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Abstract

The decline (and possible subsequent) recovery of birth rates coming along with the spreading of the gender egalitarian lifestyle can be explained with the help of diffusion dynamics. In the present paper, we illustrate by means of a diffusion model that degenerated Hopf bifurcations can occur in meaningful socio-economic problems. We show conditions for the occurrence of Hopf, Bautin and Isola bifurcations and explain the implications of these phenomena for the societal long-run development with respect to the dominating family model.

1 Introduction

One explanation of the decline in human fertility is the increasing female employment. In most of the industrialized countries the traditional family with males as breadwinners has changed to families where both partners work. McDonald (2013) illustrates that lacking support for a combination of work and family is accompanied by fewer births per woman. In an earlier paper (McDonald, 2000) argued that increased gender equity within families goes hand in hand with lower fertility.

However, somewhat surprisingly, in recent years a rebound in birth rates has been observed, particularly in Scandinavian countries as well as in France; compare, e.g. Myrskylä et al. (2009). In an interesting paper, Esping-Andersen and Billari (2015) tried to explain this phenomenon to gender equality in societies. Note that here an egalitarian couple is defined by an equal share of the men's and women's paid and domestic work.

In a recent extension, Feichtinger et al. (2013) model the dependence of the birth rate of the egalitarians on their number by a sigmoidal increasing function.The idea behind this assumption is that the spread of egalitarian values, such as the expansion of child care facilities and father leave as well as sharing domestic work between the partners, fosters parenthood.

The second crucial assumption of this model is a permanent diffusion from traditionalists to egalitarians: the incentive to adopt an egalitarian lifestyle is higher when there are many people of this kind because of a higher degree of implemented support measures. Thus, the subpopulation of egalitarians may increase in two different ways: either through the sketched diffusion process, or by its own endogenized birth rate. It is the interaction of the strength and the paces of both effects which generates a rich spectrum of possible developments of population structure and the resulting fertility path. The focus of Feichtinger et al. (2013) is to explain mechanisms responsible for the decline and possible recovery of the birth rates. The transition form a traditional to an egalitarian society is therefore studied and it is analyzed which factors might be responsible for it.

The underlying two state model is nonlinear, which leads to a complex behavior of the solution paths. In the present paper we focus on a specific property which the model may exhibit, namely persistent oscillations. This is not only interesting from a demographic point of view, but also from a mathematical one. Demographically, it might be interesting to ask for conditions under which permanent oscillations may result from the diffusion of traditionalists to egalitarians. And from a mathematical point of view we will be able to deliver one of the very first socio-economic applications of the so-called Bautin bifurcation.

We will provide an introduction to Hopf bifurcation and conditions for its occurrence. We characterize Bautin bifurcations and discuss its implications for the underlying demographic model. Within the described model we are even able to locate Isola bifurcation points, which we illustrate within the demographic model. We discuss the long-run implications for the development of society with respect to the family model which will dominate in the long run.

The paper is organized as follows. In Section 2 the model is briefly explained. Section 3 provides a short introduction into Hopf, Bautin and Isola bifurcations and illustrates these by means of a numerical analysis of the model presented in Section 2. In Section 4 some conclusions are drawn.

2 The Model

The idea of the present model is to study the impact of the transition to a gender egalitarian society on the birth rates. We assume that the number of egalitarians has an impact on the diffusion from a traditional family model to an egalitarian as well as on the birth rate.

There are two state variables. $T(t) \geq 0$ denotes the number of traditionalists and $E(t) \geq 0$ the number of egalitarians. In the subsequent we omit time argument t unless necessary. The state equations are

$$\begin{aligned}\dot{T} &= (b_T - d)T - kf(E)T, \\ \dot{E} &= (b_E(E) - d)E + kf(E)T.\end{aligned}$$

The birth rate of the traditionalists is b_T and is assumed to be fixed. The death rate of both groups is given by d. The birth rate of the egalitarians is assumed to depend on the number of egalitarians. The intuition behind this assumption is that the higher the number of egalitarians, the more their family model will be accepted. Therefore it will find stronger support within society, which might have an impact on institutional inequalities, allowing a better combination of the egalitarian lifestyle and parenthood. To capture that such a strong support might only be provided when the number of egalitarians is sufficiently high, but also that the impact is limited, we use an S-shaped function, namely

$$b_E(E) = \nu + \beta \frac{E^2}{E^2 + m},$$

where ν denotes the part of the birth rate which is not related to social interactions. Parameter β relates to the height of the S-shaped part of the function and m to the steepness.

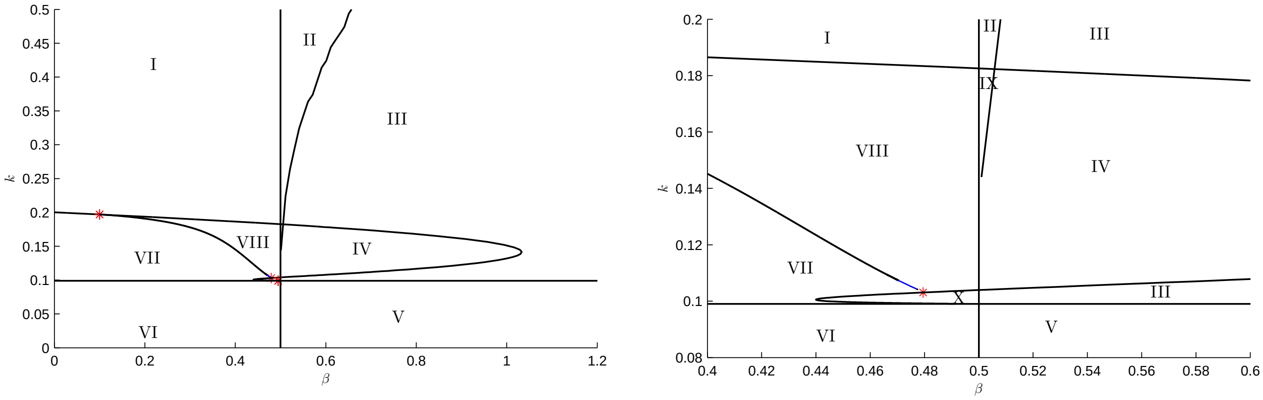

Figure 1: Bifurcation diagram showing the partition of the (k, β)-plane into domains of similar behavior, the right panel is a zooming of the left panel, see also Feichtinger et al. (2013).

Traditionalists can decide to become egalitarian[1]. This flow from T to E is also related to social interactions, as a higher number of egalitarians increase the incentive/pressure to adopt the new lifestyle. Parameter k denotes the speed of diffusion, and function $f(E)$ describes the effect of social interactions. Again we use an S-shaped function, i.e.

$$f_E = a_E + \alpha \frac{E(t)^2}{E(t)^2 + n_E}.$$

For a more detailed justification and description of the model, see Feichtinger et al. (2013). Since this paper provides a detailed analysis of the model, we here just provide some basic details. There are three steady states, one is in the interior of the admissible region (i.e. $T > 0$, $E > 0$) and is either stable or unstable, and two are at the boundary (one with $T > 0$, $E = 0$, which is either a saddle or unstable, and one with $T = 0$, $E = 0$, which is always a saddle point). The location and the stability properties of the steady states depends on the parameters.

The parameters used are shown in Tab. 1.

b_T	ν	d	m	a_E	α	n_E	k	β
1.1	0.5	1	12	0.01	1	0.5	0.16	0.4

Table 1: Parameters for the numeric calculations

Note that while there is a demographic intuition behind some of the parameters, see Feichtinger et al. (2013), many of the parameters are unobservable in practice and therefore cannot be empirically validated.

For the location of the bifurcation, we particularly considered two parameters: k, the speed of diffusion, and β which describes the impact of social interactions on the birth rate. The bifurcation diagram is shown in Fig. 1, a detailed analysis of the different regions is presented in Feichtinger et al. (2013). A Bautin bifurcation can be found for intermediate values of these two parameters, in particular between Region VII and VIII.

[1] In this model variant we do not allow a backflow to a traditionalist family model.

Note that we assume the birth rate of the traditionalists $b_T > 1$. Thus, a society where traditionalism dominates will always grow in our model. The reason now why k and β play such an important role is that if k is high, but β low the effects of the societal transition on the total birth rate are particularly big: everyone becomes egalitarian, but this lifestyle does not find support, so births suffer. If both of these parameters are high, population will simply grow. If k is low the transition from an egalitarian society will happen, if β is low, the traditionalists' high birth rate will drive population growth, if β is high the birth rate of egalitarians will do this, at least after some time.

For intermediate values of k the population might approach a steady state. If β is low, this happens independent of the initial state value. I.e. the diffusion speed then is high enough to allow a transition from traditionalism to egalitarianism, but not so fast that the lack of societal support leads to a recurrence of a population decline. Here the high birth rate of traditionalists just compensates the lower one of the egalitarians. If β is high, the long-run development depends on the initial state values. For some initial state values the population explodes due to the high societal support. For other - intermediate - state values one would approach a steady state.

3 The Hopf Bifurcation and (some of) its Degeneracies

In the present section we give a general sketch of the Hopf bifurcation and its degeneracies. As it is explained in full detail in Kuznetsov (1995), see also Guckenheimer and Holmes (1983); Grass et al. (2008), a Hopf bifurcation occurs in a dynamical system (in our example with state vector $\boldsymbol{x} = (T, E)$)

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \lambda), \tag{1}$$

if a pair of eigenvalues of the Jacobian

$$\mathbf{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}, \tag{2}$$

evaluated at an equilibrium $(\boldsymbol{x}_0, \lambda)$ of (1), crosses the imaginary axis, as the distinguished parameter λ is varied. At the Hopf bifurcation point $(\boldsymbol{x}_c, \lambda_c)$ a family of periodic solution bifurcates from the steady state, if the following conditions are satisfied:

Condition 1. 1. A pair of conjugate complex eigenvalues $(\sigma(\lambda), \overline{\sigma}(\lambda))$ with $\sigma(\lambda) = \sigma_R(\lambda) + i\sigma_I(\lambda)$ crosses the imaginary axis with non-zero velocity:

$$\sigma_R(\lambda_c) = 0, \quad \partial\sigma_R/\partial\lambda|_{\lambda=\lambda_c} \neq 0, \quad \omega \neq 0. \tag{3}$$

2. The remaining eigenvalues of the Jacobian have non-zero real part.

Using Center Manifold reduction, which in our two-dimensional example isn't necessary, because there are no non-critical components, and Normal Form simplification it is possible to transform the dynamical system close to the Hopf bifurcation point to the cubic complex equation

$$\dot{z} = (\sigma(\lambda) + a_1 z\overline{z})z, \quad z \in \mathbb{C}. \tag{4}$$

The real part c_1 of the cubic coefficient $a_1 = c_1 + id_1 \in \mathbb{C}$ is called the first Lyapunov coefficient, it governs the stability of the bifurcating periodic solutions. We introduce the unfolding parameters

$$\mu = \left.\frac{\partial \sigma_R(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_c} (\lambda - \lambda_c),$$
$$\omega_1 = \left.\frac{\partial \sigma_I(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_c} \Big/ \left.\frac{\partial \sigma_R(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_c},$$

and rewrite (4) in polar coordinates $z = r\exp(i\varphi)$

$$\dot{r} = (\mu + c_1 r^2)r, \tag{5}$$
$$\dot{\varphi} = \omega + \omega_1\mu + d_1 r^2. \tag{6}$$

Since the angular variable φ does not appear in (5) and $\omega \neq 0$, it is sufficient to investigate (5). The nonzero rotation can be calculated afterwards and does not influence the stability of the solution.

Equation (5) has two steady state solutions:

1. The equilibrium $r = 0$. It is stable for $\mu < 0$ and unstable for $\mu > 0$.

2. The periodic solution $r > 0$. It satisfies the steady state equation

$$\mu + c_1 r^2 = 0. \tag{7}$$

   If $c_1 < 0$, it exists for $\mu > 0$ and is called supercritical, otherwise it exists for $\mu < 0$ and is called subcritical. Its stability is governed by the (scalar) Jacobian of (5)

$$A = \mu + 3c_1 r^2 \overset{\text{by (7)}}{=} 2c_1 r^2.$$

   Therefore the supercritical periodic solution is asymptotically stable, whereas the subcritical one is unstable.

3.1 Generalized Hopf or Bautin Bifurcation

If only one parameter is varied in the original dynamical system, almost always a Hopf bifurcation with a nonzero first Lyapunov coefficient c_1 can be found. Varying a second parameter, one obtains a curve of Hopf bifurcation points, along which c_1 might vanish. If the first Lyapunov coefficient c_1 vanishes, the cubic terms do not provide sufficient informations and higher order terms have to be considered. Since this case occurs only for very special parameter values, we investigate the behavior of the system close to the degenerate Hopf bifurcation and introduce an unfolding parameter δ for the cubic term[2]. Calculating the Normal form up to fifth order terms and switching to polar coordinates, the amplitude equation becomes

$$\dot{r} = (\mu + \delta r^2 + c_2 r^4)r, \tag{8}$$

[2]Note that we simply set $\delta = c_1$. The use of the new symbol should indicate, that we study the bifurcation close to the Bautin point, that is for $|\delta| \ll 1$.

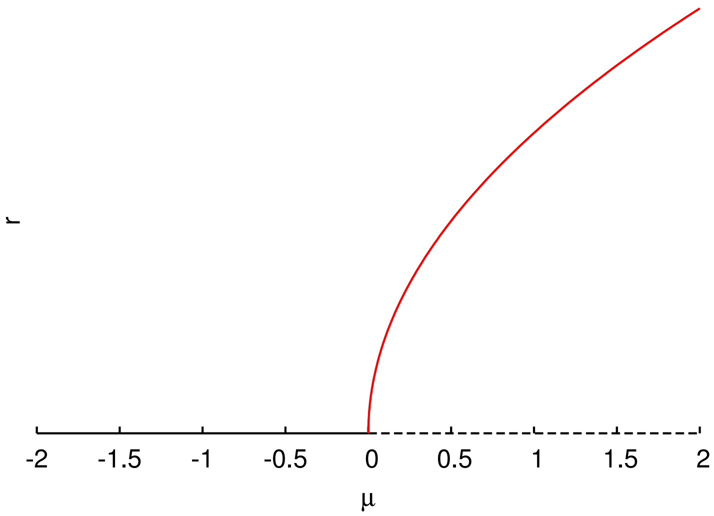

Figure 2: Bifurcation diagram for the supercritical Hopf bifurcation: At $\mu = 0$ a family of stable periodic solutions with amplitude r bifurcates from the equilibrium

where c_2 is called the second Lyapunov coefficient. Again the angular variable φ plays no role.

Again (8) has two steady state solutions:

1. The steady state $r = 0$. As before, it is stable for $\mu < 0$ and unstable for $\mu > 0$.

2. The periodic solution $r \neq 0$ now satisfies the branching equation

$$\mu + \delta r^2 + c_2 r^4 = 0. \tag{9}$$

   It's stability is governed by the Jacobian

$$A = \mu + 3\delta r^2 + 5c_2 r^4 \overset{\text{by (9)}}{=} 2\delta r^2 + 4c_2 r^4.$$

   If both δ and c_2 are negative, the bifurcation is supercritical and the periodic solutions are stable; if both coefficients are positive, the bifurcation is subcritical and the solutions are unstable. The interesting case occurs, if δ and c_2 have different signs, say $\delta > 0$ and $c_2 < 0$: For small amplitudes the term δr^2 dominates and the solution is unstable, whereas for large amplitudes the quartic term rules and causes the solution to become stable. The critical amplitude r_c is obtained from

$$0 = A = 2r_c^2(\delta + 2c_2 r_c^2) = 0 \Rightarrow r_c^2 = -\delta/(2c_2).$$

   Inserting into (9), we find the turning point

$$\mu_c = \delta^2/4c_2.$$

   For $c_2 < 0$ and $\mu \in (\mu_c, 0)$ the system has three different steady solutions: The trivial steady state, the small unstable periodic orbit and the large stable periodic orbit.

The behavior of the bifurcating periodic solutions for $c_2 < 0$ is displayed in Fig. 3: At the Bautin bifurcation point ($c_1 = 0$) the solution branch has the shape of a quartic parabola

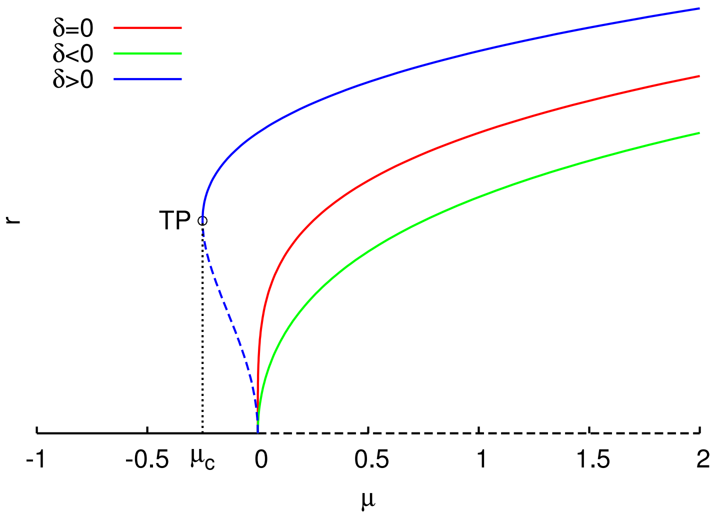

Figure 3: Bifurcation diagrams close to the Bautin bifurcation.

($\mu + c_2 r^4 = 0$). For $c_1 < 0$ also the cubic terms in the bifurcation equation contribute to the supercritical behaviour. If $c_1 > 0$ the bifurcating branch initially bends to the left, because for small amplitudes the cubic terms dominate, whereas for larger amplitudes the quintic term governs the behavior, rendering the periodic solutions stable. The stability change takes place at the turning point "TP".

For $c_2 > 0$ the situation is reversed: The small supercritical orbit is stable and the large periodic oscillation is unstable.

A typical phase portrait for parameters close to a Bautin bifurcation with $c_2 < 0$ is displayed in Fig. 4: A stable equilibrium is surrounded by a small unstable and a large stable periodic solution. The gray dashed and dot-dashed curves are the $\dot{T} = 0$ and $\dot{E} = 0$ isoclines, respectively.

The intuition behind this behavior might be that parameter k is large enough here to lead to a fast increase of egalitarians when T is intermediate and E is low. Then β is not large enough that the birth rates drive the growth of E fast enough so that the birth rate can stabilize at a constant level which is the case for intermediate E. For low E, the number of egalitarians E first only grows slightly, but the egalitarian lifestyle becomes more and more attractive and most traditionalists become egalitarian. The impact of β (and depending on the parameters also E) is not large enough to allow further population growth so after some time both the number of traditionalists and egalitarians will fall. T will recover and thus also E.

3.2 Isola Bifurcation

If the crossing condition 1 does not hold, another interesting phenomenon occurs: The pair of critical eigenvalues only touches the imaginary axis. Looking at the bifurcation diagram in the (β, k) plane, we find some points, where the Hopf bifurcation boundary has a vertical tangent, e.g. at $\beta = \beta_c \approx 0.44$ and $k \approx 0.1$, see Figure 5. Fixing β at this value and varying k, the eigenvalues behave as described. Again this situation is exceptional and we have to introduce an unfolding parameter to study the problem in a vicinity of the bifurcation point. In our example we could remove this degeneracy by shifting β a little bit. Either the Hopf bifurcation boundary and the line $\beta = \text{const}$ do

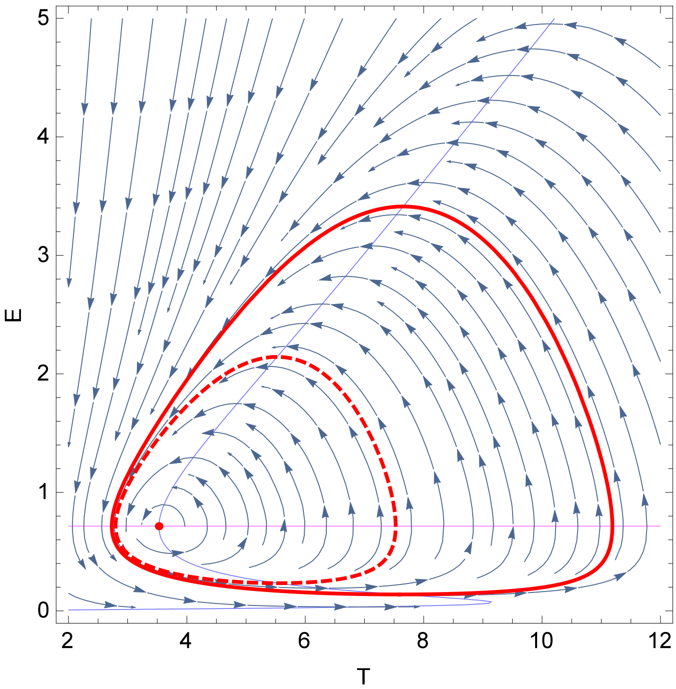

Figure 4: Phase portrait for parameters in the vicinity of a Bautin point ($k = 0.194$, $\beta = 0.15920546$): The stable fixed point is surrounded by a small unstable (dashed line) and a large stable (solid line) periodic solution.

not intersect locally, if we increase β, or they intersect twice for smaller values of β.

If we keep β fixed close to its extremal values along the S-shaped Hopf bifurcation boundary in the (β, k)-plane and vary k, the eigenvalues vary according to Figures 6 and 8, respectively: If the bifurcation boundary (solid curve) is crossed from left to right, that is, holding k fixed and increasing β, the eigenvalues cross the imaginary axis with positive speed. But keeping β fixed and increasing k, we encounter situations, where the eigenvalues just touch the imaginary axis. If we increase k along the dashed line in Fig. 5, we miss the stability boundary and the eigenvalues remain in the left half plane. Increasing k along the dotted line, we first cross the stability boundary into the unstable region and then back into the stable region.

The bifurcation boundary separates Regions VII (on the left) and X (on the right) in the bifurcation diagram Figure 1. In Region VII the interior steady state is a stable focus. There is no cycle, the steady state at the boundary is always admissible. Thus, independent of the initial number of traditionalists and egalitarians, the size of the two population groups will always become constant in the long run. In Region X the interior steady state becomes unstable, but now there is a stable limit cycle. Thus, the number of egalitarians and traditionalists will always fluctuate, but neither group will ever explode or become dominating in the long run.

At the right endpoint the situation is reversed: As displayed in Figs. 7 and 8, if we fix β at a slightly smaller value and increase k along the dashed line, we cross the stability boundary into the stable region and back into the unstable region. Fixing β a bit larger, the eigenvalues remain unstable.

In this case the bifurcation boundary separates Regions III and IV. In Region III E always diverges. Due to the high impact of social interactions of the birth rate, the number of egalitarians E always explodes in this scenario and egalitarianism becomes

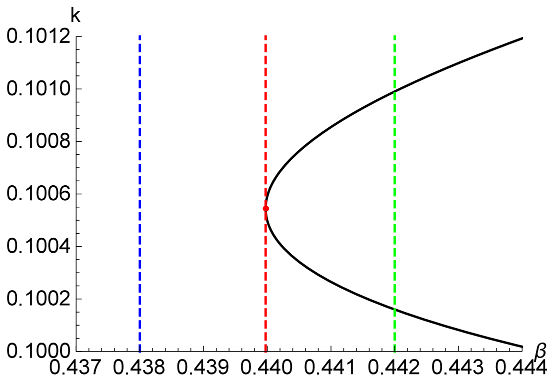

Figure 5: Enlarged stability diagram close to the left border of the stability diagram

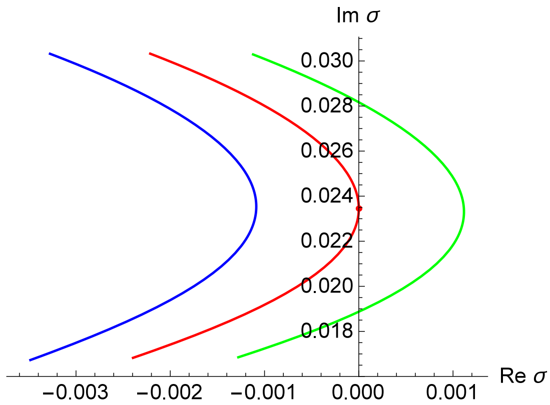

Figure 6: Evolution of the eigenvalues for varying values of k and fixed β, corresponding to the lines in Fig. 5

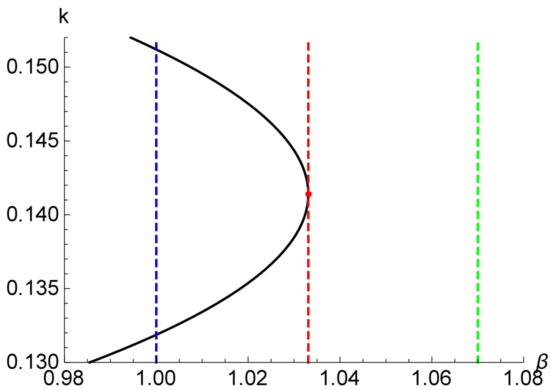

Figure 7: Enlarged stability diagram close to the right border of the stability diagram

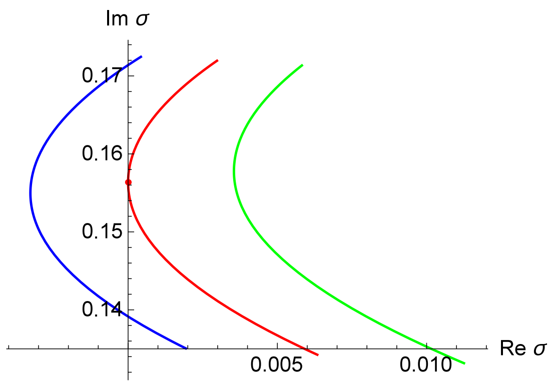

Figure 8: Evolution of the eigenvalues for varying values of k and fixed β, corresponding to the lines in Fig. 7

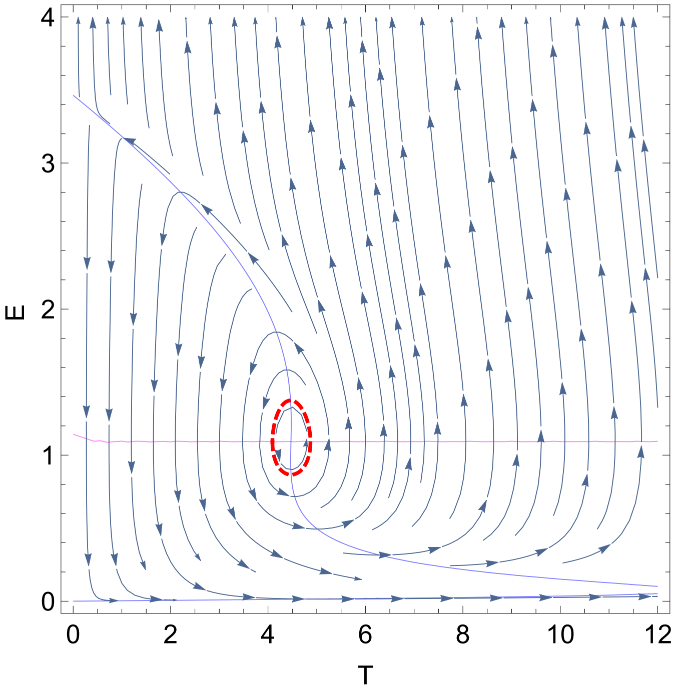

Figure 9: Phase portrait for parameters in the vicinity of the Isola bifurcation point ($k = 0.14$, $\beta = 1$): The stable fixed point is surrounded by an unstable periodic solution (dashed line).

the dominating family model. As already mentioned, the interior steady state is unstable in this region. This changes at the Hopf bifurcation boundary. Figure 9 shows a phase portrait in the vicinity of the Isola bifurcation. For most initial state values the number of egalitarians diverges, only in a small region of the state space it approaches a stable focus. The intuition behind this is as follows: when the number of traditionalists T is large, the number of egalitarians E will grow due to a high inflow from T. After some time this flow is so strong that T starts to decrease. If the number of egalitarians exceeds a certain size, i.e. it is above the 1-dimensional stable manifold of a saddle point at the boundary, the birth rate of egalitarians is so large that their size will continue to grow. If it is not, then the decrease of T means that the inflow to E becomes so small that E eventually decreases. The traditionalists' birth rate is above the replacement level, so when the flow to E lowers, their size increases again. Within the cycle, the counteracting forces that lead to an increase and decrease are more balanced than outside, thus, the size of the state variables can stabilize at a fixed level.

Concerning the eigenvalues, up to second order the real part $\sigma_R(\lambda)$ of the critical eigenvalue can be expressed as

$$\sigma_R = \gamma + \tau(\lambda - \lambda_c)^2, \tag{10}$$

where

$$\tau = \left.\frac{\partial^2 \sigma_R(\lambda)}{2\partial\lambda^2}\right|_{\lambda=\lambda_c}$$

and γ is the unfolding parameter.

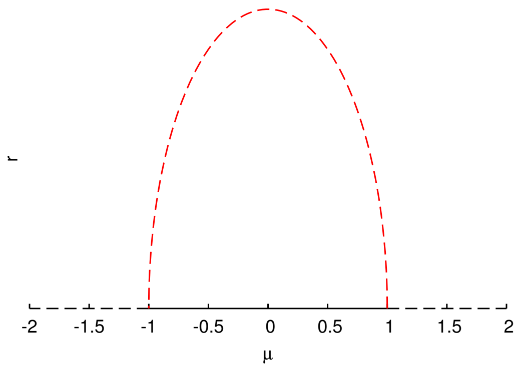

Figure 10: Bifurcation diagram of the Isola bifurcation for $c_1 > 0$ and $\gamma < 0$.

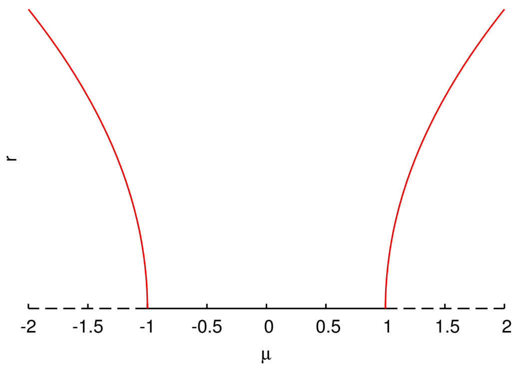

Figure 11: Bifurcation diagram of the isola bifurcation for $c_1 < 0$ and $\gamma < 0$.

If $\tau < 0$ and $\gamma > 0$, the equilibrium is unstable in the interval

$$\lambda \in (\lambda_c - \sqrt{-\gamma/\tau}, \lambda_c + \sqrt{-\gamma/\tau})$$

and stable outside. For $\tau < 0$ and $\gamma < 0$, the equilibrium is stable for all sufficiently nearby values of λ.

If $\tau > 0$, the situation is reversed: If also $\gamma > 0$, the equilibrium is throughout unstable, and for $\gamma < 0$ it becomes stable in the interval $(\lambda_c - \sqrt{-\gamma/\tau}, \lambda_c + \sqrt{-\gamma/\tau})$. This situation applies to our model, if we fix β a little bit left to the vertex at (β_c, k_c).

Let us now first investigate the case $\tau > 0$. After introducing the parameter

$$\mu = \sqrt{\tau}(\lambda - \lambda_c),$$

the amplitude equation becomes

$$\dot{r} = (\gamma + \mu^2 + c_1 r^2) r. \tag{11}$$

Compared to (5) the linear term has changed, whereas the nonlinear term is the same.

The equilibrium $r = 0$ is asymptotically stable, if $\gamma + \mu^2 < 0$. If $\gamma > 0$, it is unstable for all values of μ. Otherwise it becomes stable in the interval $(-\sqrt{-\gamma}, \sqrt{-\gamma})$.

The nontrivial equilibrium, which corresponds to a periodic solution, satisfies the branching equation

$$\gamma + \mu^2 + c_1 r^2 = 0, \tag{12}$$

which describes a conic section in the (μ, r) plane.

If $c_1 > 0$ and $\gamma > 0$, (12) has no solution at all. If $c_1 > 0$ and $\gamma < 0$, eqn. (12) describes an ellipse around the origin, connecting the two Hopf bifurcation points $\mu = \pm\sqrt{-\gamma}$. Since the Jacobian is again given by $2c_1 r^2$, the nontrivial solution is unstable.

The bifurcation diagram for this case is displayed in Fig. 10: Solid lines denote stable solutions, dashed lines unstable solutions. For increasing γ the ellipse shrinks and vanishes at $\gamma = 0$.

If $c_1 < 0$, eqn. (12) describes a hyperbola. If $\gamma < 0$, the two branches originate at the Hopf bifurcation points and coexist with the unstable equilibrium. For $\gamma > 0$ the branch of stable nontrivial solutions is disconnected from the equilibrium, but comes close to it.

The bifurcation diagrams for the different cases are shown in Figs. 11-13.

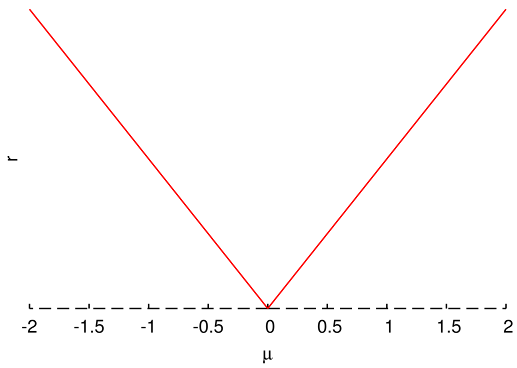

Figure 12: Bifurcation diagram of the isola bifurcation for $c_1 < 0$ and $\gamma = 0$.

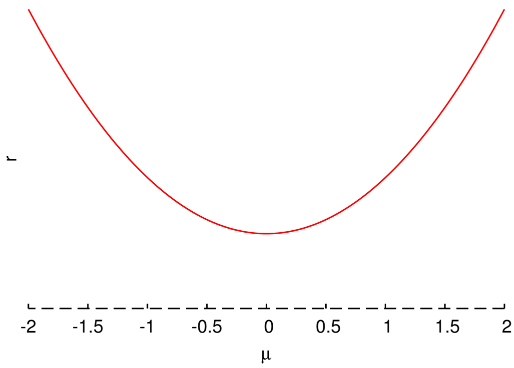

Figure 13: Bifurcation diagram of the Isola bifurcation for $c_1 < 0$ and $\gamma > 0$.

In the case $\tau < 0$, which occurs in our model at $\beta_k \approx 0.44$ and $k_c \approx 0.1$, we proceed in the same way as before. Now we set $\mu = \sqrt{-\tau}(\lambda - \lambda_c)$ and obtain the amplitude equation

$$\dot{r} = (\gamma - \mu^2 + c_1 r^2)r. \tag{13}$$

Again the nontrivial solution branches are either ellipses or hyperbolas, but the sign of γ and the stability properties are reversed. In the supercritical case $c_1 < 0$ the periodic orbits exist in the interval $\mu \in (-\sqrt{\gamma}, \sqrt{\gamma})$ and are stable. If $\gamma < 0$ the equilibrium is stable for all μ. If $c_1 > 0$, an unstable periodic orbit exists close to the equilibrium for $\gamma < 0$ and limits the domain of attraction of the equilibrium. If the initial value leaves the stable corridor, the solution grows indefinitely.

4 Conclusion

To put the main message of our paper in a nutshell: in our work we have added a further aspect of complex behavior of socio-economic model by using various degeneracies of the Hopf bifurcations.

It is not surprising that Bautin (and other degenerated Hopf) bifurcations can occur in demographic or socio-economic problems. Yet, their occurrence and implications have hardly ever been studied in detail in this context. Conducting an in-depth bifurcation analysis can provide substantial insights into the reasons for certain population developments. For example, it can provide explanations why gender egalitarianism becomes the dominating family model in some societies, why it might be widely spread within a population but cannot succeed in the long run in other societies, and why traditionalism prevails in some countries. Of course, such an analysis provides also some mathematical challenges as it is important to understand what happens at bifurcation points. While some properties of Hopf bifurcations can be derived analytically, numerical methods provide important tools for studying the implications of bifurcations.

The problem studied in this paper was purely descriptive. The next step would be to study the implications of Bautin bifurcations in an optimal control problem. There, the dependence of the optimal long-run solution path on the initial state values is even of greater importance than in a descriptive setup as decision makers might be confronted with different optimal short and long run control strategies. Even more interesting, it would be to study the occurrence of Bautin bifurcations in a differential game setup and analyze how multiple players are affected by the history dependence inherent to this type of bifurcation.

To increase the demographic importance of the paper an important task would be to empirically validate the model.

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