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Relaxation of Euler-Type Discrete-Time Control System*

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Abstract. This paper investigates what is the Hausdorff distance between the set of Euler curves of a Lipschitz continuous differential inclusion and the set of Euler curves for the corresponding convexified differential inclusion. It is known that this distance can be estimated by $O(\sqrt{h})$, where h is the Euler discretization step. It has been conjectured that, in fact, an estimation $O(h)$ holds. The paper presents results in favor of the conjecture, which cover most of the practically relevant cases. However, the conjecture remains unproven, in general.

1 Introduction

In this paper we address the problem of convexification of finite-difference inclusions resulting from Euler discretization of the differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x_0, \quad t \in [0, 1], \tag{1}$$

where $x \in I\!\!R^n$, $x_0 \in I\!\!R^n$ is given, and $F : I\!\!R^n \rightrightarrows I\!\!R^n$ is a set-valued mapping. Standing assumptions will be that F is compact-valued, bounded (by a constant denoted further by $|F|$) and Lipschitz continuous with a Lipschitz constant L with respect to the Hausdorff metric.[1]

Denote by S the set of all solutions of (1), and by $R := \{x(1) : x(\cdot) \in S\}$ the reachable set at $t = 1$. In parallel, we consider the convexified differential inclusion

$$\dot{y}(t) \in \operatorname{co} F(y(t)), \quad y(0) = x_0, \quad t \in [0, 1], \tag{2}$$

and denote by $S^{\rm co}$ and $R^{\rm co}$ the corresponding solution set and reachable set.

Now we consider the Euler discretizations of (1) and (2):

$$x_{k+1} \in x_k + hF(x_k), \quad k = 0, \ldots, N-1, \tag{3}$$

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[1] In fact, the global boundedness and Lipschitz continuity can be replaced with local ones if all solutions of (1) are contained in a bounded set. Then the formulations of some of the claims in the paper should be somewhat modified. The standing assumptions above are made simpler for more transparency.

and

$$y_{k+1} \in y_k + h \operatorname{co} F(y_k), \quad k = 0, \dots, N-1, \quad y_0 = x_0, \tag{4}$$

where N is a natural number and $h = 1/N$ is the mesh size. Denote by S_h and S_h^{co} the sets of (discrete) solutions of these inclusions, respectively, and by R_h and R_h^{co} the corresponding reachable sets.

It is well known that $S^{\text{co}} = \operatorname{cl} S$. This paper investigates what is the Hausdorff distance between S_h and S_h^{co}, and also between R_h and R_h^{co}. The former is defined as

$$H(S_h, S_h^{\text{co}}) = \sup_{(y_0, \dots y_N) \in \mathcal{S}_h^{\text{co}}} \inf_{(x_0, \dots x_N) \in \mathcal{S}_h} \max_{i=0,\dots,N} |y_i - x_i| = \sup_{y \in \mathcal{S}_h^{\text{co}}} \inf_{x \in \mathcal{S}_h} \|x - y\|_{l_\infty}.$$

Results by Tz. Donchev [1] and G. Grammel [4] imply that $H(S_h, S_h^{\text{co}}) = O(\sqrt{h})$. The unpublished author's report [9] contains the following

Conjecture: There exists a constant c such that for every natural number N

$$H(S_h, S_h^{\text{co}}) \leq ch. \tag{5}$$

This conjecture has been proved in a number of special cases (see Section 3), but not in general. It is important to clarify what the constant c depends on. A stronger form of the conjecture is that c depends only on $|F|$, L, and the dimension of the space, n. However, in some of the results presented below the constant c will depend also on some geometric properties of $F(x)$. Therefore we speak about the weak and the strong form of the conjecture. We mention that there is an even stronger form of the conjecture, where Lipschitz continuity is required for $\operatorname{co} F$ instead of F. This case will be only partly discussed in Part 2 of Section 3.

Clearly, (5) implies the same estimation for $H(R_h, R_h^{\text{co}})$, but the inverse implication does not need to be true. (Here, and at some places below we use the symbol H also for the Hausdorff distance between compact subsets of $\mathbb{R}^n$, which will be clear from the context.)

The problems mentioned above are relevant for many engineering applications, where switched systems [5] or mixed-integer control problems (see [6–8] and the references therein) arise. The mixed-integer control problems can be formulated as

$$\min_{u(\cdot)} \left\{ p(x(1)) + \int_0^1 q(x(t)) \, \mathrm{d}t \right\} \tag{6}$$

$$\dot{x}(t) = \varphi(x(t), u(t)), \quad x(0) = x_0, \quad u(t) \in U, \quad t \in [0, 1], \tag{7}$$

where some of the components of the control u are restricted in a convex set, the remaining components take values in a discrete set. Thus the set U is non-convex. The problem becomes combinatorial and due to the high dimension of its discretized counterpart (obtained, say, by the Euler method with mesh size h) is hard to be solved numerically. For this reason, in the abovementioned papers the authors propose to solve the convexified version of the problem and then from the numerically obtained optimal control to construct another, piecewise constant one, that takes values in U only, and such that the loss of performance is small (relative to the discretization step h). Such constructions will be discussed in the next sections. It is easy to see that the loss of performance (compared with the optimal performance of the convexified problem) can be estimated by $H(S_h, S_h^{\rm co}) + O(h)$, and in the case $q = 0$ by $H(R_h, R_h^{\rm co}) + O(h)$, provided that p and q are Lipschitz continuous. This gives one motivation for the question formulated above.

In the next section we present a result related to the problem posed above (but not implying validity of the conjecture), while in Section 3 we prove the conjecture under some additional conditions. The proofs use ideas from [9, 8].

2 A related result

The next result deviates from the conjecture formulated in the introduction, but has practical relevance in view of the control problem (6), (7).

Theorem 1 *There exists a constant C such that for every natural number N and for every $y = (y_0, y_1, \dots, y_N) \in S_h^{\rm co}$ there exist positive numbers $h_1, \dots h_N$ with $\sum_{k=1}^N h_k = 1$ and a solution $x = (x_0, \dots x_N)$ of*

$$x_{k+1} \in x_k + h_k F(x_k), \quad k = 0, \dots, N-1, \tag{8}$$

such that

$$\|x - y\|_{l_\infty} \le (4n+1)|F|e^L\, h.$$

Proof. Obviously $\operatorname{co} F$ is Lipschitz and bounded with the same constants as F.
Let $y = (y_0,\, y_1, \dots, y_N) \in \mathcal{S}_h^{\rm co}$. Then there exist $\xi_i \in \operatorname{co} F(y_i)$ such that

$$y_{i+1} = y_i + h\xi_i, \qquad i = 0, \dots, N-1. \tag{9}$$

We split the points $y_0, \dots, y_N$ into groups of $n+1$ successive elements, the last one containing possibly a smaller number of elements. Let m be the number of groups, not counting the last one if it contains less than $n+1$ elements. Thus m is the largest integer for which $m(n+1) \le N$.

We shall define a trajectory $(x_0, x_1, \dots, x_N)$ of (8) successively for each group of indexes. Namely, since x_0 is given and $y_0 = x_0$, we set $\Delta_0 = |x_0 - y_0| = 0$, then we assume that $x_{i(n+1)}$ is already defined, together with the corresponding steps h_j, $j = 0, \dots, i(n+1)$. Denote $\Delta_i = |x_{i(n+1)} - y_{i(n+1)}|$.

Due to (9) we have that for $j = 0, \ldots, n$

$$\xi_{i(n+1)+j} \in \operatorname{co} F(y_{i(n+1)+j}) = \operatorname{co} F\left(y_{i(n+1)} + h\sum_{s=0}^{j-1} \xi_{i(n+1)+s}\right) \subset \operatorname{co} F(y_{i(n+1)}) + hjL|F|\mathbf{B},$$

where $\mathbf{B}$ is the unit ball in $\mathbb{R}^n$. Then there exist $\tilde{\xi}_{i(n+1)+j} \in \operatorname{co} F(y_{i(n+1)})$ such that

$$|\tilde{\xi}_{i(n+1)+j} - \xi_{i(n+1)+j}| \leq hjL|F|, \qquad j = 0, \ldots, n, \tag{10}$$

where we have set $\tilde{\xi}_{i(n+1)} = \xi_{i(n+1)}$. Since $\tilde{\xi}_{i(n+1)+j} \in \operatorname{co} F(y_{i(n+1)})$, we have also that

$$\frac{1}{n+1}\sum_{j=0}^{n} \tilde{\xi}_{i(n+1)+j} \in \operatorname{co} F(y_{i(n+1)}).$$

According to the Carathéodory theorem, there exist $\tilde{\eta}_{i(n+1)+j} \in F(y_{i(n+1)})$ and $\alpha_j \geq 0$, $\sum_{j=0}^{n} \alpha_j = 1$, such that

$$\sum_{j=0}^{n} \alpha_j \tilde{\eta}_{i(n+1)+j} = \frac{1}{n+1}\sum_{j=0}^{n} \tilde{\xi}_{i(n+1)+j}. \tag{11}$$

Let us define $h_{i(n+1)+j} = \bar{h}_j := (n+1)h\alpha_j$. Due to the Lipschitz continuity of F, there exists $\eta_{i(n+1)} \in F(x_{i(n+1)})$ such that

$$|\eta_{i(n+1)} - \tilde{\eta}_{i(n+1)}| \leq L\Delta_i.$$

To extend the trajectory $x_0, \ldots, x_{i(n+1)}$ we set

$$x_{i(n+1)+1} = x_{i(n+1)} + \bar{h}_0 \eta_{i(n+1)}.$$

Since

$$H(F(x_{i(n+1)+1}), F(y_{i(n+1)})) \leq H(F(x_{i(n+1)+1}), F(x_{i(n+1)})) + H(F(x_{i(n+1)}), F(y_{i(n+1)})) \leq \bar{h}_0 L|F| + L\Delta_i,$$

there exists $\eta_{i(n+1)+1} \in F(x_{i(n+1)+1})$ such that

$$|\eta_{i(n+1)+1} - \tilde{\eta}_{i(n+1)+1}| \leq \bar{h}_0 L|F| + L\Delta_i.$$

Then we define

$$x_{i(n+1)+2} = x_{i(n+1)+1} + \bar{h}_1 \eta_{i(n+1)+1}.$$

Continuing in the same way we define for every $j = 0, \ldots, n$ the vectors $\eta_{i(n+1)+j}$ and $x_{i(n+1)+j+1}$ such that

$$\begin{aligned} &\eta_{i(n+1)+j} \in F(x_{i(n+1)+j}), \\ &x_{i(n+1)+j+1} = x_{i(n+1)+j} + \bar{h}_j \eta_{i(n+1)+j}, \\ &|\eta_{i(n+1)+j} - \tilde{\eta}_{i(n+1)+j}| \leq L|F|\sum_{k=0}^{j-1} \bar{h}_k + L\Delta_i. \end{aligned} \tag{12}$$

In this way the trajectory of (3) is extended to the discrete time $(i+1)(n+1)$. The next estimations follow from (12), (11), (10):

$$\begin{aligned}
\Delta_{i+1} &= |x_{(i+1)(n+1)} - y_{(i+1)(n+1)}| \\
&\le |x_{i(n+1)} - y_{i(n+1)}| + \left| \sum_{j=0}^{n} \bar{h}_j \eta_{i(n+1)+j} - h \sum_{j=0}^{n} \xi_{i(n+1)+j} \right| \\
&\le \Delta_i + \sum_{j=0}^{n} \bar{h}_j \left| \eta_{i(n+1)+j} - \tilde{\eta}_{i(n+1)+j} \right| + \left| \sum_{j=0}^{n} \bar{h}_j \tilde{\eta}_{i(n+1)+j} - h \sum_{j=0}^{n} \tilde{\xi}_{i(n+1)+j} \right| \\
&\quad + h \sum_{j=1}^{n} \left| \tilde{\xi}_{i(n+1)+j} - \xi_{i(n+1)+j} \right| \\
&\le \Delta_i + \sum_{j=0}^{n} \bar{h}_j L \Delta_i + L|F| \sum_{j=1}^{n} \bar{h}_j \sum_{k=0}^{j-1} \bar{h}_k \\
&\quad + \left| (n+1)h \sum_{j=1}^{n} \alpha_j \tilde{\eta}_{i(n+1)+j} - h \sum_{j=1}^{n} \tilde{\xi}_{i(n+1)+j} \right| + h \sum_{j=0}^{n} hjL|F| \\
&\le (1 + (n+1)Lh)\Delta_i + (n+1)^2 L|F|h^2 + \frac{n(n+1)}{2} L|F|h^2 \\
&\le (1 + (n+1)Lh)\Delta_i + (n+1)(2n+1)L|F|h^2.
\end{aligned}$$

Since this holds for any $i < m$ it implies in a standard way the inequality

$$\Delta_i \le (2n+1)|F|e^{i(n+1)Lh} h \le (2n+1)|F|e^{m(n+1)Lh} h \le (2n+1)|F|e^{L} h.$$

Then taking into account the errors that can be made within n intermediate steps, or in the last $N - m(n+1) \le n$ steps we obtain for the above defined solution of (8)

$$|x_k - y_k| \le (2n+1)|F|e^L h + 2n|F|h \qquad \forall k = 0, \ldots, N.$$

The proof is complete. Q.E.D.

Obviously the above theorem does not give an answer to the main question in this paper, since the time-steps in (8) need not be uniform. Although the total number of jumps is N, there could be much smaller distance between the jumps, which may be trouble for practical implementations. Moreover, as it is clear from the proof, in the terms of the control problem (6)–(7), the choice of $u_k \in U$ at step k depends on n future values of the optimal control of the convexified problem (that is, it is anticipative). However, this is in line with the model predictive control methodology used in practice. Moreover, the construction in the proof of the theorem can be viewed as an alternative of the "adaptive control grid" proposed in [7] where $2N$ jump points of the control are used (instead of N).

3 Cases in which the conjecture is proved

Part 1. First we consider the case of a constant mapping F, that is, the inclusion

$$\dot{x}(t) \in V, \quad x(0) = x_0, \;\; t \in [0,1], \tag{13}$$

where $V \subset \mathbb{R}^n$ is compact. This case will be embodied later in more general considerations.

We mention that conjecture (5) has not been proved even in this "simple" case. However, for constant mappings $F(x) = V$ it holds that

$$H(R_h, R_h^{\mathrm{co}}) \le ch,$$

where the constant c depends only on $|V|$ and n. This can be proved (and has been proved by several mathematicians in private communications with the author: Z. Artstein, M. Brokate, E. Farkhi, T. Donchev) in different ways, the simplest of which uses the Shapley-Folkmann theorem (see e.g. [3, Appendix 1]).

Now, we consider the case of a set V consisting of finite number of points:

$$V = \{v_1, \ldots, v_s\}, \quad v_i \in \mathbb{R}^n. \tag{14}$$

The proof is given in the research report [9] and is somewhat modified below.

Proposition 1. *For differential inclusion (13) with the constant mapping V specified in (14) the estimation*

$$H(S_h, S_h^{\mathrm{co}}) \le 2s|V|h,$$

holds for every $h = 1/N$, $N \in \mathbb{N}$.

Proof. Let a trajectory $y_0, \ldots, y_N$ of the convexified inclusion

$$y_{k+1} \in y_k + h \operatorname{co} V, \quad k = 0, \ldots N-1, \quad y_0 = x_0,$$

be fixed. Then

$$y_{k+1} = y_k + h\xi_k, \quad \text{with} \quad \xi_k = \sum_{j=1}^{s} \alpha_{kj} v_j,$$

where $\alpha_{kj} \ge 0$ and $\sum_{j=1}^{s} \alpha_{kj} = 1$. We have

$$y_k = y_0 + h \sum_{i=0}^{k-1} \sum_{j=1}^{s} \alpha_{ij} v_j = y_0 + h \sum_{j=1}^{s} \beta_{kj} v_j,$$

where $\beta_{kj} = \sum_{i=0}^{k-1} \alpha_{ij}$. Clearly,

$$\sum_{j=1}^{s} \beta_{kj} = \sum_{j=1}^{s} \sum_{i=0}^{k-1} \alpha_{ij} = k.$$

We shall define the sequence

$$x_{k+1} \in x_k + hV, \quad x_0 = y_0,$$

as follows. Let x_k be already defined. Denote

$$\delta_{kj} = \beta_{kj} - \gamma_{kj},$$

where γ_{kj} is the number of times v_j is chosen in the construction of x_k. Clearly,

$$\sum_{j=1}^{s} \gamma_{kj} = k, \quad \text{hence} \quad \sum_{j=1}^{s} \delta_{kj} = 0. \tag{15}$$

Denote also

$$J_k = \{j : \delta_{kj} \geq 0\}, \quad \Delta_k = \sum_{j \in J_k} \delta_{kj}.$$

Let $j = \bar{j}$ be an index for which δ_{kj} is maximal. Then we define

$$x_{k+1} = x_k + hv_{\bar{j}}.$$

We shall estimate Δ_{k+1}. First suppose that $\delta_{k+1,\bar{j}} \geq 0$, thus $\bar{j} \in J_{k+1}$. Then we have

$$\Delta_{k+1} = \sum_{j \in J_{k+1}} (\beta_{k+1,j} - \gamma_{k+1,j}) = \beta_{k\bar{j}} + \alpha_{k\bar{j}} - \gamma_{k\bar{j}} - 1 + \sum_{j \in J_{k+1} \setminus \{\bar{j}\}} (\beta_{k+1,j} - \gamma_{k+1,j})$$

$$= \delta_{k,\bar{j}} + \alpha_{k,\bar{j}} - 1 + \sum_{j \in J_{k+1} \setminus \{\bar{j}\}} (\beta_{kj} + \alpha_{kj} - \gamma_{k+1,j}) = \delta_{k,\bar{j}} + \alpha_{k,\bar{j}} - 1 + \sum_{j \in J_{k+1} \setminus \{\bar{j}\}} (\delta_{kj} + \alpha_{kj})$$

$$\leq \delta_{k,\bar{j}} + \alpha_{k,\bar{j}} - 1 + \sum_{j \in J_k \setminus \{\bar{j}\}} \delta_{kj} + \sum_{j=1}^{s} \alpha_{kj} - \alpha_{k,\bar{j}} \leq \Delta_k - 1 + \sum_{j=1}^{s} \alpha_{kj} = \Delta_k.$$

In the last line we have used that $\delta_{kj} < 0$ for $j \notin J_k$.

Now, let us consider the case $\delta_{k+1,\bar{j}} < 0$. Then

$$\beta_{k\bar{j}} + \alpha_{k\bar{j}} - \gamma_{k\bar{j}} - 1 \leq 0,$$

hence

$$\delta_{k\bar{j}} \leq 1 - \alpha_{k\bar{j}} \leq 1.$$

From the maximality of $\delta_{k\bar{j}}$ we obtain $\delta_{k,j} \leq 1$ for all j, hence

$$\Delta_k \leq s.$$

Combining the two cases we obtain that at every step k

$$\text{either} \quad \Delta_{k+1} \leq \Delta_k, \quad \text{or} \quad \Delta_k \leq s.$$

Having in mind that $\varDelta_0 = 0$ from here we conclude that

$$\varDelta_k \leq s \ \ \forall\, k = 0, \ldots, N.$$

To complete the proof we notice that

$$y_k - x_k = h \sum_{j=1}^{s} \delta_{kj} v_j = h \left(\sum_{j \in J_k} \delta_{kj} v_j + \sum_{j \notin J_k} \delta_{kj} v_j \right),$$

thus

$$|y_k - x_k| \leq h|V| \left(\sum_{j \in J_k} \delta_{kj} + \sum_{j \notin J_k} |\delta_{kj}| \right).$$

Hence, using (15) we obtain

$$|y_k - x_k| \leq h|V|(\varDelta_k + \varDelta_k) \leq 2s|V|h.$$

Q.E.D.

Notice that the proof of the above proposition is constructive and the definition of $v_k \in V$ at every step is non-anticipative with respect to the reference trajectory $y_0, \ldots, y_N$. A construction like that of v_k is named in [7, 8] as *Sum Up Rounding Strategy*.

We mention that the constant $c = 2s|V|$ in Proposition 1 depends on the number of elements of V, that is, only the weaker form of Conjecture (5) is proved (the constant c depends on the geometric properties of V). In particular, it does not help to deal with sets V for which the boundary of $\operatorname{co} V$ contains curved pieces. The next result is capable to capture some such cases.

Part 2. In this part we consider the general inclusion (1), weakening a bit the standing assumptions. Namely, instead of assuming Lipschitz continuity of F we assume that $\operatorname{co} F$ is Lipschitz continuous.

Notice that all sequences in S_h and S_h^{co} are contained in the compact set $X := \{x \in I\!R^n : |x - x_0| \leq M\}$. Let there exist functions $l_i : X \to I\!R^n$, $i = 1, \ldots, n$ such that:
(i) l_i are Lipschitz continuous;
(ii) the vectors $l_i(x)$, $i = 1, \ldots, n$, are linearly independent and $|l_i(x)| = 1$ for every $x \in X$;
(iii) for every $x \in X$, every $\bar{v} \in \operatorname{co} F(x)$ and every $\sigma_1, \ldots, \sigma_n \in \{-1, 1\}$ there exists $v \in F(x)$ such that

$$\sigma_i \alpha_i(x; v - \bar{v}) \leq 0, \ \ i = 1, \ldots, n,$$

where $\alpha_i(x; z)$ is the i-th coordinate of $z \in I\!R^n$ in the basis $\{l_i(x)\}$.

Clearly, the numbers $\alpha_i(x; z)$ are uniquely defined from

$$z = \sum_{i=1}^{n} \alpha_i(x; z)\, l_i(x). \tag{16}$$

Proposition 2. *Under the suppositions made in Part 2 there exists a constant C such that*

$$H(S_h, S_h^{\rm co}) \le Ch$$

for every $h = 1/N$, $N \in I\!N$.

Proof. Denote $b_{ij}(x) = \langle e_i, l_j(x)\rangle$, $i,j = 1,\dots,n$, where $\{e_i\}$ is the standard basis in $I\!R^n$. Then the matrix

$$B(x) = \{b_{ij}(x)\}$$

is invertible, and its inverse, $(B(x))^{-1}$, is Lipschitz continuous (with constant $\bar{L}$) and bounded (by a constant $\bar{M}$) in X with respect to the operator matrix norm. Since we have

$$\alpha(x;z) = B^{-1}(x)z,$$

the mapping $x \longrightarrow \alpha(x;z)$ is Lipschitz continuous in $x \in X$ with Lipschitz constant $\bar{L}|z|$ and $|\alpha(x;z)| \le \bar{M}|z|$ for all $z \in I\!R^n$. Moreover, due to (16)

$$|z| \le \sum_{i=1}^{n} |\alpha_i(x;z)|\,|l_i(x)| \le n\|\alpha(x,z)\|_\infty,$$

where $\|\alpha(x;z)\|_\infty := \max_i |\alpha_i(x;z)|$.

Let us fix an arbitrary $(y_0,\dots,y_N) \in S_h^{\rm co}$, and let a solution x_k of (1) be already defined until some $k \ge 0$. Denote

$$\delta_k = \|\alpha(x_k; x_k - y_k)\|_\infty.$$

We have

$$y_{k+1} = y_k + hu_k$$

for some $u_k \in {\rm co}\,F(y_k)$. There exists some $\bar{v}_k \in {\rm co}\,F(x_k)$ such that

$$|\bar{v}_k - u_k| \le L|x_k - y_k| \le nL\|\alpha(x_k; x_k - y_k)\|_\infty = nL\delta_k.$$

We define $v_k \in F(x_k)$ according to assumption (iii) applied for

$$\bar{v} = \bar{v}_k, \quad x = x_k, \quad \sigma_i = {\rm sign}(\alpha_i(x_k; x_k - y_k)).$$

Then define

$$x_{k+1} = x_k + hv_k.$$

We have

$$\begin{aligned}\alpha_i(x_{k+1}; x_{k+1} - y_{k+1}) &= \alpha_i(x_{k+1}; x_k - y_k + h\,(v_k - \bar{v}_k)) + h\,\alpha_i(x_{k+1}; \bar{v}_k - u_k)\\ &= \alpha_i(x_k; x_k - y_k) + h\,\alpha_i(x_k; v_k - \bar{v}_k)\\ &\quad +[\alpha_i(x_{k+1}; x_k - y_k + h\,(v_k - \bar{v}_k)) - \alpha_i(x_k; x_k - y_k + h\,(v_k - \bar{v}_k))]\\ &\quad +h\,\alpha_i(x_{k+1}, \bar{v}_k - u_k).\end{aligned}$$

Due to the choice of v_k we have

$$|\alpha_i(x_k; x_k - y_k) + h\,\alpha_i(x_k; v_k - \bar{v}_k)| \le \max\{\delta_k, 2hM\bar{M}\}.$$

The term in the brackets in the chain of equalities above can be estimated by

$$\bar{L}|x_{k+1} - x_k|\,|x_k - y_k + h\,(v_k - \bar{v}_k)| \le hM\bar{L}(\delta_k + 2hM),$$

and the last term by

$$h\bar{M}|\bar{v}_k - u_k| \le h\bar{M}nL\delta_k.$$

Combining the obtained estimations we obtain that

$$\begin{aligned}\delta_{k+1} &\le \max\{\delta_k, 2hM\bar{M}\} + hM\bar{L}(\delta_k + 2hM) + h\bar{M}nL\delta_k \\ &= \max\{\delta_k, 2hM\bar{M}\} + h(M\bar{L} + \bar{M}nL)\delta_k + 2h^2M^2\bar{L}.\end{aligned}$$

Now we consider two cases.

If $\delta_k \le 2hM\bar{M}$, then

$$\delta_{k+1} \le 2hM\bar{M} + 2h^2M\bar{M}(M\bar{L} + \bar{M}nL) + 2h^2M^2\bar{L} \le C_1 h,$$

with an appropriate constant C_1.

If $\delta_k > 2hM\bar{M}$, then

$$\delta_{k+1} \le (1 + h(M\bar{L} + \bar{M}nL))\delta_k + 2h^2M^2\bar{L}.$$

The above two estimations in combination imply in a standard way the claim of the proposition. Q.E.D.

We mention that the construction of the x of (3) in the above proof is non-anticipative with respect to the reference solution y of (4).

The next is a simple consequence of the above proposition.

Corollary 1. *Under the conditions of Proposition 2, let F satisfy*

$$F(x) = \partial(\mathrm{co}\,F(x)) \quad \forall x \in I\!R^n,$$

where ∂Y denotes the boundary of Y. Then the conclusion of Proposition 2 holds true.

Indeed, we may take an arbitrary fixed orthonormed basis $\{l_i(x) = l_i\}$. Let us take an arbitrary $\bar{v} \in \mathrm{co}\,F(x)$ and $\sigma_i \in \{-1, 1\}$. If $\bar{v} \notin \partial F(x)$, then moving from $\bar{v}$ along the vector $-(\sigma_1 l_1 + \ldots + \sigma_n l_n)$ we shall reach a point $v \in \partial F(x)$ for which (iii) is obviously satisfied.

One example (that was considered as non-trivial) is the inclusion (13) with V being the semi-circle in $I\!R^2$ (a semi-sphere in $I\!R^n$ can be treated in the same way):

$$V = \{(v_1, v_2) :\, (v_1)^2 + (v_2)^2 = 1,\ v_2 \ge 0\}.$$

The claim of the conjecture (5) for this example follows from Proposition 2. Indeed, one may take $l_i = e_i$ – the standard basis in $I\!R^2$. For any $\bar{v} \in \operatorname{co} V$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$ define $v_2 = \bar{v}_2$, $v_1 = -\sigma_1 \sqrt{1 - \bar{v}_2^2}$. Then $\sigma_1 \alpha_1(v) = -\sqrt{1 - \bar{v}_2^2} \leq 0$ and $\sigma_2 \alpha_2(v) = 0$. Assumption (iii) of Proposition 2 is fulfilled.

Part 3. Now, we consider a differential inclusion of the form

$$\dot{x}(t) \in G(x)V, \quad x(0) = x_0, \tag{17}$$

where $G(x)$ is an $(n \times m)$-matrix and $V \subset I\!R^m$.

Proposition 3. *Let V be compact and $G(\cdot)$ be Lipschitz continuous with constant $L > 0$, and bounded by a constant M, both with respect to the operator norm of G. Let (5) holds for the differential inclusion (13) with some constant c. Then for the differential inclusion (17) the estimation*

$$H(S_h, S_h^{\mathrm{co}}) \leq cM(1 + L)e^{L|V|}h,$$

holds for every $h = 1/N$, $N \in I\!N$.

This proposition is an extension of [8, Theorem 2] where it is assumed that G is differentiable and V is a box. The proof below is a discrete-time adoption of that in [8].

Proof. Let $\{y_k\} \in S_h^{\mathrm{co}}$. Then for every $k = 0, \ldots, N-1$ there is some $w_k \in \operatorname{co} V$ such that $y_{k+1} = y_k + hG(y_k)w_k$. According to the second assumption of the proposition, there exists a sequence $\{v_k\}$ with $v_k \in V$, such that

$$\left| \sum_{i=0}^{k} (v_i - w_i) \right| \leq c \quad \text{for every } k = 0, \ldots, N.$$

Let $\{x_k\} \in S_h$ be defined by $x_{k+1} = x_k + hG(x_k)v_k$. Denote $\delta_k = y_k - x_k$. Then $\delta_0 = 0$ and

$$\begin{aligned}
\delta_{k+1} &= h \sum_{i=0}^{k} (G(y_i)w_i - G(x_i)v_i) \\
&= h \sum_{i=0}^{k} G(y_i)(w_i - v_i) + h \sum_{i=0}^{k} (G(y_i) - G(x_i))v_i \\
&= hG(y_k) \sum_{i=0}^{k} (w_i - v_i) - h \sum_{i=0}^{k-1} (G(y_{i+1}) - G(y_i)) \sum_{j=0}^{i} (w_j - v_j) \\
&\quad + h \sum_{i=0}^{k} (G(y_i) - G(x_i))v_i,
\end{aligned}$$

where in the last equality we use the discrete analog of the integration by parts formula for Stieltjes integrals. Then

$$|\delta_{k+1}| \le hMc + hLMc + hL|V| \sum_{i=0}^{k} |\delta_i|$$
$$= hL|V| \sum_{i=0}^{k} |\delta_i| + hcM(1+L).$$

Next, we use the following simple fact that can be proved by induction.

Lemma 1. *If the sequence* $\{\Delta_k \ge 0\}$ *satisfies*

$$\Delta_{k+1} \le A \sum_{i=0}^{k} \Delta_i + B, \quad k = 0, 1, \dots, \quad \Delta_0 = 0,$$

then

$$\Delta_k \le B(1+A)^{k-1}, \quad k = 1, 2, \dots .$$

Thus,

$$|\delta_k| \le hcM(1+L)(1+hL|V|)^{k-1} \le hcM(1+L)e^{L|V|}.$$

Q.E.D.

Part 4. Following [8], one can use the above proposition to obtain an estimation as in (5) for non-affine inclusions of the form

$$\dot{x} \in f(x, U), \tag{18}$$

where $U \in I\!R^m$ consists of finite number of points; $U = \{u_1, \dots, u_s\}$, and $f(\cdot, u_i) : I\!R^n \to I\!R^n$.

Proposition 4. *Let the functions* $f(\cdot, u_i)$ *be Lipschitz continuous with constant* $L > 0$*, and bounded my a constant* M*. Then for the differential inclusion (18) the estimation*

$$H(S_h, S_h^{\text{co}}) \le 2s^{3/2} M (1 + \sqrt{s}L) e^{\sqrt{s}L}\, h, \tag{19}$$

holds for every $h = 1/N$*,* $N \in I\!N$*.*

Proof. The set of trajectories, S, of (18) coincides with that of the inclusion

$$\dot{x} \in G(x)V, \tag{20}$$

where $u = (u_1, \dots, u_s)$, $G(x) = [f(x, u_1), \dots, f(x, u_s)]$, and $V = \{e_1, \dots, e_s\}$ with e_i – the unit coordinate vectors in $I\!R^s$. The convexified version of (20) reads as

$$\dot{x} \in G(x) \operatorname{co} V. \tag{21}$$

Moreover, the Euler discretizations of (18) and $\dot{y} \in \text{co} f(y, U)$ are equivalent to those of (20) and $\dot{y} \in G(y)V$. Then it is enough to estimate $H(S_h, S_h^{\text{co}})$, where S_h and S_h^{co} are the solution sets of the Euler discretizations of (20) and (21), respectively. Then the claim of the proposition follows from Proposition 1 (saying that $c = 2s|V| = 2s$, and Proposition 3, taking into account that $|V| = 1$, the Lipschitz constant of $G(\cdot)$ is $\sqrt{s}L$, and $\|G(x)\| \leq \sqrt{s}M$, both with respect to the operator norm of G. Q.E.D.

This propositions extends the result in [8] mainly in that $f(\cdot, u)$ is not assumed differentiable. The constant in (19) depends on the number of elements of U, which means that only the weak form of Conjecture (5) is proved in the considered special case. On the other hand, the assertion of the proposition covers most of the practically interesting cases.

4 Conclusion

To the author's knowledge, the conjecture that $H(S_h, S_h^{\text{co}}) = O(h)$ is still open (both in its stronger and weaker form). We stress that the conjecture has not been proved even in the case of a constant mapping $F(x) = V \subset \mathbb{R}^n$. However, the partial results in this paper cover most of the practically important cases.

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