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Abstract

This paper is about metric regularity properties of two mappings appearing in optimal control problems. We focus first on the feasible set mapping of a nonlinear control system subject to state-control inequality constraints. Then we consider the optimality mapping for an optimal control problem with control constraints. We also show how the regularity properties the optimality mapping can be utilized for obtaining error estimates for discrete approximations.

1 Introduction

Although the concept of metric regularity can be traced back to the Banach open mapping principle and subsequent works by Lyusternik, Graves, and Robinson, its importance in the theory of extremum has been widely recognized only after the path-breaking paper by A. A. Milyutin et al [1]. Recall that a set-valued mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ when $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and there is a constant $\kappa \geq 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \text{ for all } (x, y) \in U \times V.$$

The infimum of κ over all such combinations of κ, U and V is called the *regularity modulus* for F at $\bar{x}$ for $\bar{y}$ and denoted by $\operatorname{reg}(F; \bar{x}|\bar{y})$. If X and Y are Banach spaces and F is a linear bounded (single-valued) mapping, then F is metrically regular if and only if it is surjective, or equivalently, open.

Let us fix first the basic notation and terminology. Throughout X and Y are Banach spaces. The notation $f : X \to Y$ means that f is a function while $F : X \rightrightarrows Y$ is a general mapping where the double arrow indicates that F may be set-valued (possibly

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with empty values). The *graph* of F is the set $\operatorname{gph} F = \{ (x,y) \in X \times Y \,|\, y \in F(x) \}$, the range $\operatorname{rge} F = \{ y \,|\, y \in F(x) \text{ for some } x \}$ and the *inverse* of F is the mapping $F^{-1} : Y \rightrightarrows X$ defined by $F^{-1}(y) = \{ x \,|\, y \in F(x) \}$. All norms are denoted by $\|\cdot\|$. The closed ball centered at x with radius r is denoted by $\mathbb{B}_r(x)$. The *distance* from a point x to a set C is defined as $d(x,C) = \inf_{y \in C} d(x,y)$.

A central result in the theory around metric regularity is what we now call the Lyusternik-Graves theorem, which is an extension of the Banach open mapping principle to nonlinear mappings. Roughly, it says that a metric regularity is preserved after the addition of a Lipschitz continuous mapping with a small Lipschitz constant. General versions of this theorem supplied with several different proofs can be found in particular in the recent book [5]. The most common version of the Lyusternik-Graves theorem, which is about a mapping of the form $f + F$ where the function f is strictly differentiable at the reference point.

Theorem 1 (Lyusternik-Graves)**.** *For a function $f : X \to Y$, a mapping $F : X \rightrightarrows Y$ and a point $(\bar{y}, \bar{x})$ with $\bar{y} \in f(\bar{x}) + F(\bar{x})$, suppose that f is strictly differentiable at $\bar{x}$ and that $\operatorname{gph} F$ is locally closed at $(\bar{x}, \bar{y} - f(\bar{x}))$. Then the mapping $f + F$ is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if the partially linearized mapping $x \mapsto G(x) := f(\bar{x}) + Df(\bar{x})(x - \bar{x}) + F(x)$ is metrically regular at $\bar{x}$ for $\bar{y}$. Specifically,*

$$\operatorname{reg}(f + F; \bar{x} \,|\, \bar{y}) = \operatorname{reg}(G; \bar{x} \,|\, \bar{y}).$$

For F the zero mapping, we obtain a result which comes close to the original theorems of Lyusternik and Graves: f is metrically regular if and only if its strict derivative $Df(\bar{x})$ is surjective.

Another fundamental result in variational analysis is the Robinson–Ursescu theorem, stated next, which gives a characterization of metric regularity of mappings with convex and closed graphs. It is an extension of the Banach open mapping principle to set-valued mappings.

Theorem 2 (Robinson–Ursescu)**.** *Let $F : X \rightrightarrows Y$ have closed convex graph and let $\bar{y} \in F(\bar{x})$. Then F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if $\bar{y} \in \operatorname{int} \operatorname{rge} F$.*

When a mapping $F : X \rightrightarrows Y$ is not only metrically regular at $\bar{x}$ for $\bar{y}$ but also its inverse F^{-1} localized around a point of its graph is single valued, then F is said to be *strongly metrically regular* at $\bar{x}$ for $\bar{y}$. This amounts to the existence of neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that the mapping $V \ni y \mapsto F^{-1}(y) \cap U$ is a Lipschitz continuous function. Strong metric regularity is the property exhibited in the classical inverse/implicit function theorem and its various versions. Here we state a version of the inverse function theorem going back to Robinson [8], which is in the form parallel to the Lyusternik-Graves theorem stated above.

Theorem 3 (Robinson)**.** *Consider a function $f : X \to Y$ which is strictly differentiable at $\bar{x}$, a mapping $F : X \rightrightarrows Y$ and a point $(\bar{y}, \bar{x})$ with $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Then the mapping*

$f+F$ *is strongly metrically regular at* $\bar{x}$ *for* $\bar{y}$ *if and only if the partially linearized mapping* $x \mapsto G(x) := f(\bar{x}) + Df(\bar{x})(x-\bar{x}) + F(x)$ *is strongly metrically regular at* $\bar{x}$ *for* $\bar{y}$.

Regularity properties of mappings appearing in finite-dimensional optimization are fairly well investigated. It is known, for example, that the solution mapping of a system of inequalities and equalities, e.g.,

$$x \mapsto \begin{pmatrix} g(x) \\ h(x) \end{pmatrix} + \begin{pmatrix} 0 \\ I\!R^q_+ \end{pmatrix}, \tag{1}$$

where $g : I\!R^n \to I\!R^p$ and $h : I\!R^n \to I\!R^q$ are smooth functions, is metrically regular if and only of the standard Mangasarian-Fromovitz constraint qualification holds at the reference point, see e.g. [5, Section 3.6]. There are several characterizations of metric regularity of the optimality mappings of nonlinear programming problems; some of them are displayed in [5]. For example, the optimality mapping

$$x \mapsto \nabla g(x) + N_C(x),$$

associated with the minimum problem $\min\{g(x) \,|\, x \in C\}$ for a twice continuously differentiable $g : I\!R^n \to I\!R$ and a convex polyhedral set $C \subset I\!R^n$, is strongly metrically regular at $\bar{x}$ for 0 if and only if the strong second-order sufficient optimality condition holds: $\nabla^2 g(\bar{x})$ is positive definite on the critical subspace $K - K$ where $K = \{w \in T_C(\bar{x}) | \nabla g(x) \perp w\}$ is the critical cone at the reference solution $\bar{x}$, see [5, Section 2.5] for details.

In contrast to the finite dimensions, there is not much known for metric regularity properties of mapping appearing in infinite-dimensional variational problems and in particular for optimal control problems. In this paper we wish to highlight a way to approach studying such properties. In Section 2 we focus on the feasible set mapping for a control system subject to state-control inequality constraints. Section 3 is devoted to the optimality mapping for an optimal control problem with control constraints. Section 4 illustrates how regularity properties can be utilized for obtaining error estimates of discrete approximations. Open problems are stated throughout.

2 The feasible set mapping

Given the interval $I = [0, 1]$, the control $u : I \to I\!R^m$ being an element of the space L^∞ of essentially bounded functions, the state $x : I \to I\!R^n$ being an element of $W^{1,\infty}$, the space of Lipschitz continuous functions (or, equivalently, the space of essentially bounded functions with essentially bounded derivatives), and the smooth functions $g : I\!R^{n+m} \to I\!R^n$ and $G : I\!R^{n+m} \to I\!R^l$, we consider the following control system subject to inequality state-control constraints and with fixed initial state:

$$\begin{aligned} &\dot{x}(t) = g(x(t), u(t)), \;\; x(0) = 0, \\ &G(x(t), u(t)) \le 0 \quad \text{for a.e. } t \in I. \end{aligned} \tag{2}$$

The initial condition can conveniently be replaced with the condition $x \in W_0^{1,\infty}$, where $W_0^{1,\infty} = \{x \in W^{1,\infty} |\ x(0) = 0\}$ is a closed linear subspace of $W^{1,\infty}$. (Of course, with a change of variables we may handle an arbitrary initial condition $x(0) = a$.) Then system (2) can be described as a *generalized equation* for the variable $z = (x, u) \in W_0^{1,\infty} \times L^\infty$, of the form

$$0 \in f(z) + F(z),$$

where

$$f(x,u) = \begin{pmatrix} \dot{x} - g(x,u) \\ G(x,u) \end{pmatrix}$$

and

$$F(x,u) = \begin{pmatrix} 0 \\ I\!R^l_+ \end{pmatrix}.$$

Here, with some abuse of notation, we denote by x the function describing the state trajectory with state values $x(t)$ at the moment t, and the same for the control. The set $(f+F)^{-1}(0)$ is the set of solutions of (2); therefore we call the mapping $f+F$ the *feasible set mapping*. According to the definition, if $\bar{z} = (\bar{x}, \bar{u})$ is a solution of (2), metric regularity of the mapping $f+F$ at $\bar{z}$ for 0 implies that for any $y = (p,q) \in L^\infty(I\!R^n) \times L^\infty(I\!R^l)$ close to 0 there exists a solution $z = (x,u) \in W_0^{1,\infty} \times L^\infty$ of the perturbed system

$$\begin{aligned} &\dot{x}(t) = g(x(t), u(t)) + p(t), \\ &G(x(t), u(t)) \le q(t) \quad \text{for a.e. } t \in I. \end{aligned}$$

and, moreover, the distance from z to $\bar{z}$ is proportional to the norm of y.

The following result utilizes the theorems of Lyusternik-Graves and Robinson-Ursescu, providing a characterization of metric regularity of the feasible set mapping in terms of the linearization of the system (2).

Theorem 4. *The mapping $f+F$ described above is metrically regular at $\bar{z}$ for 0 if and only if, for $A = \nabla_x g(\bar{x},\bar{u}), B = \nabla_u g(\bar{x},\bar{u}), S = \nabla_x G(\bar{x},\bar{u}), K = \nabla_u G(\bar{x},\bar{u})$, there exist a constant $\alpha > 0$, and functions $w \in W^{1,\infty}$ and $v \in L^\infty$ such that, for $i = 1, 2, \ldots, l$,*

$$\begin{aligned} &\dot{w}(t) = A(t)w(t) + B(t)v(t), \quad w(0) = 0, \\ &[G(\bar{x}(t), \bar{u}(t)) + S(t)w(t) + K(t)v(t)]_i \le -\alpha. \end{aligned} \tag{3}$$

Proof. By the Lyusternik-Graves theorem, metric regularity of the mapping $f+F$ describing (2) at $\bar{z}$ for 0 is equivalent to metric regularity at $\bar{z}$ for 0 of a linearized mapping whose "pointwise" form is

$$(x,u)(t) \mapsto \begin{pmatrix} \dot{x}(t) - \dot{\bar{x}}(t) - A(t)(x(t) - \bar{x}(t)) - B(t)(u(t) - \bar{u}(t)) \\ G(\bar{x}(t), \bar{u}(t)) + S(t)(x(t) - \bar{x}(t)) + K(t)(u(t) - \bar{u}(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ I\!R^n_+ \end{pmatrix}. \tag{4}$$

This mapping has closed and convex graph, hence we can apply (Robinson-Ursescu) Theorem 2 relative to the Banach space $W_0^{1,\infty} \times L^\infty$. Specifically, this theorem says that metric regularity is equivalent to the existence of $\delta > 0$ such that for any $y = (p, q)$ with $\|y\| \le \delta$, the following system has a solution $(x, u) \in W_0^{1,\infty} \times L^\infty$:

$$\begin{array}{l} \dot{x}(t) = \dot{\bar{x}}(t) + A(t)(x(t) - \bar{x}(t)) + B(t)(u(t) - \bar{u}(t)) + p(t), \\ G(\bar{x}(t), \bar{u}(t)) + S(t)(x(t) - \bar{x}(t)) + K(t)(u(t) - \bar{u}(t)) + q(t) \le 0. \end{array} \tag{5}$$

Taking $p = 0$, $q = (\alpha, \ldots, \alpha)$, and then $w = x - \bar{x}$, $v = u - \bar{u}$, this last condition (5) implies the condition for the system (3) in the statement of the theorem. Conversely, let (w, v) satisfy (3) for some $\alpha > 0$, let x be the solution of the differential equation in (5) corresponding to the control $u = v + \bar{u}$ and $x(0) = a$, and let $y = (p, q)$ be given. Note that $x = w + \bar{x} + Lp$ where L is a bounded linear mapping from L^∞ to $W^{1,\infty}$ (namely, $\tilde{x} = Lp$ is the solution of $\dot{\tilde{x}} = A\tilde{x} + p$, $\tilde{x}(0) = 0$). Hence,

$$G(\bar{x}, \bar{u}) + S(x - \bar{x}) + K(u - \bar{u}) + q = G(\bar{x}, \bar{u}) + Kv + S(w + Lp) + q \le -\bar{\alpha} + SLp + q \le 0$$

for (p, q) sufficiently small, where $\bar{\alpha} = (\alpha, \ldots, \alpha) \in I\!R^l$. This completes the proof. □

In the case when the inequality constraint $G(x, u) \le 0$ are separated for the state and the control, that is, of the form

$$G(x, u) = \begin{pmatrix} c(u) \\ h(x) \end{pmatrix}$$

and also the reference state and control are assumed piecewise continuous, a sufficient condition for the interiority condition (3) is obtained in [2] in terms of certain uniform linear independence condition of the gradients of the binding constraints. This condition is, however, not necessary. Obtaining a characterization of metric regularity of the feasible set mapping is an open problem. We conjecture that such a condition would resemble the Mangasarian-Fromovitz condition.

3 The optimality mapping

Consider the following optimal control problem

$$\text{minimize} \quad \int_0^1 l(x(t), u(t))\, \mathrm{d}t \tag{6}$$

subject to

$$\dot{x}(t) = g(x(t), u(t)), \quad u(t) \in U \text{ for a.e. } t \in [0, 1],$$
$$x \in W_0^{1,\infty}(I\!R^n), \quad u \in L^\infty(I\!R^m),$$

where the functions $l : I\!R^{n+m} \to I\!R$, $g : I\!R^{n+m} \to I\!R^n$ are twice continuously differentiable everywhere and U is a convex and closed set in $I\!R^m$.

Let $(\bar{x}, \bar{u})$ be a solution of problem (6) and let $W_1^{1,\infty}(I\!R^n)$ be the space of Lipschitz continuous functions ψ with values in $I\!R^n$ and such that $\psi(1) = 0$. In terms of the Hamiltonian

$$H(x, \psi, u) = l(x, u) + \psi^\mathsf{T} g(x, u),$$

it is well known that the first-order necessary conditions (a Pontryagin maximum principle) for a weak minimum at the solution $(\bar{x}, \bar{u})$ can be expressed in the following way: there exists $\bar{\psi} \in W_1^{1,\infty}(I\!R^n)$, such that $(\bar{x}, \bar{\psi}, \bar{u})$ is a solution of the following two-point boundary value problem coupled with a variational inequality satisfied for a.e. $t \in [0, 1]$:

$$\begin{array}{rcl} \dot{x}(t) & = & g(x(t), u(t)),\ x(0) = 0, \\ \dot{\psi}(t) & = & -\nabla_x H(x(t), \psi(t), u(t)),\ \psi(1) = 0, \\ 0 & \in & \nabla_u H(x(t), \psi(t), u(t)) + N_U(u(t)), \end{array} \tag{7}$$

where $N_U(u)$ is the normal cone to the set U at the point u. Denote $X = W_0^{1,\infty}(I\!R^n) \times W_1^{1,\infty}(I\!R^n) \times L^\infty(I\!R^m)$ and $Y = L^\infty(I\!R^n) \times L^\infty(I\!R^n) \times L^\infty(I\!R^m)$. Further, for $z = (x, \psi, u)$ let

$$f(z) = \begin{pmatrix} \dot{x} - g(x, u) \\ \dot{\psi} + \nabla_x H(x, \psi, u) \\ \nabla_u H(x, \psi, u) \end{pmatrix} \tag{8}$$

and

$$F(z) = \begin{pmatrix} 0 \\ 0 \\ N_U(u) \end{pmatrix}. \tag{9}$$

Note that the notation $N_U(u)$ is somewhat overloaded. In fact, in (9) we mean the normal cone $N_{\mathcal{U}}(u)$ to the set $\mathcal{U} = \{\tilde{u} \in L^\infty |\ \tilde{u}(t) \in U \text{ for a.e. } t \in [0, 1]\}$. Since in the context of the generalized equation (10) below only L^∞-selections of $N_{\mathcal{U}}(u)$ are of interest, we may comfortably represent the relevant subset of $N_{\mathcal{U}}(u)$ (using the same notation) in the pointwise form $N_{\mathcal{U}}(u) = \{v \in L^\infty |\ v(t) \in N_U(u(t)) \text{ for a.e. } t \in [0, 1]\}$. Then the optimality conditions (7) can be equivalently rewritten as

$$0 \in f(z) + F(z). \tag{10}$$

That is, the *optimality mapping* describing the optimality system (7) is $f + F : X \rightrightarrows Y$, which fits into the format of the Lyusternik-Graves theorem. Thus metric regularity of the mapping $f + F$ described in (8) and (9) is equivalent to metric regularity of the linearized mapping

$$(\zeta, \varphi, \eta)(t) \mapsto \begin{pmatrix} \dot{\zeta}(t) - A(t)\zeta(t) - B(t)\eta(t) \\ \dot{\varphi}(t) + A(t)^\mathsf{T}\varphi(t) + Q(t)\eta(t) + D(t)\zeta(t) \\ R(t)\eta(t) + Q(t)^\mathsf{T}\zeta(t) + B(t)^\mathsf{T}\varphi(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ N_U(\bar{u}(t) + \eta(t)) \end{pmatrix},$$

where, skipping the dependence on t, we denote $\zeta = x - \bar{x}$, $\varphi = \psi - \bar{\psi}, \eta = u - \bar{u}$ and the matrices A, B, Q, D and R are suitably defined by the second derivatives of H with respect to x, ψ and u, e.g., $A = \nabla_x f = \nabla_{\psi x} H$ etc. Note that this mapping is associated with the optimality system for a linear-quadratic optimal control problem which is actually an approximation of the nonlinear problem (6). The same could be said for the *strong metric regularity* of the optimality mapping by using instead (Robinson's) Theorem 3.

To the author's knowledge, there are no results characterizing metric regularity of the mapping $f + F$ for a system of the type (7). On the other hand there have been quite a few developments on *strong metric regularity* in the last several decades, see e.g. [2], [4] and [7]. Finding sharp conditions for (strong) metric regularity in optimal control is an important open problem.

4 Regularity and discrete approximations

It was observed in authors' paper [6] that metric regularity of the optimality mapping described in (7) (or in the equivalent form (10)) could be quite instrumental for obtaining *a priori* error estimates for a discrete approximation to the problem at hand. In this section we present a slightly stronger version of that result.

Specifically, suppose that the optimality system (7) is solved inexactly by means of a numerical method applied to a discrete approximation provided by the Euler scheme. Specifically, let N be a natural number, let $h = 1/N$ be the mesh spacing, and let $t_i = ih$. Denote by $PL_0^N(I\!R^n)$ the space of piecewise linear and continuous functions x_N over the grid $\{t_i\}$ with values in $I\!R^n$ and such that $x_N(0) = 0$, by $PL_1^N(I\!R^m)$ the space of piecewise linear and continuous functions ψ_N over the grid $\{t_i\}$ with values in $I\!R^n$ and such that $\psi_n(1) = 0$, and by $PC^N(I\!R^m)$ the space of piecewise constant and continuous from the right functions over the grid $\{t_i\}$ with values in $I\!R^m$. Then introduce the products $X^N = PL_0^N(I\!R^n) \times PL_1^N(I\!R^n) \times PC^N(I\!R^m)$ as an approximation space for the triple (x, ψ, u). We identify $x \in PL_0^N(I\!R^n)$ with the vector $(x^0, \ldots, x^N)$ of its values at the mesh points (and similarly for ψ), and $u \in PC^N(I\!R^m)$ – with the vector $(u^0, \ldots, u^{N-1})$ of the values of u in the mesh subintervals.

Suppose that, as a result of the computations, for certain natural N a function $\tilde{z} = (x_N, \psi_N, u_N) \in X^N$ is found that approximately satisfies the discretized optimality system

$$\left\{ \begin{array}{lcl} \dot{x}^i & = & g(x^i, u^i), \quad x^0 = 0, \\ \dot{\psi}^i & = & -\nabla_x H(x^i, \psi^{i+1}, u^i), \quad \psi^N = 0, \\ 0 & \in & \nabla_u H(x^i, \psi^i, u^i) + N_U(u^i) \end{array} \right. \tag{11}$$

for $i = 0, 1, \ldots, N-1$ where $\dot{x}^i$ is the first divided difference of x at t_i. We state here the following result proven in [6]:

Theorem 5. *Assume that the mapping describing the optimality system (7) is metrically regular at* $\bar{\xi} = (\bar{x}, \bar{u}, \bar{\psi})$ *for* 0. *Then there exist constants* a *and* c *such that for every solution* $\xi_N = (x_N, u_N, \psi_N)$ *of the discretized system (11) contained in* $\mathbb{B}_a(\bar{\xi})$*, there exists a solution* $\bar{\xi}^N = (\bar{x}^N, \bar{u}^N, \bar{\psi}^N)$ *of the continuous system (4) such that*

$$\|\bar{x}^N - x_N\|_{W^{1,\infty}} + \|\bar{u}^N - u_N\|_{L^\infty} + \|\bar{\psi}^N - \psi_N\|_{W^{1,\infty}} \leq ch.$$

Furthermore, if the mapping of the optimality system (7) is strongly metrically regular at $\bar{\xi}$ *for* 0, *then the above claim holds with* $\bar{\xi}^N = \bar{\xi}$*; that is, every sequence* $\xi_N = (x_N, u_N, \psi_N)$ *of approximate solutions to the discretized system (11) contained in* $\mathbb{B}_a(\bar{\xi})$ *converges to* $\bar{\xi}$ *with rate proportional to the mesh size* h.

In other words, metric regularity *alone* is enough to obtain an a priori estimate for the discrete approximation considered, and a stronger estimate is obtained under strong metric regularity. Extending this approach to more general variational models, e.g. involving PDEs as well as more elaborate finite element approximations, is a challenging avenue for further research (see [9] for an extension for a class of distributed optimal control problems).

Theorem 5 concerns the case of first order approximation, where the discrete solution (x^i, ψ^i) and the discrete control u^i can be comfortably embedded in the continuous-time spaces X^N: the embedding creates a first order (with respect to h) residual in (10). If a higher order scheme is applied to the optimality system (7), as in [3], then an embedding into a continuous-time space that gives a higher than first order residual in (10) might be problematic. The main trouble is caused by the last inclusion in (7) due to the irregularity of the normal cone mapping. In the paper [3], a second order error estimate is obtained under conditions that are sufficient for (even strong) metric regularity of the solution mapping $f + F$, but are not necessary. It is a challenging problem to obtain higher than first order a priori error estimates in the spirit of Theorem 5, based on Runge-Kutta or other higher order discretization schemes assuming only metric regularity.

One can obtain *a posteriori* error estimates provided that the mapping of discretized system (11) is metrically regular, uniformly in N. Observe that the system (11) is a variational inequality but with different spaces than the continuous one, namely, spaces of piecewise linear functions for the state and costate and piecewise constant functions for the control. It is an open question, however, whether the metric regularity of the discretized system (11) can be obtained from the metric regularity of the original system (7).

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