a second application we analyze the smallest feasible delivery price. In both cases we aim at almost surely nonnegative end value. Finally we relax this requirement and consider the minimum capital requirement and the acceptability price. The last section, 4, concludes the paper.

2. No-Arbitrage Conditions

In the following all relevant risk factors (in particular prices) and the related decisions are considered as stochastic processes, defined on a filtered probability space $\mathfrak{Y} = \left(\Omega, \mathcal{F}, \mathfrak{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ in discrete time $t = 0, 1, \ldots, T$. For simplicity this setup assumes constant time increments, e.g. hours, days or weeks. However, all following statements can easily be adapted to more general time models. Time zero represents here and now and the related σ-algebra $\mathcal{F}_0$ is the trivial σ-algebra $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Time T denotes the end of the planning horizon. In order to simplify notation we use the sets $\mathcal{T} = \{0, 1, \ldots, T\}$, $\mathcal{T}_0 = \{0, 1, \ldots, T-1\}$, $\mathcal{T}_1 = \{1, \ldots, T\}$ and $\mathcal{T}_1^{T-1} = \{1, \ldots, T-1\}$.

In the simplest model the basic risk factors are fuel prices and electricity prices, represented by real valued stochastic processes $X_t^f(\omega)$ and $X_t^e(\omega)$, both adapted to the filtration $\mathfrak{F}$. In fact, fuel prices are assumed to be almost surely nonnegative, whereas electricity prices may take also negative values with positive probability. Both, fuel and electricity are measured in MWh and prices are given in currency units per MWh. Note that it is not assumed that the filtration $\mathfrak{F}$ is generated by the price processes X_t^f and X_t^e. Additional information like prices of further fuels, weather, or general business activity may play a role.

Consider now a producer, using a generator (e.g. a turbine) with efficiency η. For simplicity we measure both, electric energy and quantities of fuel (the related energy content) in MWh. The amount of fuel burned for producing an amount y_t [MWh] of electricity at time t then is given by $\eta^{-1}y_t$ [MWh]. The producer buys z_t [MWh] from the fuel market at price X_t^f currency units per MWh in order to generate electricity. Amounts of Fuel s_t [MWh] can be stored between time t and $t+1$. In addition, the producer manages a cash account c_t with interest rate $r \geq 0$ (per period). We also use the notation $R = (1+r)$

The decision processes y_t and z_t as well as the decision processes c_t and s_t are considered as real valued random processes defined on $\mathfrak{Y}$ and are adapted to the filtration $\mathfrak{F}$. This means that decisions at time t have to rely on information available at time t. Furthermore, fuel z_t is bought at time t at a known price X_t^f. On the other hand we assume that electricity production over the period $[t, t+1]$ is planned in advance at time t but the electricity price X_{t+1}^e(currency units per MWh) at which the planned amount is sold is revealed only at the end, $t+1$, of the period. Keep in mind that c_0, s_0, y_0, z_0 are deterministic, as $\mathcal{F}_0$ is assumed to be the trivial σ-algebra. For physical reasons, electricity production and fuel storage are almost surely restricted to nonnegative values.

The basic setup will be extended when contractual delivery and delivery price are introduced. For the moment however, we use it as it is to derive no-arbitrage conditions. Extending the usual definitions (see e.g. [4] definition 2.14, 2.15) we say that

Definition 1. In the basic model a strategy $\{y_t, z_t\}_{t\geq 0}$ with cash position c_t and fuel storage s_t, where $y_t, s_t \geq 0$ is self financing if the following conditions hold almost surely for all $t \in \mathcal{T}_1$

$$c_t = \left(c_{t-1} - z_{t-1}X_{t-1}^f\right) R + y_{t-1}X_t^e, \tag{2.1}$$

$$s_t = s_{t-1} - y_{t-1}\eta^{-1} + z_{t-1}. \tag{2.2}$$

At time t the value of a strategy is given by

$$V_t^\eta = c_t + X_t^f \cdot s_t$$

Definition 2. An η-arbitrage for market $\left\{X_t^e, X_t^f\right\}$ is a self financing strategy $\{y_t, z_t\}_{t\geq 0}$ with

$$V_0^\eta \leq 0, \tag{2.3}$$

$$\mathbb{P}\left(V_T^\eta \geq 0\right) = 1. \tag{2.4}$$

$$\mathbb{P}\left(V_T^\eta > 0\right) > 0. \tag{2.5}$$

We call a market $\left\{X_t^e, X_t^f\right\}$ η-arbitrage free, if no η-arbitrage exists. A market $\left\{X_t^e, X_t^f\right\}$ is arbitrage free if it is η-arbitrage free for any $0 \leq \eta \leq \eta_{max}$, where η_{max} denotes the maximum efficiency available to the producers.

Clearly an η-arbitrage free market is η'-arbitrage free for any $\eta' \leq \eta$.

Based on these definitions and assuming that c_t, y_t, s_t and the products $X_T^f s_T$ and $z_t X_t^f$ are integrable, the following optimization problem can be used to detect arbitrage strategies.

$$\max_{y,z,c,s} \mathbb{E}^{\mathbb{P}}\left[c_T + X_T^f s_T\right] \tag{2.6}$$

$$\begin{aligned}
&\text{subject to:}\\
&(t \in \mathcal{T}_1) : c_t = \left(c_{t-1} - z_{t-1}X_{t-1}^f\right) R + y_{t-1}X_t^e\\
&(t \in \mathcal{T}_1) : s_t = s_{t-1} - y_{t-1}\eta^{-1} + z_{t-1}\\
&\qquad c_0 + X_0^f s_0 \leq 0\\
&\qquad c_T + X_T^f s_T \geq 0\\
&(t \in \mathcal{T}\ ) : s_t \geq 0\\
&(t \in \mathcal{T}_0) : y_t \geq 0
\end{aligned}$$

Remark 1. In order to avoid too much numbering, we may refer to parts of the contraint sets as “contrain groups”. As an example, the first line of contraints in (2.6) will be referred to as “constraint group 1”.

The following observation is a key to characterizing η-arbitrage.

Lemma 1. *An η-arbitrage for a market $\left\{X_t^e, X_t^f\right\}$ exists if and only if (2.6) is unbounded.*

Proof. The first two constraints of (2.6) correspond to conditions (2.1) and (2.2) for a self financing portfolio. The third and fourth constraints enforce conditions (2.3) and (2.4). The last two constraints are the nonnegativity constraints on electricity production and fuel storage. Because of (2.4) , $\mathbb{E}^{\mathbb{P}}\left[c_T + X_t^f s_T\right] > 0$ if and only if $c_T + X_T^f s_T > 0$ on a set with positive probability. Furthermore, because the objective function and the constraints are positively homogeneous in the decision variables $\{y_t, z_t, c_t, s_t\}$, the optimal value is unbounded if and only if a positive expectation can be fulfilled by a feasible strategy. Hence conditions (2.3)-(2.5) can be achieved by a self financing strategy if and only if problem (2.6) is unbounded. □

The concept of η-arbitrage deserves some additional discussion. By weakening the notion of no-arbitrage we account for the fact that (depending on technical progress) arbitrage possibilities in the above sense may exist for very efficient (high η) generators, at least for some time. Two effects, in contrast to purely financial markets, may prevent immediate or even quick return to a completely arbitrage free situation: First it is not possible to realize infinite profits in reality, because production capacity y_t and fuel storage s_t are not only bounded to be nonnegative, but also restricted by some upper capacity of production and storage. Therefore profits are bounded as well. On the other hand, over time others might build similarly efficient generators as well and profit will fade away eventually. However this process will take some time, because investment costs are high and arbitrage profits are restricted by the production capacity and can be realized only gradually over time.

Given this argument for the concept of η-arbitrage, it might sound strange that test problem (2.6) is formulated without upper bounds on storage and production. However, because of positive homogeneity, a strategy which leads to a positive end value with positive probability can be scaled in a way such that either the scaled solution leads to an infinite expectation without upper bounds or such that all upper bounds are observed and at least one upper bound is reached with positive probability at some point in time. Therefore for a pure test of η-efficiency the upper bounds are not relevant.

Using test problem (2.6), it is now possible to derive general no-arbitrage conditions.

Lemma 2. *A market $\left\{X_t^e, X_t^f\right\}$ is η-arbitrage free in the described setup if and only if there exist adapted stochastic processes $\{\xi_t, \lambda_t\}$ with the following properties:*

A1: For each $t \in \mathcal{T}_1$ the random variables ξ_t and λ_t are essentially bounded.

A2: $\xi_T > 0$ and $\lambda_T \geq X_T^f \cdot \xi_T$

A3: $R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t$ for $t = 1, \ldots, T-1$, and $R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = 1$

A4: $\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} X_{t+1}^e|\mathcal{F}_t\right] \leq R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot \eta^{-1} X_t^f$ for $t \in \mathcal{T}_0$

A5: $\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\, E\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot X_t^f$ for $t \in \mathcal{T}_0$

A6: $\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t$ for $t \in \mathcal{T}_1^{T-1}$

Proof. The Lagrangian of (2.6) is given by

$$\begin{aligned}
L(y, z, c, s; \xi, \lambda, \zeta, \gamma) = {} & \mathbb{E}^{\mathbb{P}}\left[c_T + X_T^f s_T\right] \\
& + \mathbb{E}^{\mathbb{P}}\left[\zeta\left(c_T + X_T^f s_T\right)\right] \\
& + \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}\left[\xi_t\left(R c_{t-1} - c_t + X_t^e y_{t-1} - R\, X_{t-1}^f z_{t-1}\right)\right] \\
& + \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}\left[\lambda_t\left(s_{t-1} - s_t - y_{t-1}\eta^{-1} + z_{t-1}\right)\right] \\
& - \gamma\left(c_0 + X_0^f s_0\right),
\end{aligned} \tag{2.7}$$

where $\gamma \geq 0$ is a real number, $\zeta \geq 0$ a $\mathcal{F}_T$-measurable essentially bounded random variable, ξ_t and λ_t are $\mathcal{F}_t$-measurable and essentially bounded.

After some reordering, (2.7) can be rearranged as follows:

$$\begin{aligned} L(y,z,c,s;\xi,\lambda,\zeta,\gamma) = {} & \mathbb{E}^{\mathbb{P}}\left[c_T\left(1+\zeta-\xi_T\right)\right] + \mathbb{E}^{\mathbb{P}}\left[s_T\left(X_T^f(1+\zeta)-\lambda_T\right)\right] \\ & + c_0\left(\mathbb{E}^{\mathbb{P}}\left[\xi_1\right]R-\gamma\right) + s_0\left(\mathbb{E}^{\mathbb{P}}\left[\lambda_1\right]-\gamma X_0^f\right) \\ & + y_0\mathbb{E}^{\mathbb{P}}\left[\xi_1 X_1^e - \lambda_1\eta^{-1}\right] + z_0\mathbb{E}^{\mathbb{P}}\left[\lambda_1 - R\,\xi_1 X_0^f\right] \\ & + \sum_{t=1}^{T-1}\mathbb{E}^{\mathbb{P}}\left[c_t\left(R\xi_{t+1}-\xi_t\right)\right] + \sum_{t=1}^{T-1}\mathbb{E}^{\mathbb{P}}\left[s_t\left(\lambda_{t+1}-\lambda_t\right)\right] \\ & + \sum_{t=1}^{T-1}\mathbb{E}^{\mathbb{P}}\left[y_t\left(\xi_{t+1}X_{t+1}^e-\eta^{-1}\lambda_{t+1}\right)\right] \\ & + \sum_{t=1}^{T-1}\mathbb{E}^{\mathbb{P}}\left[z_t\left(\lambda_{t+1}-R\,\xi_{t+1}X_t^f\right)\right] \end{aligned} \tag{2.8}$$

Using (2.8), the tower property of conditional expectation and keeping in mind $y_t, s_t \geq 0$, the dual function

$$\max_{y\geq 0,z,c,s\geq 0} L(y,z,c,s;\xi,\lambda,\zeta,\gamma), \tag{2.9}$$

is bounded (in fact zero almost surely) if and only if the following conditions hold:

$$\zeta \geq 0 \tag{2.10}$$

$$\xi_T = 1+\zeta \tag{2.11}$$

$$\lambda_T \geq (1+\zeta)\,X_T^f \tag{2.12}$$

$$\gamma \geq 0 \tag{2.13}$$

$$\mathbb{E}^{\mathbb{P}}\left[\xi_1\right]R = \gamma \tag{2.14}$$

$$\mathbb{E}^{\mathbb{P}}\left[\lambda_1\right] \leq \gamma X_0^f \tag{2.15}$$

$$\mathbb{E}^{\mathbb{P}}\left[\xi_1 X_1^e\right] \leq \eta^{-1}\mathbb{E}^{\mathbb{P}}\left[\lambda_1\right] \tag{2.16}$$

$$\mathbb{E}^{\mathbb{P}}\left[\lambda_1\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_1\right]X_0^f \tag{2.17}$$

$$R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \qquad \text{for } t=1,\ldots,T-1 \tag{2.18}$$

$$\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t \qquad \text{for } t=1,\ldots,T-1 \tag{2.19}$$

$$\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e|\mathcal{F}_t\right] \leq \eta^{-1}\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \qquad \text{for } t=1,\ldots,T-1 \tag{2.20}$$

$$\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]X_t^f \qquad \text{for } t=1,\ldots,T-1. \tag{2.21}$$

Hence, the dual problem is a pure feasibility problem and it follows that the original problem is unbounded if and only if conditions (2.10)-(2.21) hold.

Equations (2.16) and (2.20), respectively (2.17) and (2.21) can be consolidated by

$$\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e|\mathcal{F}_t\right] \leq \eta^{-1}\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \qquad \text{for } t\in\mathcal{T}_0 \tag{2.22}$$

$$\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]X_t^f \qquad \text{for } t\in\mathcal{T}_0, \tag{2.23}$$

if the fact that $\mathcal{F}_0$ is the trivial σ-algebra is taken into account.

From (2.10) and (2.11) we can infer $\xi_T \geq 1 > 0$. Applying (2.18) recursively and keeping in mind $R>0$, it follows that

$$\xi_t > 0 \tag{2.24}$$

for $t\in\mathcal{T}_1$. Immediately we get

$$\gamma > 0 \tag{2.25}$$

from (2.14).

Therefore it is possible to divide inequalities by γ without changing their direction. It follows that processes ξ' and λ' fulfill the system (2.10)-(2.21) if and only if the processes $\xi_t = \frac{\xi_t'}{\gamma}$ and

$\lambda_t = \frac{\lambda'_t}{\gamma}$ fulfill the modified system

$$\zeta \geq 0 \tag{2.26}$$

$$\xi_T = \frac{1+\zeta}{\gamma} \tag{2.27}$$

$$\lambda_T \geq \frac{(1+\zeta)}{\gamma} X_T^f \tag{2.28}$$

$$1 \geq 0 \tag{2.29}$$

$$\mathbb{E}^{\mathbb{P}}[\xi_1] R = 1 \tag{2.30}$$

$$\mathbb{E}^{\mathbb{P}}[\lambda_1] \leq X_0^f \tag{2.31}$$

$$\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} X_{t+1}^e | \mathcal{F}_t\right] \leq \eta^{-1} \mathbb{E}^{\mathbb{P}}[\lambda_{t+1}|\mathcal{F}_t] \qquad \text{for } t = 0, \ldots, T-1 \tag{2.32}$$

$$\mathbb{E}^{\mathbb{P}}[\lambda_{t+1}|\mathcal{F}_t] = R\,\mathbb{E}^{\mathbb{P}}[\xi_{t+1}|\mathcal{F}_t] X_t^f \qquad \text{for } t = 0, \ldots, T-1 \tag{2.33}$$

$$R\mathbb{E}^{\mathbb{P}}[\xi_{t+1}|\mathcal{F}_t] = \xi_t \qquad \text{for } t = 1, \ldots, T-1 \tag{2.34}$$

$$\mathbb{E}^{\mathbb{P}}[\lambda_{t+1}|\mathcal{F}_t] \leq \lambda_t \qquad \text{for } t = 1, \ldots, T-1. \tag{2.35}$$

(2.29) is superfluous. Clearly, from (2.27) and (2.28) we have $\lambda_T \geq \xi_t X_t^f$. Furthermore, (2.26) and (2.25) are equivalent to $\xi_T > 0$. Together this gives property A2 of the Lemma. Equations (2.30) and (2.34) lead to property A3 and (2.35) is property A6.

It turns out that (2.31) is automatically fulfilled and can therefore be omitted: Using (2.33) it is possible to replace $\mathbb{E}^{\mathbb{P}}[\lambda_1]$ by $\mathbb{E}^{\mathbb{P}}[\xi_1] X_0^f$ in (2.31). Furthermore

$$\mathbb{E}^{\mathbb{P}}[\xi_1] = \frac{1}{R} \tag{2.36}$$

can be inferred from (2.34). Together this leads to

$$X_0^f \leq R X_0^f,$$

which is a tautology, given that $R \geq 1$ and X_0^f are nonnegative.

Finally, we see that property A4 and (2.32) are equivalent if (2.33) holds. Hence all properties, A1-A6, of the Lemma can be derived from the Lagrangian of the test problem.

Note also that

$$\lambda_t \geq 0 \tag{2.37}$$

for $t = 0, \ldots, T-1$ (and $\lambda_T > 0$) because of $X_t^f \geq 0$ and A2, A6. □

Lemma 2 can also be restated in probabilistic terms in the following way:

Proposition 1. *A market* $\left\{X_t^e, X_t^f\right\}$ *is* η*-arbitrage free if and only if there exist an equivalent measure* $\mathbb{Q}$ *and a process* $\{\lambda_t\}$ *with the following properties:*

B1: $\mathbb{E}^{\mathbb{Q}}\left[X_{t+1}^e|\mathcal{F}_t\right] \leq R\eta^{-1} X_t^f$ *for* $t \in \mathcal{T}_0$.

B2: $\lambda_T \geq X_T^f$

B3: $\frac{1}{R}\mathbb{E}^{\mathbb{Q}}[\lambda_{t+1}|\mathcal{F}_t] \leq \lambda_t$ *for* $t \in \mathcal{T}_1^{T-1}$

B4: $\frac{1}{R}\mathbb{E}^{\mathbb{Q}}[\lambda_{t+1}|\mathcal{F}_t] = X_t^f$ *for* $t \in \mathcal{T}_0$

Proof. Assume that the market is η-arbitrage free. Hence there exist processes ξ', λ' fulfilling conditions A1-A6 of Lemma 2. Define now

$$\xi_t = \frac{\xi'_t}{\mathbb{E}^{\mathbb{P}}[\xi'_t]} \text{ and } \lambda_t = \frac{\lambda'_t}{\xi'_t}. \tag{2.38}$$

Furthermore note that from (2.36) we can infer

$$\mathbb{E}^{\mathbb{P}}[\xi'_t] = \frac{1}{R^t}$$

by recursively applying property A3 and taking expectation. Immediately it follows from A2 and A3 that ξ_t is a (almost surely) positive martingale and $\mathbb{E}^{\mathbb{P}}[\xi_t] = 1$.

Together this means that ξ_t is a process of densities and we can construct an equivalent measure $\mathbb{Q}$ with density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi_T. \tag{2.39}$$

From $\lambda'_T \geq X_T^f \cdot \xi'_T$ (property A2. of Lemma 2) it is possible to conclude $\lambda_T \geq X_T^f$, i.e. B2.

The equation

$$\mathbb{E}^{\mathbb{P}}\left[\xi'_{t+1}X^e_{t+1}|\mathcal{F}_t\right] \le R\,\mathbb{E}^{\mathbb{P}}\left[\xi'_{t+1}|\mathcal{F}_t\right]\cdot\eta^{-1}X^f_t,$$

property A4 of Lemma 2, is equivalent to $\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X^e_{t+1}|\mathcal{F}_t\right] \le R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\cdot\eta^{-1}X^f_t$, and finally (using the abstract Bayes rule, see e.g. [4], Proposition B.41) to

$$\mathbb{E}^{\mathbb{Q}}\left[X^e_{t+1}|\mathcal{F}_t\right] \le R\,\eta^{-1}X^f_t,$$

which is B1.

In similar manner, starting with $\mathbb{E}^{\mathbb{P}}\left[\lambda'_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi'_{t+1}|\mathcal{F}_t\right]\cdot X^f_t$ (property A5 of Lemma2) we get

$$\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}\xi_{t+1}\mathbb{E}^{\mathbb{P}}\left[\xi'_{t+1}\right]|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}\mathbb{E}^{\mathbb{P}}\left[\xi'_{t+1}\right]|\mathcal{F}_t\right]\cdot X^f_t$$

and therefore

$$\mathbb{E}^{\mathbb{Q}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,X^f_t,$$

i.e. property B4 of the Proposition.

Finally, property A6 of Lemma 2, $\mathbb{E}^{\mathbb{P}}\left[\lambda'_{t+1}|\mathcal{F}_t\right] \le \lambda'_t$ is equivalent to

$$\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}\xi_{t+1}\mathbb{E}^{\mathbb{P}}\left[\xi'_{t+1}\right]|\mathcal{F}_t\right] \le \lambda_t\xi'_{t+1}.$$

Applying property A3 of of Lemma 2 another time gives

$$\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}\xi_{t+1}|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{P}}\left[\xi'_{t+1}\right] \le \lambda_t R\mathbb{E}^{\mathbb{P}}\left[\xi'_{t+1}|\mathcal{F}_t\right] = \lambda_t R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{P}}\left[\xi'_{t+1}\right],$$

and hence (again by the Bayes rule)

$$\mathbb{E}^{\mathbb{Q}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \le \lambda_t R,$$

which is property B3.

For the converse start with a measure $\mathbb{Q}$ and a process λ fulfilling conditions B1-B4 of the Proposition. Then use the density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$, see (2.39), to define a random variable ξ_T and the positive martingale

$$\xi_t = \mathbb{E}\left[\xi_{t+1}|\mathcal{F}_t\right]$$

as well as the processes

$$\xi'_t = \frac{\xi_t}{R^t} \text{ and } \lambda'_t{=}\lambda_t\xi'_t$$

Using the same equivalent transformation as above one can show that ξ', λ' fulfills 1-6 of Lemma (2) if ξ, λ fulfills B1-B4 of the Proposition. □

B1 can be interpreted easily: under the equivalent measure $\mathbb{Q}$ the expected proceeds of selling one MWh of electricity at the end of the period must be less or equal to the compounded costs $R\,\eta^{-1}X^f_t$ for producing one MWh electricity over the time period. Properties B2-B4 can be restated as $\lambda_t \ge X^f_t$ together with B4: under the measure $\mathbb{Q}$ the discounted expectation of λ equals the fuel price and at each point in time λ is not less than the fuel price. In the light of (2.38) the conditions can be interpreted as no arbitrage conditions between the shadow price of storage and the fuel price. Moreover, conditions B2-B4 can be used to derive further properties of the fuel price under the measure $\mathbb{Q}$:

Corollary 1. *If a market* $\left\{X^e_t, X^f_t\right\}$ *is* η*-arbitrage free then there exists an equivalent measure such that B1 holds together with*

$$\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[X^f_{t+1}|\mathcal{F}_t\right] \quad\le\quad X^f_t \text{ for } t\in\mathcal{T}_0. \tag{2.40}$$

Proof. From B3, B4 one can infer $X^f_t \le \lambda_t$ for $t\in\mathcal{T}_0$. Taking expectation and using B4 a second time leads to

$$\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[X^f_{t+1}|\mathcal{F}_t\right] \le \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = X^f_t \tag{2.41}$$

for$t\in\mathcal{T}_0$.

In similar manner (2.40) can be derived for $t = T-1$ from B2 and B4. □

The necessary condition (2.41), in fact stating that the discounted fuel price must be a supermartingale under any feasible $\mathbb{Q}$, ensures that the expected profit from storing fuel does not exceed the proceeds from immediately selling the fuel, if interest is taken into account.

The equivalent systems A1-A6 of Lemma 2 and B1-B4 of Proposition 1 include inequalities. Depending on the processes X^f_t and X^e_t, those systems will not in general lead to unique solutions ξ_t respectively $\mathbb{Q}$. Moreover, keep in mind that $\mathbb{Q}$ is not required to be a martingale measure like in purely financial models: neither the (discounted) fuel price, nor the electricity price are bounded to be martingales under $\mathbb{Q}$. Nevertheless it can be shown that the existence of a martingale measure for the fuel price process together with consistency between fuel and electricity price is sufficient for the exclusion of arbitrage.

Corollary 2. *If there exists an equivalent measure $\mathbb{Q}$ such that B1 holds together with*

$$\frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[X^f_{t+1}|\mathcal{F}_t\right] = X^f_t, \tag{2.42}$$

then the market is η-arbitrage free.

Proof. B1 holds by assumption. Set $\lambda_t = X^f_t$ for all t. This choice fulfills B2. Substituting λ_{t+1} for X^f_{t+1} at the left side of (2.42) leads to B4. Finally, using the same substitution on both sides, and observing $r > 0$ leads to B3. □

As a last remark in this section we note that Lemma 2 and Proposition 1 still hold (in the following sense), if several production units $i \in \{1, \ldots, K\}$ with different efficiencies are used by market participants. If y^i_t denotes the energy produced by unit i and η_i denotes the efficiency of unit i it is possible to reformulate the basic accounting equations (2.1) and (2.2) as

$$c_t = \left(c_{t-1} - z_{t-1}X^f_{t-1}\right) R + \sum_{i=1}^{K} y^i_{t-1}X^e_t, \tag{2.43}$$

$$s_t = s_{t-1} - \sum_{i=1}^{K} y^i_{t-1}\eta_i^{-1} + z_{t-1}, \tag{2.44}$$

which leads to a corresponding reformulation of the test problem (2.6). Analyzing the dual then leads to the following result.

Corollary 3. *Let η_{max} denote the largest production efficiency available to market participants. Then the market is η-arbitrage free for any $\eta_i \leq \eta_{max}$ if and only if conditions A1-A6 (respectively B1-B4) hold for $\eta = \eta_{max}$.*

Proof. Taking the same steps as in the proof of Lemma (2), but ensuring boundedness for each y^i_t separately, the reformulated test problem leads to conditions A4': $\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X^e_{t+1}|\mathcal{F}_t\right] \leq R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot \eta_i^{-1}X^f_t$ for $t \in \mathcal{T}_0$ and $i \in \{1, \ldots, K\}$. This is equivalent to $\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X^e_{t+1}|\mathcal{F}_t\right] \leq R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot \eta_{max}^{-1}X^f_t$ for $t \in \mathcal{T}_0$. □

3. Contract Pricing and Valuation

Consider now a delivery contract for electric energy, agreed between a producer and a customer and specified in the following way: the producer has the obligation to deliver an amount D_t [MWh] of electric energy at a price of price K currency units per MWh over each period $[t, t+1]$, $t \in \mathcal{T}_0$. The demand D_t is a stochastic process adapted to the filtration $\{\mathcal{F}_t\}$ while K is a fixed price, agreed in advance. Two natural questions arise, when designing the contract:

(1) What is the value of the contract, when the delivery price K is given.
(2) What is an adequate delivery price?

Given the no-arbitrage conditions derived above, in general there is little hope that a unique arbitrage free price (or valus) could be derived from a unique equivalent measure, even if the market is η-arbitrage free. Therefore, still taking into account the previously derived arbitrage conditions, we analyze valuation and pricing from the standpoint of a producer. A variant of the first question then can be stated as: What is the minimum initial asset value or upfront-payment $V_0 = c_0 + s_0X^f_0$ such that the producer is able to fulfill all contractual obligations and the distribution of the asset value at the end of the planning horizon, i.e. $V_T = c_T + s_TX^f_T$, still stays "acceptable". The second question can be restated in similar manner: Given an asset value of zero at the beginning, what is the minimum delivery price K such that the producer is able to fulfill all contractual obligations and the distribution of the asset value at the end of the planning horizon, i.e. $V_T = c_T + s_TX^f_T$, still stays "acceptable". Both, the minimum upfront payment and the minimum delivery price are not market prices or market values. However, when market value or market price exist and fall below the respective firms value or price, then the producer has to take some risk.

In the following we consider two types of acceptability. The first interpretation requires that the asset value is almost surely not negative at the end. This strict approach is closely related to financial superhedging and gives only a boundary on the price or value. In the second case, the producer measures acceptability by applying an acceptability functional (or a risk measure) to the end distribution and accepts only if the value exceeds some acceptability bound. In what follows, we call this approach "acceptability pricing".

3.1. **Superhedging and the minimum upfront payment.** The basic setup is the same as in the previous section. However, in contrast to the pure no-arbitrage arguments, an individual producer can not neglect that its fuel storage and production capacity is restricted. So S will denote the upper bound on storage and P_t is an $\{\mathcal{F}_t\}$-adapted process of upper bounds on the production of a generator with efficiency η. While from a technical point of view the generator has a fixed production capacity, the usage of an (adapted) process $P_t > 0$ of production capacities allows for considering the effects of reserve-requirements or preferential demand, not related to the contract under consideration. Furthermore, random outages can be handled in this way. Throughout the following subsections we assume that the producer is a price-taker at both the fuel and the electricity market.

Because of the contractual obligations only parts of the produced energy y_t can be sold at the electricity spot market after entering into the agreement. Moreover, it can be useful for the producer to buy electricity from the market in order to meet obligations, either because such trades are expected to be cheaper than producing or because the deliverable amount exceeds the production boundary. The amount of electricity sold at the market is an $\{\mathcal{F}_t\}$-adapted stochastic process of decisions that can take values in $\mathbb{R}$ and will be denoted by w_t. If it is negative, this means that an amount of energy is bought.

The following optimization problem describes the decision of a producer with efficiency η, trying to fulfill the contractual obligations, i.e. delivery of amounts amounts of energy $D_t > 0$ over the period $(t, t+1]$ at price K, and starting with a minimal amount of wealth $V_0^* = c_0 + X_0^f s_0$ at the beginning:

$$
\begin{aligned}
& V_0^*(K, D, \eta) = && (3.1)\\
& \qquad \min_{y,z,c,s} c_0 + X_0^f s_0 \\
& \qquad \text{subject to} \\
& \qquad (t \in \mathcal{T}_1) : c_t = \left(c_{t-1} - z_{t-1} X_{t-1}^f\right) R + y_{t-1} X_t^e - D_{t-1} X_t^e + K D_{t-1} \\
& \qquad (t \in \mathcal{T}_1) : s_t = s_{t-1} - y_{t-1}\eta^{-1} + z_{t-1} \\
& \qquad\qquad c_T + X_T^f s_T \geq 0 \\
& \qquad (t \in \mathcal{T}) : 0 \leq s_t \leq S \\
& \qquad (t \in \mathcal{T}_0) : 0 \leq y_t \leq P_t.
\end{aligned}
$$

Again, we assume that $c_t, y_t, s_t, z_t X_t^f$ and $X_T^f s_T$ are integrable. Moreover $D_t X_{t+1}^e$and D_t are assumed to be integrable. The optimal value V_0^* can be interpreted as the smallest up-front payment, the producer would accept for a contract with delivery D_t and delivery price D if he aims at an almost surely nonnegative end value.

Note that under the contractual obligations the cash position develops according to

$$
c_t = \left(c_{t-1} - z_{t-1} X_{t-1}^f\right) R + w_{t-1} X_t^e + K D_{t-1}. \tag{3.2}
$$

Here we assume that the delivery price K is payable at the end of each delivery period. Produced energy splits into energy traded at the market and contractual energy delivery, i.e. $w_t + D_t = y_t$. Therefore the first constraint of 3.1 follows by substituting $y_t - D_t$ for w_t in the basic cash equation 3.2.

The pricing problem (3.1) and the no-arbitrage test problem (2.6) look quite different at first glance. However it turns out that the related dual problems have quite similar constraints as the following Lemma shows.

Lemma 3. *The Lagrange dual of the valuation problem (3.1) is given by*

$$U_0^*(K,D,\eta) = \max_{\xi,\lambda,\mu,\nu} \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}\left(X_{t+1}^e - K\right) D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\right] \tag{3.3}$$

subject to

$$\begin{aligned}
& \xi_T \geq 0 \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \xi_T \\
& R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \text{ for } t = 1,\ldots,T-1 \text{ and } R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = 1 \\
(t \in \mathcal{T}_0) :\ & \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} X_{t+1}^e|\mathcal{F}_t\right] \leq R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot \eta^{-1} X_t^f + \mu_t \\
(t \in \mathcal{T}_0) :\ & \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot X_t^f \\
\left(t \in \mathcal{T}_1^{T-1}\right) :\ & \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t + \nu_t \\
(t \in \mathcal{T}) :\ & \mu_t \geq 0, \quad \nu_t \geq 0,
\end{aligned}$$

where ξ_t,λ_t, μ_t *and* ν_t *are essentially bounded and measurable w.r.t.* $\mathcal{F}_t$.

Proof. The Lagrangian of (3.1) is given by

$$\begin{aligned}
L(y,z,c,s;\xi,\lambda,\zeta,\mu,\nu) = {} & c_0 + X_0^f s_0 \\
& - \mathbb{E}^{\mathbb{P}}\left[\zeta\left(c_T + X_t^f s_T\right)\right] \\
& - \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}\left[\xi_t\left(Rc_{t-1} - c_t + X_t^e y_{t-1} - D_{t-1} X_t^e - R\, X_{t-1}^f z_{t-1} + K D_{t-1}\right)\right] \\
& - \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}\left[\lambda_t\left(s_{t-1} - s_t - y_{t-1}\eta^{-1} + z_{t-1}\right)\right] \\
& + \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t\left(y_t - P_t\right)\right] + \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\left(s_t - S\right)\right],
\end{aligned} \tag{3.4}$$

where $\zeta \geq 0$ is a $\mathcal{F}_T$-measurable random variable, and ξ_t, λ_t , $\mu_t \geq 0, \nu_t \geq 0$ are chosen $\mathcal{F}_t$-measurable and essentially bounded.

After reordering, (3.4) yields

$$\begin{aligned}
L(y,z,c,s;\xi,\lambda,\zeta,\gamma) = {} & \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}\left(X_{t+1}^e - K\right) D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\right] \\
& + \mathbb{E}^{\mathbb{P}}\left[c_T\left(\xi_T - \zeta\right)\right] + \mathbb{E}^{\mathbb{P}}\left[s_T\left(\lambda_T + \nu_t - X_T^f \zeta\right)\right] \\
& + c_0\left(1 - \mathbb{E}^{\mathbb{P}}\left[\xi_1\right] R\right) + s_0\left(X_0^f + \nu_0 - \mathbb{E}^{\mathbb{P}}\left[\lambda_1\right]\right) \\
& + y_0 \mathbb{E}^{\mathbb{P}}\left[\lambda_1 \eta^{-1} + \mu_0 - \xi_1 X_1^e\right] + z_0 \mathbb{E}^{\mathbb{P}}\left[R\,\xi_1 X_0^f - \lambda_1\right] \\
& + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}}\left[c_t\left(\xi_t - R\xi_{t+1}\right)\right] + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}}\left[s_t\left(\lambda_t - \lambda_{t+1}\right)\right] \\
& + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}}\left[y_t\left(\eta^{-1}\lambda_{t+1} + \mu_t - \xi_{t+1} X_{t+1}^e\right)\right] \\
& + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}}\left[z_t\left(\xi_{t+1} X_t^f R - \lambda_{t+1}\right)\right].
\end{aligned} \tag{3.5}$$

The dual problem then reads

$$\max_{\xi,\lambda,\mu,\nu} \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}\left(X_{t+1}^{e}-K\right)D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\right] \tag{3.6}$$

subject to

$$\begin{aligned}
&\xi_T = \zeta \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \zeta \\
&X_0^f + \nu_0 \geq \mathbb{E}^{\mathbb{P}}\left[\lambda_1\right] \\
&R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \text{ for } t = 1,\ldots,T-1 \text{ and } R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = 1 \\
(t\in\mathcal{T}_0) :\ &\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^{e}|\mathcal{F}_t\right] \leq \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right]\cdot\eta^{-1} + \mu_t \\
(t\in\mathcal{T}_0) :\ &\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\cdot X_t^f \\
\left(t\in\mathcal{T}_1^{T-1}\right) :\ &\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t + \nu_t \\
(t\in\mathcal{T}) :\ &\zeta\geq 0, \quad \mu_t \geq 0, \quad \nu_t\geq 0,
\end{aligned}$$

where the constraints ensure that the Lagrange function $\min_{y\geq 0,z,c,s\geq 0} L(y,z,c,s;\xi,\lambda,\zeta,\gamma)$ stays bounded.

Constraint group 3-6 of (3.6) already coincide with the constraint groups 2-5 of (3.3). Using $\xi_T = \zeta \geq 0$ the multiplier ζ can be eliminated, which (using the first line of constraints in (3.3)) leads to $\xi_T \geq 0$ and $\lambda_T + \nu_T \geq X_T^f \cdot \xi_T$. This resembles the first line of constraints in (3.3). The second constraint of (3.6) can be reformulated as

$$X_0^f + \nu_0 \geq \mathbb{E}^{\mathbb{P}}\left[\lambda_1\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_1\right]X_0^f = X_0^f,$$

and taking into account $\nu_t \geq 0$ shows that it is superfluous. □

Absence of η-arbitrage ensures feasibility and the existence of inner solutions of (3.3), which leads to the following assertion:

Corollary 4. *If a market $\left\{X_t^e, X_t^f\right\}$ is η-arbitrage free, then*

$$V_0^*(K,D,\eta) = U_0^*(K,D,\eta). \tag{3.7}$$

Proof. A1-A6 of Lemma 2 defines a subset of the feasible set of (3.3). By absence of η-arbitrage the set defined by A1-A6 must be nonempty (Lemma (2)), which implies that the feasible set of (3.3) has nonempty interior. Observe now that starting with a feasible solution $\xi_t > 0, \lambda_t > 0$ and $\mu_t = \nu_t = 0$, it is always feasible to increase μ_t and ν_t, which means that the relative interior of the feasible set is not empty. Then $V_0^*(K,D,\eta) = U_0^*(K,D,\eta)$ follows from strong duality. □

Remark 2. If the market is not η-arbitrage free, still (3.3) is the dual of (3.1). However, the arbitrage argument can not be used to ensure strong duality. Therefore, for all results based on equality between primal and dual optimal value, one has to find other arguments for showing that the feasible set of (3.11) is nonempty. Alternatively one may try to base the argumentation on other constraint qualifications.

Lemma 3, together with Corollary 4, shows that the process ξ (actually the shadow price related to the cash position) can be interpreted as a process of stochastic discount factors applied to the opportunity costs of delivering at price K. The processes μ and ν valuate the available production and storage capacities. On the other hand, the process λ (the shadow price for storage accounting) seems to be harder to interprete. However, note that it is possible to derive the relation $\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = \xi_t X_t^f$, which shows that the expectation of λ_{t+1} equals the (stochastically) discounted fuel price at time t.

If the market is η-arbitrage free, the minimum upfront payment can be calculated by the dual program. In this case the optimal multiplicators ξ,λ,μ and ν are interpretable as stochastic discount factors, applied to the processes $\left(X_{t+1}^e - K\right)D_t$, P_t and $-S$. We can characterize them even more precisely: the relation

$$\mathbb{E}^{\mathbb{P}}\left[\xi_t\right] = \frac{1}{R^t} \tag{3.8}$$

can be derived from the constraints of the dual program in the same way as (2.36) was derived from the no-arbitrage conditions. Furthermore, constraint groups three and five of (3.3) implie

$$\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}\right] = \mathbb{E}^{\mathbb{P}}\left[X_t^f\right] + Cov(\xi_t, X_t^f).$$

Finally, given the stochastic discount factors one can derive the following decomposition of the minimum upfront payment.

Corollary 5. *If a market is η-arbitrage free, and given the stochastic discount factors ξ, λ, μ, ν the superhedging value can be decomposed as follows:*

$$U_0^*(K, D, \eta) = \sum_{t=0}^{T-1} \frac{1}{R^{t+1}} \mathbb{E}^{\mathbb{P}} \left[\left(X_{t+1}^e - K \right) D_t \right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}} \left[\mu_t \right] \mathbb{E}^{\mathbb{P}} \left[P_t \right] - S \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}} \left[\nu_t \right] + \sum_{t=0}^{T-1} Cov^{\mathbb{P}} \left[\xi_{t+1}, \left(X_{t+1}^e - K \right) D_t \right] - \sum_{t=0}^{T-1} Cov^{\mathbb{P}} \left[\mu_t, P_t \right]. \quad (3.9)$$

Proof. For η-arbitrage free market, corollary (4) can be applied. The assertion then can be obtained easily from the objective function of (3.3), by applying the relation $Cov(X,Y) = \mathbb{E}^{\mathbb{P}}[XY] - \mathbb{E}^{\mathbb{P}}[X]\,\mathbb{E}^{\mathbb{P}}[Y]$ and (3.8). □

The first termat the right hand side of (3.9) is the expected present value of a pure trader, fulfilling the contractual obligations by electricity bought at the market. Further expectation terms correct for the effects of (limited) storage and production outages. The first covariance term can be interpreted as a risk loading for the pure trader, while the second covariance term is related to the effects of uncertain production capacities. Because the covariances may have any sign and the effects of production and storage are subtracted from the expectation, all kinds of contango and backwardation may arise.

From the objective function of (3.11) it can be seen that the minimum up-front payment increases with increasing demands D_t and decreases with increasing delivery price K and with increasing capacities P_t, S .

Corollary 6. *If the market $\left\{X_t^e, X_t^f\right\}$ is η-arbitrage free, then the optimal value function $V_0^*(K, D, \eta)$ is convex (separately) in K and in D.*

Proof. Let F denote the (convex) set of feasible $\xi,\ \lambda,\ \mu,\ \nu$ for formulation (3.12) and define

$$\delta_F(\xi, \lambda, \mu, \nu) = \begin{cases} 0 & (\xi, \lambda) \in F \\ \infty & \text{else.} \end{cases}$$

We then have

$$U_0^*(K, D, \eta) = \max_{\xi, \lambda, \mu, \nu} \left\{ \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}} \left[\xi_{t+1} \left(X_{t+1}^e - K \right) D_t \right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}} \left[\mu_t P_t \right] - S \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}} \left[\nu_t \right] + \delta_F(\xi, \lambda) \right\} \quad (3.10)$$

which is equal to $V_0^*(K, D, \eta)$ by Proposition 2. For fixed K the combined objective in (3.10) is convex in D. Hence,$U_0^*(K, D, \eta)$ is obtained as a pointwise maximum over an infinite set of convex functionals, which shows that $U_0^*(K, D, \eta)$ is convex in D. A similar argument can be applied to show convexity in K. □

An important difference between the constraints of (3.3) and the no-arbitrage conditions A1-A6 is the fact that in the first case we have $\xi_T \geq 0$, whereas in the second case the stricter $\xi_T > 0$ is demanded. For η-arbitrage free markets the dual problem can be restated by using equivalent measures in the following way:

Proposition 2. *If the market* $\left\{X_t^e, X_t^f\right\}$ *is* η*-arbitrage free, then the superhedging value* $V_0^*(K,D)$ *can be calculated as*

$$V_0^*(K,D,\eta) = U_0^*(K,D,\eta) = \sup_{\mathbb{Q},\lambda,\mu,\nu} \mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{t=0}^{T-1} \frac{1}{R^{t+1}}\left(X_{t+1}^e - K\right) D_t\right)\right] \tag{3.11}$$
$$- \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{Q}} \frac{1}{R^t}\left[\mu_t P_t\right] - S \sum_{t=0}^{T} \frac{1}{R^t} \mathbb{E}^{\mathbb{Q}}\left[\nu_t\right]$$

subject to

$$\begin{aligned}
(t \in \mathcal{T}_0) &: \mathbb{E}^{\mathbb{Q}}\left[X_{t+1}^e | \mathcal{F}_t\right] \leq R\eta^{-1} X_t^f + \mu_t \\
& \lambda_T + \nu_T \geq X_T^f \\
\left(t \in \mathcal{T}_1^{T-1}\right) &: \mathbb{E}^{\mathbb{Q}}\left[\lambda_{t+1} | \mathcal{F}_t\right] \leq \left(\lambda_t + \nu_t\right) R \\
(t \in \mathcal{T}_0) &: \mathbb{E}^{\mathbb{Q}}\left[\lambda_{t+1} | \mathcal{F}_t\right] = R X_t^f \\
& \mathbb{P} \sim \mathbb{Q}.
\end{aligned}$$

Proof. Starting from (3.3) the dual problem can be reformulated as

$$U_0^*(K,D,\eta) = \sup_{\xi,\lambda,\mu,\nu} \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}\left(X_{t+1}^e - K\right) D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\right] \tag{3.12}$$

subject to

$$\xi_T > 0 \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \xi_T \tag{3.13}$$
$$\begin{aligned}
(t \in \mathcal{T}_1) &: R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} | \mathcal{F}_t\right] = \xi_t \text{ and } R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = 1 \\
(t \in \mathcal{T}_0) &: \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} X_{t+1}^e | \mathcal{F}_t\right] \leq R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} | \mathcal{F}_t\right] \cdot \eta^{-1} X_t^f + \mu_t \\
(t \in \mathcal{T}_0) &: \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1} | \mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} | \mathcal{F}_t\right] \cdot X_t^f \\
\left(t \in \mathcal{T}_1^{T-1}\right) &: \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1} | \mathcal{F}_t\right] \leq \lambda_t + \nu_t \\
(t \in \mathcal{T}) &: \mu_t \geq 0, \quad \nu_t \geq 0,
\end{aligned}$$

If the market is η-arbitrage free then by Lemma 2 the feasible set of (3.12) is not empty. Let $\xi', \lambda', \mu', \nu'$ be feasible for (3.12) and define again

$$\xi_t = \frac{\xi_t'}{\mathbb{E}^{\mathbb{P}}\left[\xi_t'\right]} \text{ and } \lambda_t = \frac{\lambda_t'}{\xi_t'}. \tag{3.14}$$

In addition, now set

$$\mu_t = \frac{\mu'}{\xi'} \text{ and } \nu_t = \frac{\nu_t'}{\xi_t'}. \tag{3.15}$$

Clearly we then have $\mu_t \geq 0$ $\nu_t \geq 0$. The arguments used in the proof of Proposition 1 can be applied again to derive the following dual feasibility conditions for λ, μ, ν together with and an equivalent measure $\mathbb{Q}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi_T$:

C1: $\mathbb{E}^{\mathbb{Q}}\left[X_{t+1}^e | \mathcal{F}_t\right] \leq R\,\eta^{-1} X_t^f + \mu_t$ for $t \in \mathcal{T}_0$
C2: $\lambda_T + \nu_T \geq X_T^f$
C3: $\mathbb{E}^{\mathbb{Q}}\left[\lambda_{t+1} | \mathcal{F}_t\right] \leq \left(\lambda_t + \nu_t\right) R$ for $t \in \mathcal{T}_1^{T-1}$
C4: $\mathbb{E}^{\mathbb{Q}}\left[\lambda_{t+1} | \mathcal{F}_t\right] = R\,X_t^f$ for $t \in \mathcal{T}_0$
C5: $\mu_t \geq 0$,$\nu_t \geq 0$ for $t \in \mathcal{T}$

The second group of constraints in (3.3), i.e. $R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} | \mathcal{F}_t\right] = \xi_t$ is equivalent to $\xi_{t+1} = R^{T-t-1}\mathbb{E}^{\mathbb{P}}\left[\xi_T | \mathcal{F}_{t+1}\right]$. Together with definitions (3.15) this fact can be used to reformulate the dual objective function as

$$\sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}'\left(X_{t+1}^e - K\right) D_t\right] = \mathbb{E}^{\mathbb{P}}\left[\xi_T' \sum_{t=0}^{T-1} R^{T-t-1}\left(X_{t+1}^e - K\right) D_t\right]$$
$$- \sum_{t=0}^{T-1} R^{T-t}\mathbb{E}^{\mathbb{P}}\left[\xi_T' \mu_t P_t\right] - S \sum_{t=0}^{T} R^{T-t}\mathbb{E}^{\mathbb{P}}\left[\xi_T' \nu_t\right]$$

Applying now (3.14) and the relation $\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi_T$ it is possible to reformulate the objective function again and one gets the reformulated dual

$$\begin{array}{cc} \sup\limits_{\mathbb{Q},\lambda,\mu,\nu} & \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{R^{t+1}}\left(X_{t+1}^e - K\right) D_t\right] - \sum_{t=0}^{T-1} \frac{1}{R^t}\mathbb{E}^{\mathbb{Q}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T} \frac{1}{R^t}\mathbb{E}^{\mathbb{P}}\left[\nu_t\right] \\ s.t. & C1 - C5 \\ & \mathbb{Q} \sim \mathbb{P}. \end{array}$$

If the market is η-arbitrage free we have $V_0^*(K,D,\eta) = U_0^*(K,D,\eta)$ by Corollary (4). □

3.2. **Superheding prices.** We keep the basic setup for a producer with bounded storage and production. In order to find a superhedging price ensuring that all contractual obligations can be fulfilled and the end wealth stays nonnegative, specification (3.1) is modified in the following way: the objective now is to minimize the delivery price and the asset value starts at zero at time zero.

$$\begin{aligned} K^*(D,\eta) = & \min_{y,z,c,s} K \\ & \text{subject to} \\ & (t \in \mathcal{T}_1): c_t = \left(c_{t-1} - z_{t-1}X_{t-1}^f\right) R + y_{t-1}X_t^e - D_{t-1}X_t^e + KD_{t-1} \\ & (t \in \mathcal{T}_1): s_t = s_{t-1} - y_{t-1}\eta^{-1} + z_{t-1} \\ & \qquad c_0 + X_0^f s_0 = 0 \\ & \qquad c_T + X_T^f s_T \geq 0 \\ & (t \in \mathcal{T}): 0 \leq s_t \leq S \\ & (t \in \mathcal{T}_0): 0 \leq y_t \leq P_t. \end{aligned} \tag{3.16}$$

Again, in order to analyze (3.16) we use the related dual representation.

Proposition 3. *The Lagrange dual of the pricing problem (3.16) is given by*

$$G^*(D,\eta) = \max_{\xi,\lambda,\mu,\nu,\gamma} \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\right] \tag{3.17}$$

subject to

$$\xi_T \geq 0 \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \xi_T \tag{3.18}$$

$$R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \text{ for } t = 1,\dots,T-1, \text{ and } R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = \gamma$$

$$(t \in \mathcal{T}_0): \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e|\mathcal{F}_t\right] \leq R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot \eta^{-1}X_t^f + \mu_t$$

$$(t \in \mathcal{T}_0): \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot X_t^f$$

$$\left(t \in \mathcal{T}_1^{T-1}\right): \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t + \nu_t$$

$$\sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}D_t\right] = 1 \tag{3.19}$$

$$(t \in \mathcal{T}): \mu_t \geq 0, \quad \nu_t \geq 0,$$

where for each $t = 1,\dots,T$ the random variables ξ_t,λ_t, μ_t and ν_t are essentially bounded and measurable w.r.t. $\mathcal{F}_t$.

If the market is η-arbitrage free then

$$K^*(D,\eta) = G^*(D,\eta).$$

Proof. The Lagrangian of (3.17) is

$$
\begin{aligned}
L(y,z,c,s;\xi,\lambda,\zeta,\mu,\nu) = & \qquad\qquad (3.20)\\
& K\\
& - \mathbb{E}^{\mathbb{P}}\left[\zeta\left(c_T + X_t^f s_T\right)\right]\\
& - \sum_{t=1}^{T}\mathbb{E}^{\mathbb{P}}\left[\xi_t\left(Rc_{t-1} - c_t + X_t^e y_{t-1} - D_{t-1}X_t^e - R\,X_{t-1}^f z_{t-1} + KD_{t-1}\right)\right]\\
& - \sum_{t=1}^{T}\mathbb{E}^{\mathbb{P}}\left[\lambda_t\left(s_{t-1} - s_t - y_{t-1}\eta^{-1} + z_{t-1}\right)\right]\\
& + \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\mu_t\left(y_t - P_t\right)\right] + \sum_{t=0}^{T}\mathbb{E}^{\mathbb{P}}\left[\nu_t\left(s_t - S\right)\right]\\
& + \gamma\left(c_0 + X_0^f s_0\right), \qquad\qquad (3.21)
\end{aligned}
$$

where $\zeta \geq 0$ is an essentially bounded $\mathcal{F}_T$-measurable random variable, and ξ_t, λ_t , $\mu_t \geq 0$, $\nu_t \geq 0$ are chosen $\mathcal{F}_t$-measurable and essentially bounded.

After reordering (3.20) yields

$$
\begin{aligned}
L(y,z,c,s;\xi,\lambda,\zeta,\gamma) = & \qquad\qquad (3.22)\\
& \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e D_t\right] - \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T}\mathbb{E}^{\mathbb{P}}\left[\nu_t\right]\\
& + \mathbb{E}^{\mathbb{P}}\left[c_T\left(\xi_T - \zeta\right)\right] + \mathbb{E}^{\mathbb{P}}\left[s_T\left(\lambda_T + \nu_t - X_T^f\zeta\right)\right]\\
& + c_0\left(\gamma - \mathbb{E}^{\mathbb{P}}\left[\xi_1\right]R\right) + s_0\left(\gamma X_0^f + \nu_0 - \mathbb{E}^{\mathbb{P}}\left[\lambda_1\right]\right)\\
& + y_0\mathbb{E}^{\mathbb{P}}\left[\lambda_1\eta^{-1} + \mu_0 - \xi_1 X_1^e\right] + z_0\mathbb{E}^{\mathbb{P}}\left[R\,\xi_1 X_0^f - \lambda_1\right]\\
& + \sum_{t=1}^{T-1}\mathbb{E}^{\mathbb{P}}\left[c_t\left(\xi_t - R\xi_{t+1}\right)\right] + \sum_{t=1}^{T-1}\mathbb{E}^{\mathbb{P}}\left[s_t\left(\lambda_t - \lambda_{t+1}\right)\right]\\
& + \sum_{t=1}^{T-1}\mathbb{E}^{\mathbb{P}}\left[y_t\left(\eta^{-1}\lambda_{t+1} + \mu_t - \xi_{t+1}X_{t+1}^e\right)\right]\\
& + \sum_{t=1}^{T-1}\mathbb{E}^{\mathbb{P}}\left[z_t\left(\xi_{t+1}X_t^f R - \lambda_{t+1}\right)\right]\\
& + K\left(1 - \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}D_t\right]\right). \qquad\qquad (3.23)
\end{aligned}
$$

The Lagrange dual problem then can be derived as

$$\max_{\xi,\lambda,\mu,\nu} \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^{e}D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\right] \tag{3.24}$$

subject to

$$\begin{aligned} & \xi_T = \zeta \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \zeta \qquad (3.25)\\ & \gamma X_0^f + \nu_0 \geq \mathbb{E}^{\mathbb{P}}\left[\lambda_1\right] \\ & R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \text{ for } t = 1,\dots,T-1, \text{ and } R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = \gamma \\ (t\in\mathcal{T}_0): & \ \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^{e}|\mathcal{F}_t\right] \leq \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right]\cdot\eta^{-1} + \mu_t \text{ for } t = 0,\dots,T-1 \\ (t\in\mathcal{T}_0): & \ \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\cdot X_t^f \text{ for } t = 0,\dots,T-1 \\ \left(t\in\mathcal{T}_1^{T-1}\right): & \ \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t + \nu_t \text{ for } t = 0,\dots,T-1 \\ & \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}D_t\right] = 1 \\ & \zeta \geq 0 \text{ and } \mu_t \geq 0, \quad \nu_t \geq 0 \text{ for } t\in\mathcal{T}, \end{aligned}$$

where the constraints ensure that the Lagrange function

$$\min_{y\geq 0,z,c,s\geq 0} L(y,z,c,s;\xi,\lambda,\zeta,\gamma) \tag{3.26}$$

stays bounded.

Constraint groups 3-6 of (3.24) already coincide with constraint groups 2-5 of (3.3). Using $\xi_T = \zeta \geq 0$ we can eliminate ζ which leads to $\xi_T \geq 0$ and $\lambda_T + \nu_T \geq X_T^f \cdot \xi_T$ from the first line of constraints of (3.17). This resembles the first line of constraints in (3.17). Finally, the second constraint of (3.24) can be reformulated as

$$\gamma X_0^f + \nu_0 = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] X_0^f + \nu_0 \geq \mathbb{E}^{\mathbb{P}}\left[\lambda_1\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] X_0^f,$$

Taking into account $\nu_t \geq 0$ shows that this constraint is superfluous. Altogether we have the dual (3.24).

Now, because of $R > 0$ and $R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = \gamma$ the γ is nonnegative. Moreover, $\gamma = 0$ implies $\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = 0$ and hence $\xi_1 = 0$ with probability one, which (using the second constraint and $\xi_t \geq 0$) implies $\xi_t = 0$ a.s. for all t. This contradicts $\sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}D_t\right] = 1$ and hence we can conclude$\gamma > 0$. If the market is η-arbitrage free, conditions A1-A6 define a nonempty set of processes ξ, λ. Multiplying all equations and inequalities by $\gamma > 0$, one sees that the reskaled processes $\kappa\xi, \kappa\lambda$ fulfill A1-A6 but with $R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = 1$ replaced by $R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = \kappa$. Because it is always feasible to increase μ_t and ν_t, it follows that the relative interior of the feasible set of (3.24) is not empty. This implies strong duality. □

From the dual problem one sees that the minimum delivery price can be obtained by applying the stochastic discount factor process ξ to the process of opportunity costs, the revenues from selling the contractual delivery amounts at the electricity spot market. The resulting expected present value is augmented by a valuation for the effects of restricted production and storage. There is also a decomposition into expected present values and risk premia.

Corollary 7. *If a market* $\left\{X_t^e, X_t^f\right\}$ *is* η*-arbitrage free, and given the stochastic discount factors* ξ, λ, μ, ν *and* γ*, the superhedging price can be decomposed into*

$$\begin{aligned} K^*(D,\eta) = \sum_{t=0}^{T-1} \frac{1}{R^{t+1}} \mathbb{E}^{\mathbb{P}}\left[X_{t+1}^{e}D_t\right] \\ - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t\right]\mathbb{E}^{\mathbb{P}}\left[P_t\right] - S\sum_{t=0}^{T}\mathbb{E}^{\mathbb{P}}\left[\nu_t\right] \\ + \sum_{t=0}^{T-1} Cov^{\mathbb{P}}\left[\xi_{t+1}, X_{t+1}^{e}D_t\right] - \sum_{t=0}^{T-1} Cov^{\mathbb{P}}\left[\mu_t, P_t\right]. \end{aligned}$$

Proof. Again, this is a straightforward application of $Cov(X,Y) = \mathbb{E}^{\mathbb{P}}\left[XY\right] - \mathbb{E}^{\mathbb{P}}\left[X\right]\mathbb{E}^{\mathbb{P}}\left[Y\right]$. □

The interpretation of the stochastic discount factor ξ in Proposition (3) is quite different from the interpretation of ξ in the case of superhedging. While $\mathbb{E}[\xi_t] = \frac{1}{R^t}$ for valuations, this is not true in Proposition (3) . Instead, ξ is normalized to ensure $\sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}[\xi_{t+1} D_t] = 1$, a valuation of the demand process. Nonetheless, it is possible to reformulate the dual such that the stochastic discount factors can be interpreted in the same way as for valuation. However this leads away from the linear objective function in (3.17): the superhedging price can be interpreted as expected (and adjusted) present value of the opportunity costs divided by the expected and discounted demand, where both expectations use ξ as stochastic discount factors.

Corollary 8. *The dual problem (3.17) is equivalent to the fractional optimization problem*

$$G^*(D,\eta) = \max_{\xi,\lambda,\mu,\nu,\gamma} \frac{\sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} X_{t+1}^e D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}[\mu_t P_t] - S \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}[\nu_t]}{\sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}[\xi_{t+1} D_t]} \tag{3.27}$$

subject to

$$\begin{aligned}
&\xi_T \geq 0 \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \xi_T \\
&R\mathbb{E}^{\mathbb{P}}[\xi_{t+1}|\mathcal{F}_t] = \xi_t \text{ for } t = 1,\dots,T-1, \text{ and } R\mathbb{E}^{\mathbb{P}}[\xi_1] = 1 \\
(t \in \mathcal{T}_0) :&\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} X_{t+1}^e|\mathcal{F}_t\right] \leq R\,\mathbb{E}^{\mathbb{P}}[\xi_{t+1}|\mathcal{F}_t] \cdot \eta^{-1} X_t^f + \mu_t \\
(t \in \mathcal{T}_0) :&\mathbb{E}^{\mathbb{P}}[\lambda_{t+1}|\mathcal{F}_t] = R\,\mathbb{E}^{\mathbb{P}}[\xi_{t+1}|\mathcal{F}_t] \cdot X_t^f \\
\left(t \in \mathcal{T}_1^{T-1}\right) :&\mathbb{E}^{\mathbb{P}}[\lambda_{t+1}|\mathcal{F}_t] \leq \lambda_t + \nu_t \\
(t \in \mathcal{T}) :&\mu_t \geq 0, \quad \nu_t \geq 0,
\end{aligned}$$

Proof. Suppose $\xi', \lambda', \mu', \nu'$ and γ fulfill the constraints of the dual problem (3.17). Define

$$\xi_t = \frac{\xi_t'}{\gamma},\ \lambda_t = \frac{\lambda_t'}{\gamma} \text{ and } \mu_t = \frac{\mu_t'}{\gamma}\ \nu_t = \frac{\nu_t'}{\gamma}.$$

Dividing all constraints of the dual (3.17) by γ and using the rescaled variables ξ, λ, μ, ν, one gets the constraints of (3.27) plus the constraint

$$\sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}[\xi_{t+1} D_t] = \frac{1}{\gamma}. \tag{3.28}$$

On the other hand, the dual objective function can be rewritten with scaled multiplicators:

$$\begin{aligned}
&\sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}' X_{t+1}^e D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}[\mu_t' P_t] - S \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}[\nu_t'] \\
&\qquad = \gamma \cdot \left(\sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}[\xi_{t+1} D_t] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}[\mu_t P_t] - S \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}[\nu_t] \right).
\end{aligned} \tag{3.29}$$

Substituting γ from (3.28) into (3.29) leads to the objective of (3.27). □

3.3. **Capital Requirement and Acceptability pricing.** Superhedging-based approaches lead to price bounds for delivery contracts, comparable with the rough no-arbitrage bounds for options. The related strategy is riskless from the producer's perspective but can be very expensive from the customer's point of view. Usually the contract can not be sold a tthe superhedging price. For a reasonable price (or value) the producer definitely has to take some risk.

In the following we consider a producer who deviates from complete superhedging but nevertheless wants to control the risk: instead of aiming at an almost surely nonnegative value of the strategy at the end, the producer accepts some risk but bounds the acceptability of the final wealth distribution. In terms of optimization, he replaces the third constraint group of the original problem (3.1) (or the fourth of (3.16)) by a constraint

$$\mathcal{A}\left(c_T + X_T^f s_T\right) \geq \alpha, \tag{3.30}$$

where $\mathcal{A}$ denotes an acceptability functional which maps integrable random variables to the real line and α is a real number. In the following we analyze the two main issues, valuation and pricing in this new context. If we search for the smallest capital such that (3.30) holds together with the production and trading constraints of (3.1), this is the problem of *acceptability valuation (capital requirement).* If we search instead for the minimum delivery price, using (3.30) within (3.16), we call this an *acceptability pricing* problem.

Acceptability valuation and pricing in the strict sense aim at the case $\alpha = 0$. This is a relaxation of the superhedging problem, hence leads to smaller feasible capital values and prices. However, $\alpha \neq 0$ may also be a sensible choice in certain situations. In particular, electricity producers often optimize their production and trading activities. See e.g. [24, 23, 25, 22] for stochastic optimization models and surveys. Such a producer therefore is able to maximize the acceptability of his portfolio without the contract and then to calculate a capital requirement or acceptability price for the new contract (keeping all existing production and trading possibilities) with an acceptability not below the optimal value, calculated previously. This idea is often called *indifference pricing* (see [5, 19] for the general idea, and [15, 1] for some applications to electricity production and trading).

Different notions of acceptability exist in the literature. In the following we use acceptability functionals and mappings (see e.g. [20]}), also known as monetary utility functions (e.g. [13]). Acceptability functionals are mappings $\mathcal{A} : \Xi \subseteq L_1(\Omega, \mathfrak{F}, \mathbb{P}) \to \overline{\mathbb{R}}$ (where Ξ is a subspace), satisfying the following properties

- (MA1) **Concavity.** The functional $Y \mapsto \mathcal{A}(Y)$ is concave.
- (MA2) **Monotonicity.** If $X, Y \in L_1(\Omega, \mathfrak{F}, \mathbb{P})$ and $X \leq Y$ holds a.s., then $\mathcal{A}(X) \leq \mathcal{A}(Y)$.
- (MA3) **Translation Equivariance.** If $X \in L_1(\Omega, \mathfrak{F}, \mathbb{P})$ and $a \in \mathbb{R}$ then $\mathcal{A}(X + a) = \mathcal{A}(X) + \beta a$.

It should be kept in mind that - up to sign - such functionals are identical with convex risk measures, e.g. [11]. For simplicity we restrict the analysis to positive homogeneous functionals centered at zero, i.e.

- (MA4) If $X \in L_1(\Omega, \mathfrak{F}, \mathbb{P})$ and $a \in \mathbb{R} \geq 0$ then $\mathcal{A}(aX) = a\mathcal{A}(X)$.
- (MA5) $\mathcal{A}(0) = 0$

Recall that any positive homogeneous acceptability functional functional $\mathcal{A}$ on Ξ can be rewritten in terms of the conjugate representation

$$\mathcal{A}(X) = \mathbf{inf}\left\{\mathbb{E}\left(\zeta X\right) : \zeta \in \Upsilon_{\mathcal{A}}\right\}, \tag{3.31}$$

for some set $\Upsilon_{\mathcal{A}} \subseteq \left\{\zeta : \ Z \in \Xi^* \text{and } \mathbb{E}^{\mathbb{P}}\left[\zeta\right] = \beta, \ \ \zeta \geq 0\right\}$ where Ξ^* is the dual space of Ξ. In this formulation β can be interpreted as a discount factor. The infimum is attained. See e.g. [20], 2.22 for more details and many examples of relevant functionals and [9] for a related class of risk functionals used in electricity planning.

We denote the set of stochastic processes fulfilling A1-A5 by $\mathbb{A}$, which is nonempty if the market is η-arbitrage free. Based on the above setup, capital requirements calculated by acceptability valuation lead to the following modification of the superhedging results:

Proposition 4. *Consider an acceptability valuation problem based on a positive homogeneous acceptability functional $\mathcal{A}$ with conjugate representation (3.31) and $\beta = \frac{1}{R^T}$. The dual problem is*

$$U_0^*(K, D, \eta) = \tag{3.32}$$
$$\max_{\xi,\lambda,\mu,\nu} \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}\left(X_{t+1}^e - K\right) D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\right] + \alpha$$

subject to

$$\begin{aligned} & \xi_T \in \Upsilon_{\mathcal{A}} \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \xi_T \\ & R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \text{ for } t = 1, \ldots, T, \text{ and } R\,\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = 1 \\ (t \in \mathcal{T}_0) : \ & \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1} X_{t+1}^e|\mathcal{F}_t\right] \leq R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot \eta^{-1} X_t^f + \mu_t \\ (t \in \mathcal{T}_0) : \ & \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] \cdot X_t^f \\ \left(t \in \mathcal{T}_1^{T-1}\right) : \ & \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t + \nu_t \\ (t \in \mathcal{T}) : \ & \mu_t \geq 0, \quad \nu_t \geq 0. \end{aligned} \tag{3.33}$$

If the market $\left\{X_t^e, X_t^f\right\}$ is arbitrage free and $\Upsilon_{\mathcal{A}} \cap \mathbb{X} \neq \emptyset$, then the capital requirement $V_0^*(K, D, \eta)$ equals the dual value $U_0^*(K, D, \eta)$.

Proof. Define a generalized Lagrangian of (3.1) by

$$\begin{aligned} L(y,z,c,s;\xi,\lambda,\mu,\nu,\kappa,\zeta) = {} & c_0 + X_0^f s_0 \\ & - \mathbb{E}^{\mathbb{P}}\left[\zeta\left(c_T + X_t^f s_T\right)\right] + \kappa\alpha \\ & - \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}\left[\xi_t\left(Rc_{t-1} - c_t + X_t^e y_{t-1} - D_{t-1}X_t^e - R\,X_{t-1}^f z_{t-1} + KD_{t-1}\right)\right] \\ & - \sum_{t=1}^{T} \mathbb{E}^{\mathbb{P}}\left[\lambda_t\left(s_{t-1} - s_t - y_{t-1}\eta^{-1} + z_{t-1}\right)\right] \\ & + \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t\left(y_t - P_t\right)\right] + \sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\left(s_t - S\right)\right], \end{aligned} \tag{3.34}$$

where $\kappa \geq 0$ is a real number, $\zeta = \kappa\zeta'$ for some $\zeta' \in \Upsilon_{\mathcal{A}}$, and the processes ξ_t, λ_t , $\mu_t \geq 0$, $\nu_t \geq 0$ are adapted and essentially bounded. Recall that ζ' is the (attained) minimizer in (3.31) . Because of $-\mathbf{inf}\{\mathbb{E}(\zeta X) : \zeta \in \Upsilon_{\mathcal{A}}\} = \sup\{-\mathbb{E}(\zeta X) : \zeta \in \Upsilon_{\mathcal{A}}\}$ the optimal value $V_0^*(K,D,\eta)$ of the acceptability pricing problem is $V_0^*(K,D,\eta) = \min_{y,z,c,s\xi.\lambda,\mu,\nu,\gamma,\zeta} \max L(y,z,c,s;\xi,\lambda,\zeta,\mu,\nu,\gamma,\zeta)$ w.r.t. the above restrictions.

After reordering, (3.34) yields

$$\begin{aligned} L(y,z,c,s;\xi,\lambda,\zeta,\gamma) = {} & \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}\left(X_{t+1}^e - K\right)D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\right] + \kappa\alpha \\ & + \mathbb{E}^{\mathbb{P}}\left[c_T\left(\xi_T - \zeta\right)\right] + \mathbb{E}^{\mathbb{P}}\left[s_T\left(\lambda_T + \nu_t - X_T^f\zeta\right)\right] \\ & + c_0\left(1 - \mathbb{E}^{\mathbb{P}}\left[\xi_1\right]R\right) + s_0\left(X_0^f + \nu_0 - \mathbb{E}^{\mathbb{P}}\left[\lambda_1\right]\right) \\ & + y_0\mathbb{E}^{\mathbb{P}}\left[\lambda_1\eta^{-1} + \mu_0 - \xi_1 X_1^e\right] + z_0\mathbb{E}^{\mathbb{P}}\left[R\,\xi_1 X_0^f - \lambda_1\right] \\ & + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}}\left[c_t\left(\xi_t - R\xi_{t+1}\right)\right] + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}}\left[s_t\left(\lambda_t - \lambda_{t+1}\right)\right] \\ & + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}}\left[y_t\left(\eta^{-1}\lambda_{t+1} + \mu_t - \xi_{t+1}X_{t+1}^e\right)\right] \\ & + \sum_{t=1}^{T-1} \mathbb{E}^{\mathbb{P}}\left[z_t\left(\xi_{t+1}X_t^f R - \lambda_{t+1}\right)\right]. \end{aligned} \tag{3.35}$$

Ensuring boundedness, the Lagrange dual then is given by

$$\max_{\xi,\zeta,\kappa,\lambda,\mu,\nu} \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}\left(X_{t+1}^e - K\right)D_t\right] - \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T} \mathbb{E}^{\mathbb{P}}\left[\nu_t\right] + \kappa\alpha \tag{3.36}$$

subject to

$$\begin{aligned} & \xi_T = \zeta,\ \zeta = \kappa\zeta'\ \zeta' \in \Upsilon_{\mathcal{A}} \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \zeta \\ & X_0^f + \nu_0 \geq \mathbb{E}^{\mathbb{P}}\left[\lambda_1\right] \\ & R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \text{ for } t = 1,\dots,T, \text{ and } R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = 1 \\ (t \in \mathcal{T}_0) : {} & \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e|\mathcal{F}_t\right] \leq \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right]\cdot\eta^{-1} + \mu_t \\ (t \in \mathcal{T}_0) : {} & \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\cdot X_t^f \\ \left(t \in \mathcal{T}_1^{T-1}\right) : {} & \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t + \nu_t \\ (t \in \mathcal{T}) : {} & \zeta \geq 0, \quad \mu_t \geq 0, \quad \nu_t \geq 0. \end{aligned} \tag{3.37}$$

Using $\mathbb{E}^{\mathbb{P}}\left[\zeta'\right] = \frac{1}{R^T}$ (which follows from translation equivariance of $\mathcal{A}$) and $\mathbb{E}^{\mathbb{P}}\left[\xi_t\right] = \frac{1}{R^t}$, we can conclude $\kappa\frac{1}{R^T} = \kappa\cdot\mathbb{E}^{\mathbb{P}}\left[\zeta'\right] = \mathbb{E}^{\mathbb{P}}\left[\zeta\right] = \mathbb{E}^{\mathbb{P}}\left[\xi_T\right] = \frac{1}{R^T}$, hence $\kappa = 1$ and therefore $\xi_T = \zeta \in \Upsilon_{\mathcal{A}}$. The same arguments as at the end of the proof for Lemma 3 then lead to (3.32).

If $\Upsilon_{\mathcal{A}} \cap \mathbb{A} \neq \emptyset$, then the arguments of Corollary 4 can be used to show that the relative interior of the feasible set is nonempty and strong duality follows. □

Furthermore, it becomes apparent that acceptability valuation can be considered as a transfer of the idea of good deal bounds to energy markets. Good-deal bounds were used in several papers for pricing of standard financial contracts at incomplete markets. The upper bound is given by the maximum contract price using positive stochastic discount factors that price the basic assets (which leads to the no-arbitrage bound) and fulfill some additional restrictions. A special case with a restriction on the variance of discount factors was derived as the dual of minimizing the variance of discount factors that correctly price a set of assets in [12]. [6] proposes good deal bounds as a general method for pricing at incomplete markets and also coined the term. Different restrictions on the discount factors were proposed e.g. in [3]. In the present context of electricity markets with production, the problem of acceptability valuation leads to an upper good deal bounds with the appropriate no-arbitrage conditions and a restriction on the stochastic discount factord which is derived from the used acceptability functional.

Acceptability pricing can be treated in the same way.

Proposition 5. *Consider an acceptability pricing problem based on a positive homogeneous acceptability functional $\mathcal{A}$ with conjugate representation (3.31) and $\beta = \frac{1}{R^T}$. The dual problem is*

$$G_0^*(K,D,\eta) = \tag{3.38}$$

$$\max_{\xi,\lambda,\mu,\nu} \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e D_t\right] - \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T}\mathbb{E}^{\mathbb{P}}\left[\nu_t\right] + \alpha$$

subject to

$$\begin{aligned}
&\xi_T \in \Upsilon_{\mathcal{A}} \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \xi_T \qquad (3.39)\\
&R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \text{ for } t = 1,\dots,T, \text{ and } R\,\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = \gamma\\
(t \in \mathcal{T}_0):\ &\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e|\mathcal{F}_t\right] \leq R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\cdot\eta^{-1}X_t^f + \mu_t\\
(t \in \mathcal{T}_0):\ &\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\cdot X_t^f\\
\left(t \in \mathcal{T}_1^{T-1}\right):\ &\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t + \nu_t\\
&\sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}D_t\right] = 1\\
(t \in \mathcal{T}):\ &\mu_t \geq 0, \quad \nu_t \geq 0.
\end{aligned}$$

If $\Upsilon_{\mathcal{A}} \cap \mathbb{A} \neq \emptyset$, then the minimum upfront payment based on acceptability pricing, $K_0^*(K,D,\eta)$, equals the optimal dual value $G_0^*(K,D,\eta)$.

Proof. Defining a generalized Lagrangian like in the proof of proposition 4 leads to the dual problem

$$\max_{\xi,\lambda,\mu,\nu} \sum_{t=0}^{T-1} \mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e D_t\right] - \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T}\mathbb{E}^{\mathbb{P}}\left[\nu_t\right] + \kappa\alpha \tag{3.40}$$

subject to

$$\begin{aligned}
&\xi_T = \zeta,\ \zeta = \kappa\zeta'\ \zeta' \in \Upsilon_{\mathcal{A}} \text{ and } \lambda_T + \nu_T \geq X_T^f \cdot \zeta \qquad (3.41)\\
&X_0^f + \nu_0 \geq \mathbb{E}^{\mathbb{P}}\left[\lambda_1\right]\\
&R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \text{ for } t = 1,\dots,T, \text{ and } R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = \gamma\\
(t \in \mathcal{T}_0):\ &\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X_{t+1}^e|\mathcal{F}_t\right] \leq \mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right]\cdot\eta^{-1} + \mu_t\\
(t \in \mathcal{T}_0):\ &\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\cdot X_t^f\\
\left(t \in \mathcal{T}_1^{T-1}\right):\ &\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t + \nu_t\\
&\sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}D_t\right] = 1\\
(t \in \mathcal{T}):\ &\zeta \geq 0, \quad \mu_t \geq 0, \quad \nu_t \geq 0.
\end{aligned}$$

Using $\mathbb{E}^{\mathbb{P}}\left[\xi_t\right] = \frac{\gamma}{R^t}$, we then can conclude

$$\kappa\frac{1}{R^T} = \kappa\cdot\mathbb{E}^{\mathbb{P}}\left[\zeta'\right] = \mathbb{E}^{\mathbb{P}}\left[\zeta\right] = \mathbb{E}^{\mathbb{P}}\left[\xi_T\right] = \frac{\gamma}{R^T},$$

hence $\kappa = \gamma$ and therefore $\xi_T = \gamma\zeta'$. Again, it can be shown that $\gamma > 0$. Using rescaled variables like in the proof of proposition 8, leads to a dual objective function with scaled multiplicators

$$\gamma\left(\sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}D_t\right] - \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T}\mathbb{E}^{\mathbb{P}}\left[\nu_t\right] + \alpha\right),$$

and the first line of contraints reads $\xi_T \in \Gamma_{\mathcal{A}}$.

The further arguments of the proofs for propositions 8 and 4 then can be applied in a direct way to derive (3.38) and the further statement of the current proposition. □

Proposition also (5) implies a fractional representaion. The proof repeats the arguments of Corollary (8).

Corollary 9. *The dual problem (3.17) is equivalent to the fractional optimization problem*

$$G^*(D,\eta) = \max_{\xi,\lambda,\mu,\nu,\gamma} \frac{\sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X^e_{t+1}D_t\right] - \sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right] - S\sum_{t=0}^{T}\mathbb{E}^{\mathbb{P}}\left[\nu_t\right] + \alpha}{\sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}D_t\right]} \tag{3.42}$$

subject to

$$\begin{aligned}
&\xi_T \geq 0 \text{ and } \lambda_T + \nu_T \geq X^f_T \cdot \xi_T\\
&R\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right] = \xi_t \text{ for } t = 1,\dots,T-1, \text{ and } R\mathbb{E}^{\mathbb{P}}\left[\xi_1\right] = 1\\
(t \in \mathcal{T}_0) :&\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}X^e_{t+1}|\mathcal{F}_t\right] \leq R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\cdot\eta^{-1}X^f_t + \mu_t\\
(t \in \mathcal{T}_0) :&\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] = R\,\mathbb{E}^{\mathbb{P}}\left[\xi_{t+1}|\mathcal{F}_t\right]\cdot X^f_t\\
\left(t \in \mathcal{T}_1^{T-1}\right) :&\mathbb{E}^{\mathbb{P}}\left[\lambda_{t+1}|\mathcal{F}_t\right] \leq \lambda_t + \nu_t\\
(t \in \mathcal{T}) :&\mu_t \geq 0, \quad \nu_t \geq 0,
\end{aligned}$$

Interpreting propositions 4 and 5 one can see that compared to superhedging, acceptability values and prices show two effects: The acceptability restriction $\xi_T \in \Upsilon_{\mathcal{A}}$restricts the dual feasible set anddecreases the capital requirement or acceptability price. On the other hand α goes into the objective function and increases the value or price, hence can be interpreted as a premium for unhedged risk.

Remark 3. We assumed that the producer starts with zero capital at the beginning. This expresses the view that the producer does not relate any risk capital to the analyzed delivery contract. However, in many situations risk capital is assigned either by internal risk management or because of regulatory requirements. Furthermore, [19] uses initial capital for indifference pricing, which expresses the view that the financial position of a planner influences the feasible prices. In the current framework, initial capital E_0 leads to the slight modification that the objective function in (3.38) is augmented by E_0.

Remark 4. Given Proposition 4, one can easily derive that Propositions 2 and 8 are also valid for the acceptability pricing problem, if one adds the constraint $\xi_T \in \Upsilon_{\mathcal{A}}$ and augments the objective byα.

Remark 5. It is also possible to analyze valuation and pricing of delivery contracts if the producer has available several generators. Using the setup of Corollary, (3) the constraints of Propositions (3), (8),(4) and (5) have to be reformulated with $\eta = \eta_{max}$. Moreover, each expression $\sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\mu_t P_t\right]$ in the respective objective functions has to be replaced with $\sum_{i=1}^{K}\sum_{t=0}^{T-1}\mathbb{E}^{\mathbb{P}}\left[\mu^i_t P^i_t\right]$, where P^i_t denotes the production restriction of generator i and μ^i is the related process of (essentially bounded) shadow prices.

4. Conclusion

The paper at hand analyzes basic differences regarding the characterization of arbitrage and the prizing of OTC contracts between electricity markets and financial markets, using tools from convex analysis and duality theory in a stochastic optimization framework. While keeping a fundamentally financial view, it accounts for typical frictions like storage restrictions, production efficiency and asymmetric production possibilities that are important for energy markets but not relevant for financial markets.

Arbitrage (more precisely: η-arbitrage) can be characterized by the feasibility of an equation system ensuring consistency between fuel and electricity prices and the possibilities of storage, cash accumulation and production. While this characterizing equations can be stated in terms of equivalent measures, the existence of equivalent martingale measures are neither a necessary nor a sufficient condition like in the classical financial results.

We also derived (dual) optimization problems,based on stochastic discount factors, for superhedging values and prices. The feasible discount factors are restricted by those discount factors consistent with no-arbitrage. Moreover the objective contains terms that give a value to the maximum sizes of storage and production. All kinds of contango and backwardation can be obtained in this way.

Finally, we considered capital requirements and acceptability prices, where the acceptability of the wealth distribution at the end was measured by concave, positive homogeneous acceptability functionals. It shows that the dual representation of those quantities is quite similar to the superhedging quantities, but that the set of feasible discount factors has to be restricted even further.

While those results show the similarities and dissimilarities between financial and energy markets quite well, further frictions like storage costs, the usage of several fuels and production dependent efficiencies will be considered in future research.

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