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Optimal Management of Ecosystem Services with Pollution Traps: The Lake Model Revisited[1]

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Abstract

In this paper, optimal management of the lake model and common-property outcomes are reconsidered when the lake model is extended with a slowly changing variable (describing sedimentation and recycling of phosphorus), which is considered fixed in the simplified version that has been used in the literature up to now. Some intuitive results are reconfirmed but new optimal trajectories are found that were hidden in the simplified analysis. Moreover, it is shown that in the case of common-property, two Nash equilibria exist in a certain area of initial states where the one leading to the steady state with a high level of ecological services dominates the other one. However, for larger initial states, only the Nash equilibrium steady state with a low level of ecological services exists. This implies that at the separating states a significant drop in welfare occurs, and the users of the lake

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are trapped in the bad Nash equilibrium when this line is crossed (a pollution trap). Finally, it is shown what the consequences are of implementing the optimal phosphorus loadings from the simplified version into the full lake model. The welfare losses can be considerable, and it can happen that the lake ends up in the wrong basin of attraction, but in most cases the welfare losses are small because of the fast-slow dynamics. The analytical tool that is used in this paper is an advanced extension of a Matlab solver for boundary value problems.

Key words: ecosystem services, lakes, multiple equilibria, pollution trap, optimal control, games, fast-slow dynamics

JEL codes: Q20, Q25, C61, C63, C73

1 Introduction

Resources that are embedded in ecological systems often have characteristics that complicate management. A famous example is the lake system that may have multiple equilibria for certain phosphorus loadings due to non-linear feedbacks in the eutrophication process in the lake (Carpenter and Cottingham, 1997, Scheffer, 1997). Gradually increasing the release of phosphorus on the lake may lead to a tipping point where the lake system flips from an oligotrophic state, with a high level of ecological services, to a eutrophic state, with a low level of services. These services include, for example, water, fish and several amenities. After a flip it may be costly (a hysteresis effect) or even impossible to restore the oligotrophic conditions. Many other systems such as coral reefs, grasslands and climate have similar characteristics.

The release of phosphorus on the lake is a decision variable and a dynamic trade-off has to be made between the agricultural benefits that are connected to the release of phosphorus and the loss of ecological services that are due to the accumulation of phosphorus in the water of the lake (Brock and Starrett, 2003, Mäler et al., 2003). Optimal management of this ecological system may have multiple steady states, that are comparable to the multiple equilibria for fixed loadings above. This may lead to history dependence in the sense that a so-called Skiba point or indifference point exists that divides the initial conditions of the lake into an area from where the optimal trajectory converges to an oligotrophic steady state and an area from where this trajectory converges to a eutrophic steady state.

Many other papers have been written on optimal management of the lake (e.g., Wagener, 2003, Dechert and O'Donnell, 2006, Kiseleva and Wagener, 2010, Kossioris et al., 2008; 2011) but all the work up to now has used a one-dimensional representation of the lake system in terms of the phosphorus accumulation in the water of the lake. However, the basic ecological model (Carpenter, 2005) has two differential equations (and an equation for the loading of phosphorus from the soil around the lake into the water which in a management setting is transformed into the control variable). The other differential equation describes the accumulation of phosphorus in the sediment of the lake (the so-called "mud" equation). The two differential equations interact. Part of the stock of phosphorus in the water ends up in the sediment, and the accumulated phosphorus in the sediment affects the maximum rate of the non-linear feedbacks of phosphorus into the water. The dynamics in the sediment of the lake are found to be much slower than the dynamics in the water of the lake (Janssen and Carpenter, 1999), so that this system has fast-slow dynamics.[2] A system is characterized by fast-slow dynamics when there is time scale separation between a set of slowly changing variables (sediment in our case) and a set of rapidly changing variables (phosphorous in the water of the lake).

The one-dimensional lake management models consider the stock of phosphorus in the sediment as fixed and thus the maximum rate of the non-linear feedbacks into the water as a parameter, and in this way ignore the slow dynamics in the lake. In this paper we will analyze optimal management of the full lake system, with fast-slow dynamics, in which this parameter becomes a slowly changing variable. We will show what the long-run effects are but we will also show that a different type of Skiba point may arise. Such a Skiba point does not indicate indifference between moving to one or the other steady state, as in the one-dimensional representation of the lake, but indicates indifference between one or the other optimal trajectory towards the same long-run steady state. To distinguish these points from

[2]This depends on the type of the lake. If the lake is a non-thermally stratified shallow lake, then phosphorus release from the sediments may be rapid because the sediments are exposed to waves and thus phosphorous stored in them will be released fast to the surface waters. On the other hand, on thermally stratified deep lakes phosphorous release takes place when stratification breaks down and the lower layer of the lake water is depleted of dissolved oxygen during the year. This means that the rate of phosphorous release is slower in deep lakes than in shallow lakes (Carpenter et al. 1999; Carpenter 2003).

the traditional Skiba points, which separate different basins of attraction, we will call them *weak* Skiba points and the containing manifold a *weak* Skiba manifold. It may happen in such a case, for example, that the optimal trajectory can either move into the oligotrophic area of the lake immediately or can move into the eutrophic area first and flip to the oligotrophic area later, with the same total discounted net benefits. In case taxes have to be used in order to internalize the pollution externality, the tax trajectories and the tax burden are very different on these two optimal trajectories. This could still yield a preferred trajectory.

In this paper we will show what will happen for different values of the parameter that weighs the benefits and the costs, and for different initial values of the stocks of phosphorus in the water and in the sediment of the lake. For a high weight on the loss of ecological services, there is one long-run steady state in the oligotrophic area of the lake but a weak Skiba manifold of initial points appears from which there are different options for the optimal trajectory towards the steady state. Similarly, for a low weight on the loss of ecological services, there is also one long-run steady state but in the eutrophic area of the lake, with a weak Skiba manifold of initial points from where there are different options for the optimal trajectory towards the steady state. For intermediate values of the parameter that weighs the benefits and the costs, two stable steady states appear with a traditional Skiba manifold that separates the basins of attraction, but possibly also with a weak Skiba manifold with different options for optimal trajectories towards the same steady state.

Resources embedded in ecological systems are usually common-pool resources. An important result on the management of the one-dimensional lake model is the qualitative change that can occur when comparing cooperation and non-cooperation (Mäler et al., 2003). In case full cooperation has only one saddle-point-stable steady state in the oligotrophic area of the lake, non-cooperation can lead to a situation with two saddle-point-stable steady states (one in the oligotrophic area and one in the eutrophic area), and an unstable one in between. This implies that the lake can end up in the eutrophic area of the lake. We extend this analysis to the full lake system. The non-cooperative equilibrium between multiple users of the lake is characterized by the open-loop Nash equilibrium of this differential game (Başar and Olsder, 1982). The possible trajectories of the open-loop Nash equilibrium are the same as for the optimal management problem in which the parameter that weighs the benefits and the costs is divided by the number of users. Mäler et al. (2003) perform the analysis for the one-dimensional representation of the lake, but they are not fully correct in the interpretation of the result. They claim the existence of a Skiba indifference point, whereas we will show that three areas of initial conditions exist. The middle area has two Nash equilibria, one leading to the oligotrophic steady state and the other one leading to the eutrophic steady state. The first one dominates the second one in terms of net benefits. To the left and to the right of this middle area there are areas with one Nash equilibrium, leading to the oligotrophic and to the eutrophic steady state, respectively. It may be argued that in the middle area, the users of the lake will coordinate on the Nash equilibrium with the higher net benefits. This implies that the initial conditions are divided again into areas that either lead to the oligotrophic or to the eutrophic steady state. However, the separation point is not an indifference point, but a point with a substantial drop in net benefits. We will call this a pollution trap. It should be noted that this pollution trap which emerges at an open-loop Nash equilibrium and which exists both in the one- and two-dimensional lake models was not identified by Mäler et al. (2003). The identification of this pollution trap is a contribution of the present paper relative to the existing literature.

This paper also assesses the consequences of ignoring the slow dynamics. If the slow variable is considered to be constant, the optimal management strategies can be calculated with the one-dimensional representation of the lake system, as in the previous literature. If these strategies are implemented in simulations of the two-dimensional lake system, the consequences of ignoring the slow dynamics can be studied. It is shown that the welfare losses can be considerable, and that the lake can end up in the wrong basin of attraction. However, in most cases the welfare losses are small. In these cases, the stock of phosphorus in the water does not change much on the slow part of the optimal trajectory, so that the net benefits are not strongly affected either.

Although our paper focuses exclusively on lake fast-slow dynamics and the implications for optimal management of the extended lake model, we believe that the approach and the results obtained in terms of Skiba manifolds and pollution traps have wider implications since the interactions between fast and slow processes and the separation of time scales is an integral part of ecosystem analysis (e.g. Holling, 2001; Gunderson and Pritchard, 2002; Walker et al., 2012; Levin et al., 2013) and appears in a wide variety of ecosystems modeling. Slow variables in ecosystems include variables such as land use, soil properties, freshwater dynamics, or mutation dynamics. On the other hand, slow variables can be identified in social drivers such as education, social norms or urbanization. When time scale separation is ignored, slow variables are treated as fixed parameters. This could create a policy failure because policy instruments will be designed based on a model with incomplete or misspecified dynamics.

In environmental and resource economics there have been a few attempts to study ecosystems in separate time scales. In particular, Huffaker and Hotchkiss (2006) analyze the economic dynamics of reservoir sedimentation management using the hydrosuction-dredging sediment-removal system. Grimsrud and Huffaker (2006) incorporate time-scale separation in a bio-economic model to investigate the optimal management of pest resistance to pesticidal crops. Crepin (2007) presents a general framework to handle systems with fast and slow variables, and illustrates the approach by using a model of coral reefs subject to fishing pressure, while Crepin et al. (2011) explore how non-convexities and slow-fast dynamics affect coupled human-nature systems and apply the methodology to pest control management.

Crepin (2007) and Crepin et al. (2011) study the polar cases of the slow system, by assuming that the fast system is already in equilibrium, or the fast system, by assuming that the slow variable is fixed, but not the two time scales simultaneously. Huffaker and Hotchkiss (2006) and Grimsrud and Huffaker (2006) undertake singular perturbation analysis by approximating the slow manifold using Fenichel's (1979) invariant manifold theorem. They approximate the slow manifold for an arbitrary time scale separation, but not the paths of the fast variables.

The main difference with this literature is that our approach determines the slow manifold for the actual time scale separation of the lake model, as well as the actual paths of the fast variables for different initial conditions. Through this approach we uncover Skiba manifolds and solution paths, which are policy relevant and which are not identifiable by studying polar cases or approximations of the slow manifold. This is our contribution relative to the existing literature in fast/slow dynamics.

Turning to an area of particular interest, the importance of fast-slow dynamics for climate models was recognized decades ago. Hasselmann (1976) points out that in climate models, fast responding components such as weather variables should be distinguished from slowly responding components such as sea surface temperature, ice coverage or land foliage. Over the past decade, the framework of fast response and slow feedback has emerged as a useful conceptual framework for understanding climate system response to a change in external forcing such as atmospheric carbon dioxide concentration or solar irradiance (e.g. Bala et al., 2010; Cao et al., 2015). Fast response usually refers to rapid climate adjustment to an imposed external forcing that occurs before substantial change in surface temperature, and slow feedback refers to climate response that is associated with a change in surface temperature. It is interesting to note that in our lake model the mud equation provides the slow feedback. Held et al. (2010) model fast and slow components in a model of global warming, by the surface and deeper ocean layers respectively. Other slow evolving variables, such as sea and land ice, permafrost warming, boreal forest expansion or forest dieback, can also be identified in the context of climate science and the economics of climate change.

In climate science the analysis is based on approximation methods for constructing a reduced model for slow variables of multi-scale dynamics with linear or non-linear coupling (e.g. Abramov, 2012, 2013). Regression analysis is also used to identify fast adjustments by the intercept, and slow response by the slope of a linear regression of climate variables against changes in surface temperature (e.g. Bala et al. 2010). Given the significance of time scale separation in climate science, it is surprising that, as far as we know, no significant effort has been undertaken to incorporate fast/slow dynamics in the economics of climate change. We think that our approach of solving the full model and identifying the fast and slow manifolds could be useful in analyzing coupled models of the economy and the climate with time scale separation.

Our contribution relative to existing fast/slow dynamics literature is that we solve the more realistic full-lake model, we identify the different potential tax paths, the policy implications from ignoring slow dynamics, and the pollution trap at the open-loop Nash equilibrium. We acknowledge that due to the complexity of the model we cannot obtain analytical results, but this is something common to models dealing with non-linear dynamics. Our numerical results are based on a parametrization which corresponds to the watershed in Lake Mendota, Wisconsin USA. We use this parametrization because it corresponds to a real world example and also because we can compare our results with one-dimensional lake models that used the same parametrization. Our approach however allows the use of different parametrizations depending on the specific problem. Furthermore, the conceptual framework can be used to study different issues with non-linear feedbacks and time scale separation such as climate change.

Section 2 extends the one-dimensional representation of the lake model to the full two-dimensional lake model and discusses the parameter values. Section 3 presents the optimality conditions for optimal management of the lake, and for the symmetric open-loop Nash equilibrium. Section 4 yields the results for optimal management or full cooperation and Section 5 for the open-loop Nash equilibria. Section 6 studies the implications of ignoring the slow dynamics, and Section 7 concludes.

2 The Lake Model

The lake model as described by Carpenter (2005) is a system of differential equations for phosphorus density in soil, lake water (denoted by P) and surface sediment (denoted by M for mud). The phosphorus in soil generates run-off into the lake water (denoted by L for loading). The phosphorus densities in the water and the sediment of the lake interact according to the system of differential equations that is given by:

$$\dot{P}(t) = L(t) - (s+h)P(t) + rM(t)f(P(t)), P(0) = P_0 \qquad (1)$$

$$\dot{M}(t) = sP(t) - bM(t) - rM(t)f(P(t)), M(0) = M_0 \qquad (2)$$

with

$$f(P) = \frac{P^q}{P^q + m^q}. \qquad (3)$$

The parameter s denotes the sedimentation rate, h the rate of outflow from the lake system, and b the permanent burial rate. The non-linear term in (1) and (2), given by (3), describes the recycling of phosphorus from the surface sediment into the lake water, and this depends on the level of M, with r denoting the maximum recycling rate. The parameter s determines the rate at which phosphorus is removed from the water and stored into the sediment, while r determines the rate at which phosphorus is recycled back from the sediment to the water. More precisely, the term sP determines sedimentation, while the term $rMf(P)$ determines phosphorus recycling. The more P we have, the more will end up in the sediment and thus the more will be recycled from the sediment to the water. Furthermore, m denotes the level of P where recycling is equal to $0.5r$, and q is a parameter that reflects the steepness of $f(P)$ near m. The lake model in Carpenter (2005) has higher powers in the non-linear term (3) but we take quadratic forms, in order to simplify the analysis and in order to be able to compare the results with previous work that assumed quadratic forms, i.e. $q = 2$. We have found that this simplification does not affect the qualitative structure of the results. The other parameters are set equal to the ones that followed from observations on the watershed of Lake Mendota in Wisconsin, USA: $s = 0.7$, $h = 0.15$, $b = 0.001$, $r = 0.019$ and $m = 2.4$. Janssen and Carpenter (1999) note that the dynamics in M is slow as compared to the dynamics in P. The reason is that small values of the parameters r and b and large values of the parameter q turn M into a slow variable (see Appendix A).

The main difference with the model in Carpenter (2005) is that we consider the loading L to be subject to control and thus the result of the optimization of discounted net benefits over an infinite time horizon. The runoff L results from agricultural activities on land and a trade-off occurs between the benefits of agricultural production and the damage of phosphorus accumulation in the water of the lake. Following Mäler et al. (2003), we consider a lake around which $i = 1, ...n$ communities receive two types of benefits: benefits from agricultural production and benefits from the ecosystem services generated by the lake system. Community i, by being able to release phosphorous $L_i(t)$ at time t on the lake, receives agricultural benefits equal to $\ln L_i(t)$. Phosphorous in the lake water $P(t)$ at time t decreases the flow of services generated by the lake ecosystem by $cP(t)^2$. Thus the net flow of benefits accruing to community i is $\ln L_i(t) - cP(t)^2$. The parameter c indicates the relative importance of the costs versus the benefits. First we will consider the optimal management problem for a single user of the lake, because this forms the basis for all other outcomes. This problem can be written as:

$$\max_{L(\cdot)} \left\{ \int_0^\infty \mathrm{e}^{-\rho t} \left[\ln L(t) - cP(t)^2 \right] dt \right\}, \text{ s.t. (1) and (2).} \tag{4}$$

The traditional analysis of the lake management problem corresponds to the study of only the fast sub-model or:

$$\max_{L(\cdot)} \left\{ \int_0^\infty \mathrm{e}^{-\rho t} \left[\ln L(t) - cP(t)^2 \right] dt \right\}, \text{ s.t} \tag{5}$$

$$\dot{P}(t) = L(t) - (s+h) P(t) + rM_0 f(P(t)), P(0) = P_0 \tag{6}$$

$$\dot{M}(t) = 0, M(0) = M_0. \tag{7}$$

Thus, in the traditional analysis of lake management slow dynamics are ignored and the amount of phosphorous in the sediment is assumed to be a fixed parameter at a level M_0, so that equation (2) effectively drops out. By setting $x = P/m$, $a = L/rM_0$ and $\hat{b} = (s+h)\, m/rM_0$, and by changing the time scale to $(rM_0 t)/m$, equation (6) can be rewritten as $x(t) = a(t) - \hat{b}x(t) + \frac{x(t)^2}{x(t)^2+1}$ (with $q = 2$) which is the basic equation describing the lake dynamics in the traditional analysis of lake management. Note that the change in the time scale implies that the discount rate is changed to $(rM_0\rho)/m$. Mäler et al. (2003) take $\hat{b} = 0.6$, so that the lake has tipping points with hysteresis when moving the fixed loading a up and then down again. The discount rate is set equal to 0.03. The parameter $\hat{b}$ becomes 0.6 for $M_0 = 179$, so that this initial condition for M allows for comparing the results below with the results of the traditional analysis. Furthermore, in the sequel we set the discount rate ρ equal to 0.0425, since we do not change the time scale.

3 Optimal Management, Full Cooperation and Open-Loop Nash Equilibria

In this section we will determine the necessary conditions for optimal management of the lake, in terms of a modified Hamiltonian system, and we will show that the same type of modified Hamiltonian system determines the necessary conditions for the full-cooperative outcome as well as the symmetric open-loop Nash equilibrium.

The optimal management problem for a single user of the lake, that was defined in the previous section, has the current-value Hamiltonian function:

$$\mathcal{H}(L,P,M,\lambda_1,\lambda_2) = \ln L - cP^2 + \lambda_1 (L - (s+h)P + rMf(P)) + \lambda_2 (sP - bM - rMf(P)). \tag{8}$$

The necessary conditions resulting from the application of the maximum principle determine the optimal phosphorous loading, $L(t)$, for interior solutions by:

$$\frac{\partial \mathcal{H}}{\partial L} = \frac{1}{L} + \lambda_1 = 0, \tag{9}$$

yielding

$$L(t) = -\frac{1}{\lambda_1(t)}. \tag{10}$$

Setting $f'(P) = \frac{\partial}{\partial P}\left(\frac{P^q}{P^q+m^q}\right) = \frac{qm^q P^{q-1}}{(P^q+m^q)^2}$, and dropping t to ease notation, the modified Hamiltonian system becomes:

$$\begin{aligned}
\dot{P} &= -\frac{1}{\lambda_1} - (s+h)P + rMf(P) && (11)\\
\dot{M} &= sP - bM - rMf(P) && (12)\\
\dot{\lambda}_1 &= 2cP + \left[\rho + s + h - rMf'(P)\right]\lambda_1 - \left[s - rMf'(P)\right]\lambda_2 && (13)\\
\dot{\lambda}_2 &= (\rho + b + rf(P))\lambda_2 - rf(P)\lambda_1. && (14)
\end{aligned}$$

Using (9), $\lambda_1 = \frac{-1}{L}$, so that the time derivative $\dot{\lambda}_1 = \frac{1}{L^2}\dot{L}$, we obtain the modified Hamiltonian system as:

$$\begin{aligned}
\dot{P} &= L - (s+h)P + rMf(P) && (15)\\
\dot{M} &= sP - bM - rMf(P) && (16)\\
\dot{L} &= \left[rMf'(P) - \rho - s - h\right]L + && (17)\\
&\quad \left[2cP - \left[s - rMf'(P)\right]\lambda_2\right]L^2 &&\\
\dot{\lambda}_2 &= (\rho + b + rf(P))\lambda_2 + \frac{rf(P)}{L}. && (18)
\end{aligned}$$

For n communities with the net benefit functions $\ln L_i(t) - cP(t)^2, i = 1, 2, ..., n$, as described in the previous section, we can distinguish full cooperation and a non-cooperative Nash equilibrium. When using the maximum principle, we find the so-called open-loop Nash equilibrium (Başar and Olsder, 1982).

In case of full cooperation the lake problem becomes:

$$\max_{L_i(\cdot),i=1,..,n}\left\{\int_0^\infty \mathrm{e}^{-\rho t}\left[\sum_{i=1}^{n}\ln L_i\left(t\right)-ncP\left(t\right)^2\right]dt\right\},\text{s.t. (1) and (2).} \quad (19)$$

Note that symmetry and the logarithmic form in the objective function effectively imply that the full-cooperative solution is independent of the number of communities n, because the problem can be restated as:

$$\max_{L(\cdot)}\left\{\int_0^\infty \mathrm{e}^{-\rho t}n\left[\ln L\left(t\right)-cP\left(t\right)^2-\ln n\right]dt\right\},\text{s.t. (1) and (2),} \quad (20)$$

where $L=\sum_{i=1}^{n}L_i\left(t\right)$ denotes the total loading of phosphorous. From this it can be seen that the solution for P, M and for total loading L is the same as for optimal management above, with total loading L as the control variable. It follows that the modified Hamiltonian system under full cooperation is given by (15)-(18) and does not depend on n. However, the resulting total welfare level (20), of course, depends on n.

The open-loop Nash equilibrium is the solution, derived with the maximum principle, of the set of optimal control problems:

$$\max_{L_i(\cdot)}\left\{\int_0^\infty \mathrm{e}^{-\rho t}\left[L_i\left(t\right)-cP\left(t\right)^2\right]dt\right\},i=1,2,...,n,\text{s.t. (1) and (2).} \quad (21)$$

We will focus on the symmetric open-loop Nash equilibrium. Each community i maximizes (21) by taking the loadings of all other communities $j\neq i$ as given. The current-value Hamiltonian function for community i becomes:

$$\begin{aligned}&\mathcal{H}\left(L_i,P,M,\mu_1,\mu_2\right)=\ln L_i-cP^2+\\&\mu_1(L-\left(s+h\right)P+rMf\left(P\right))+\mu_2(sP-bM-rMf\left(P\right))\end{aligned} \quad (22)$$

The necessary conditions resulting from the application of the maximum principle for the symmetric equilibrium determine phosphorous loading, $L_i\left(t\right)$, for interior solutions by:

$$\frac{\partial\mathcal{H}}{\partial L_i}=\frac{1}{L_i}+\mu_1=0, \quad (23)$$

yielding

$$L_i\left(t\right) = -\frac{1}{\mu_1\left(t\right)},\ \text{for all } i. \tag{24}$$

Using (23), $\mu_1 = \frac{-n}{L}$, so that the time derivative $\dot{\mu}_1 = \frac{n}{L^2}\dot{L}$, and setting $\mu_3 = \frac{\mu_2}{n}$,we obtain the modified Hamiltonian system as:

$$\dot{P} = L - \left(s+h\right)P + rMf\left(P\right) \tag{25}$$

$$\dot{M} = sP - bM - rMf\left(P\right) \tag{26}$$

$$\dot{L} = \left[rMf'\left(P\right) - \rho - s - h\right)\right]L + \left[\frac{2cP}{n} - \left[s - rMf'\left(P\right)\right]\mu_3\right]L^2 \tag{27}$$

$$\dot{\mu}_3 = \left(\rho + b + rf\left(P\right)\right)\mu_3 + \frac{rf(P)}{L}, \tag{28}$$

which is exactly the same system as (15)-(18), except for the term $2cP/n$ in (27). It follows that the set of possible trajectories of the symmetric open-loop Nash equilibrium can be found by solving the modified Hamiltonian system of the optimal management problem with relative cost parameter c/n instead of c. This is intuitively clear because in the non-cooperative equilibrium, each user only cares about the damage they experience themselves and not about the total damage experienced by all the users. Note, however, that these are only necessary conditions and that the individual welfare indicators (21) are different. This implies that the Nash equilibrium trajectories for some c and n may differ from the optimal trajectories for c/n. Moreover, multiple Nash equilibria may exist. This will become clear in Section 5.

Using the open-loop Nash equilibrium as the non-cooperative equilibrium is only one option. Usually one prefers to use the feedback Nash equilibrium (Başar and Olsder, 1982) because it is reasonable to assume that the state of the system is observed, and this equilibrium is Markov-perfect and thus robust against unexpected changes in the state of the system. Kossioris et al. (2008) and Dockner and Wagener (2014) derive the feedback Nash equilibrium for the one-dimensional representation of the lake. However, they use solution techniques that only apply to one-dimensional systems. They start with the Hamilton-Jacobi-Bellman (HJB) equation that characterizes the feedback Nash equilibrium. Kossioris et al. (2008) transfer this equation into a differential equation in the equilibrium phosphorus loadings. Dockner and Wagener (2014) transfer this equation into a two-dimensional auxiliary system that is relatively easy to solve. Both papers find a multiplicity of feedback Nash equilibria. Kossioris et al. (2008) find only local equilibria, which means that the equilibrium loadings are not defined on the whole state space. Dockner and Wagener (2014) add a global equilibrium, but this one has a discontinuity. The problem is that these solution techniques have not yet been extended to higher-dimensional systems.

In our two-dimensional problem with fast/slow dynamics a way of approaching the feedback Nash equilibrium would be to assume that loading strategies in a time-stationary symmetric feedback Nash equilibrium are given by $L_i = g(P, M)$. That is, loading strategies depend on the current stocks of phosphorus and sediment in the lake. Assuming symmetry, the HJB equation for community $i = 1, ..., n$ becomes

$$\begin{aligned}\rho V(P, M) = \max_{L_i} \{ &\ln L_i - cP^2 + V'_P [L_i - (n-1) g(P, M) - (s+h) P + rMf(P)] \\ &+ V'_M [sP - bM - rMf(P)] \}, \end{aligned} \tag{29}$$

where $V(P, M)$ is the value function for the problem. First-order conditions yield the policy function $g(P, M)$:

$$\frac{1}{L_i} = -V'_P(P, M) \Rightarrow L_i := g(P, M) = -\frac{1}{V'_P(P, M)}, \; V'_P(P, M) < 0. \tag{30}$$

Substituting the policy function from (30) into (29), the HJB equation becomes a non-linear partial differential equation which characterizes the feedback equilibrium for the two-dimensional lake problem:

$$\begin{aligned}\rho V(P, M) = \Bigg\{ &\ln \frac{-1}{V'_P(P, M)} - cP^2 + V'_P(P, M) \left[\frac{-n}{V'_P(P, M)} - (s+h) P + rMf(P) \right] \\ &+ V'_M(P, M) [sP - bM - rMf(P)] \Bigg\}. \end{aligned} \tag{31}$$

Solution of this partial differential equation will determine the feedback Nash equilibrium. Such a solution, however, is a formidable task, especially in the presence of multiply equilibria, which is beyond the scope of the present paper. Determining feedback Nash equilibria under fast/slow dynamics should be a very interesting area for further research.

Summarizing, we can study the management of the lake by studying the modified Hamiltonian system (15)-(18) and by varying the cost parameter c that reflects the relative importance of the costs versus the benefits. We can vary c, but we can also fix c at some value and then divide c by the number of communities n in order to move from the optimal management or full-cooperative outcome to the symmetric open-loop Nash equilibrium. By implementing optimality or the Nash equilibrium conditions we can identify the optimal and Nash equilibrium trajectories.

We can compare the results with the results for the one-dimensional representation of the lake system by taking $c = 1/m^2 = 0.1736$. This corresponds to $c = 1$ in the analysis of Mäler et al. (2003) who use the single lake equation as described in the previous section, with the transformation $x = P/m$. Mäler et al. (2003) find one saddle-point stable steady state in the oligotrophic area of the lake under full cooperation. In the open-loop Nash equilibrium they find two saddle-point stable steady states, one in the oligotrophic area and one in the eutrophic area of the lake, and an unstable steady state in between. More specifically, using the one-dimensional representation of the lake, with a fixed parameter M_0, the steady states are given by the solutions to the set of equations:

$$0 = L - (s+h)\,P + rM_0 f\,(P) \tag{32}$$

$$0 = \left[rM_0 f'\,(P) - \rho - s - h)\right] + 2cPL. \tag{33}$$

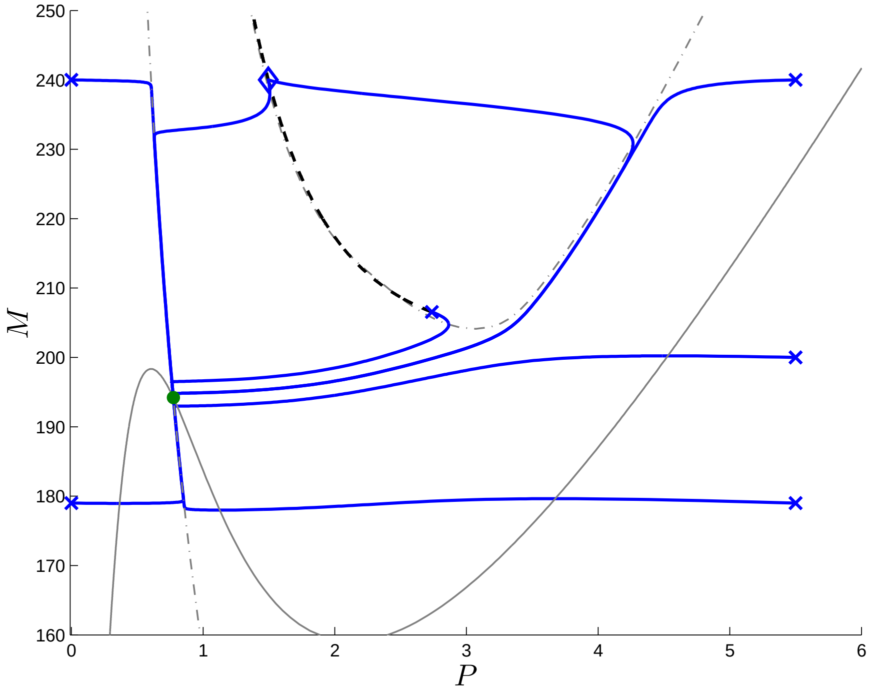
For the parameter values given in section 2, with $c = 0.1736$, the full-cooperative outcome has only one saddle-point stable steady state $P^C = 0.84793$ in the oligotrophic area of the lake (the superscript C denotes cooperation). The open-loop Nash equilibrium for two communities has two saddle-point stable steady states $P_{21}^N = 0.94318$ and $P_{22}^N = 3.8023$ (the superscript N denotes Nash and the first subscript 2 denotes two communities). Depending on the initial condition of the lake, the open-loop Nash equilibrium trajectory will either move to P_{21}^N in the oligotrophic area of the lake or to P_{22}^N in the eutrophic area of the lake. The question is what happens in the full lake model, with fast-slow dynamics. This will be investigated in the next sections.

4 Results for Optimal Management or Full Cooperation

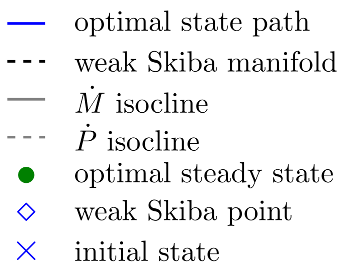
Using the full lake model (1)-(3), in which the phosphorus density in the surface sediment changes, we have to solve the modified Hamiltonian system (15)-(18), which has fast-slow dynamics. First we will look at the optimal management problem for three values of the relative cost parameter: a high value $c_H = 0.1736$, a medium value $c_M = 0.0868$ and a low value $c_L = 0.057867$. The solutions are also the full-cooperative outcomes for these parameter values, because the modified Hamiltonian systems are the same and because the welfare levels are the same (up to a constant depending on n). The modified Hamiltonian systems for $c_M = 0.0868$ and $c_L = 0.057867$ are also the modified Hamiltonian systems for the open-loop Nash equilibria for the relative cost parameter $c_H = 0.1736$ with two and three communities, respectively. However, the open-loop Nash equilibrium for $c_H = 0.1736$ and $n = 2$ is different, of course, from the full-cooperative outcome for $c_M = 0.0868$. We will look at the open-loop Nash equilibria in the next section.

The optimal control of the two-dimensional system yields as a necessary condition a four-dimensional modified Hamiltonian system. This is a two-point boundary value problem, with initial conditions on P and M, and transversality conditions on L and the shadow value λ_2. We use the toolbox OCMat[3] which is especially designed for this type of problems, within Matlab, using solvers for fixed boundary value problems (see Appendix B for details). The system adjusts quickly on the fast dimension and then the trajectory moves slowly with the changes on the slow dimension.

[3]See http://orcos.tuwien.ac.at/research/ocmat_software



(a) Phase portrait



(b) Legend for the phase portraits

Figure 1: Phase portrait of the full-cooperative outcome for $c_H = 0.1736$ on the (P, M)-plane.

Figure 1 presents the projection of the full-cooperative outcome for the high-value cost parameter $c_H = 0.1736$ on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M. All the trajectories converge to $(P_H^C, M_H^C) = (0.77420, 194.19)$ (i.e. the green point) in the oligotrophic area of the lake (the subscript H refers to c_H). It is the intersection point of the isoclines $\dot{P} = 0$ (the dashed grey curves) and $\dot{M} = 0$ (the grey curve) under optimal loading L^*. The phosphorus density in the lake water becomes somewhat lower than in the one-dimensional analysis of the lake. Starting for example in $(P_0, M_0) = (0, 179)$ or in $(P_0, M_0) = (5.5, 179)$, there is first a relatively fast adjustment of P to approximately the steady-state value of P in the one-dimensional analysis of the lake, followed by a slow adjustment of M upwards and P downwards (along the optimal isocline $\dot{P} = 0$).[4] This is not surprising.

However, starting in higher values of M_0, a manifold of a different type of Skiba points occurs (the black dashed curve). Those Skiba points do not separate different basins of attraction towards different steady states but separate different optimal trajectories towards the same long-run steady state. To distinguish these points from the traditional Skiba points, which separate different basins of attraction, we will call them *weak* Skiba points and the containing manifold a *weak* Skiba manifold. More specifically, starting in the diamond on that Skiba manifold yields two optimal trajectories towards the steady state, with the same total discounted net benefits. The first one moves into the oligotrophic area of the lake immediately and then slowly down towards the long-run steady state (along a part of the optimal isocline $\dot{P} = 0$). The other one moves fast into the eutrophic area of the lake first, then slowly down and in the direction of the oligotrophic area of the lake (along another part of the optimal isocline $\dot{P} = 0$), and finally quickly towards the long-run steady state. Furthermore, starting in $(P_0, M_0) = (0, 240)$ or in $(P_0, M_0) = (5.5, 240)$, there is first a relatively fast adjustment of P, followed by a slow adjustment of M and P towards the branches of the two optimal trajectories just described. Finally, starting at the end of the black Skiba manifold, in $(P_0, M_0) = (2.7376, 206.56)$, there is only one optimal trajectory towards the long-run steady state. Actually, starting anywhere below that point (for example in $(P_0, M_0) = (5.5, 200)$), only one optimal trajectory occurs.

In the traditional analysis of lake management, with a fixed amount of phosphorus in the sediment of the lake equal to $M_0 = 240$, the following would happen. The parameter $\hat{b}$ in Mäler et al. (2003) becomes 0.447 (see section 2). In this case optimal management has two saddle-point stable steady states $P_1 = 0.60053$ and $P_2 = 4.6486$, and in between a Skiba indif-

[4]These movements can be regarded as a kind of turnpike behavior. In growth theory the turnpike property means that the optimal path converges to the golden rule path. In our case the optimal path for the fast variable converges fast from different initial P values to the optimal path for the slow variable, and eventually rests at a steady state for the slow variable.

ference point arises. Optimal loading L^* can either start with a moratorium on loading and then move up to the low steady-state level, or it can jump and move up to the high steady-state level. However, in the present analysis the phosphorus in the sediment of the lake M slowly decreases, so that from a one-dimensional point of view the parameter $\hat{b}$ increases. At some point the qualitative properties change and from a one-dimensional point of view there is only one saddle-point stable steady state, in the oligotrophic area of the lake. This implies that at some point a bifurcation occurs. It follows that the optimal loading that moved the trajectory into the oligotrophic area of the lake only has to adjust to the changes in M, but the optimal loading that moved the trajectory into the eutrophic area of the lake has to be reduced significantly at some point in order to move towards the long-run steady state in the oligotrophic area of the lake. The present analysis shows that still a Skiba indifference point arises at the initial level $M_0 = 240$ which means that a similar start-up as above plus the continuation of the optimal loadings L^* towards the long-run steady state have the same total discounted net benefits. One can think of the processes as a sequence of trajectories for different fixed values of M starting at $M_0 = 240$ and ending at $M = 194.19$. Each trajectory can be regarded as corresponding to something like "short-run steady states" of the one-dimensional model for the stock P, for fixed M. As M changes slowly, under the slow dynamics, we move to different trajectories and different short-run steady states. When M is above 206.56, we still have two steady states for fixed $M \in [206.56, 240]$ and the two trajectories move towards them, but below 206.56 there is only one steady state. Thus at $M = 194.19$ there is one optimal steady state and the two trajectories meet at this "long-run steady state". In this respect the one-dimensional model can be regarded as incomplete by not accounting for the slow dynamics. Figure 2 shows these two possible optimal loading policies. The black line is the optimal loading L^* that moves the trajectory to the oligotrophic area and then to the steady state, while the blue line is the optimal loading L^* that moves the trajectory to the eutrophic area first and then to the steady state in the oligotrophic area. The large downward jump at some point is needed to get to the oligotrophic area.

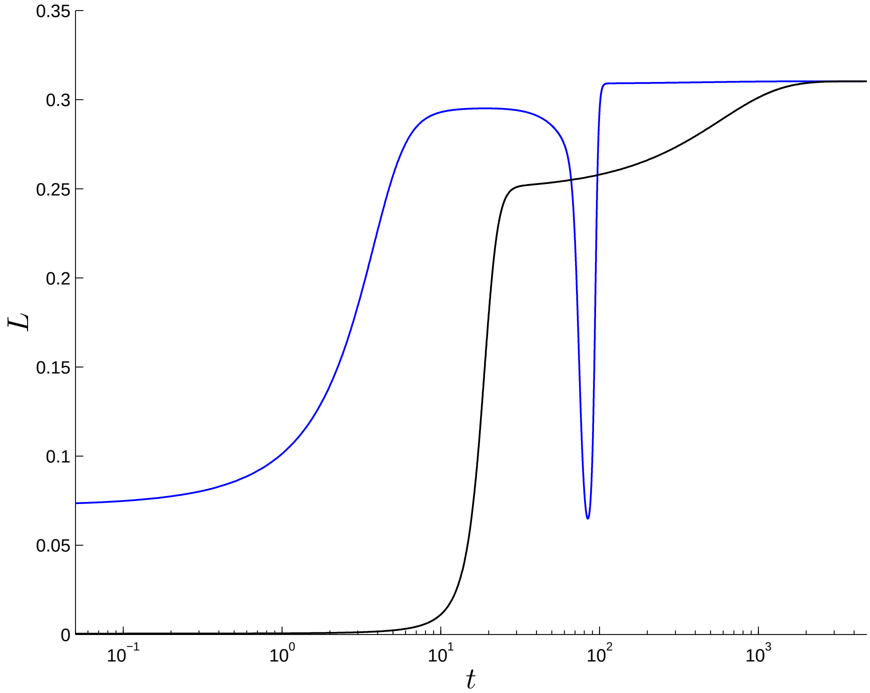


Figure 2: Two possible optimal loading policies.

From equation (10) we know that the optimal loading L^* is the inverse of the shadow price of an extra unit of phosphorus in the lake water $-\lambda_1$. Figure 3 shows the corresponding two paths for the shadow price $-\lambda_1$, yielding the same total discounted net benefits.

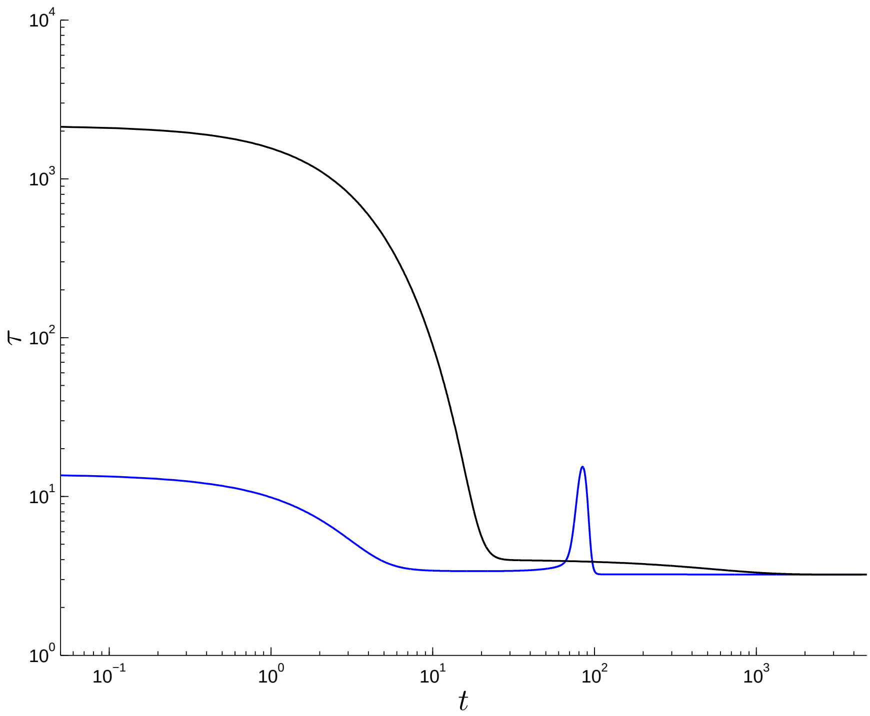


Figure 3: Two paths for the shadow price $-\lambda_1$ or tax τ.

We can interpret these shadow prices as the time-dependent Pigouvian taxes on phosphorus loadings in a decentralized regulatory system with $\tau(t) = -\lambda_1(t)$. It follows that the existence of these weak Skiba points implies indifference for the regulator between the two tax paths, in terms of aggregated discounted welfare. However, the tax effort between the two paths is different. The black line requires a high tax in the beginning, followed by a lower almost constant tax for most of the remaining time. The blue line requires a lower tax in the beginning but at some point the tax has to be increased (temporarily) in order to decrease the phosphorus loadings and move to the oligotrophic area of the lake. Even though the tax revenues can be returned lump-sum to the economy, the black line requires a higher regulating tax effort than the blue line. Actually, the total discounted tax burden under the black line amounts to 11912, whereas under the blue line it amounts to 206. Thus, it could be argued that firms would favor low taxes, but these taxes will have to be paired with a tax increase for a certain period of time to avoid eutrophic outcomes. On the other hand, from the perspective of the costs of changing taxes, the black line is preferred since the blue line requires a temporary increase in taxes in order to lower the loadings of phopshorus and return to the oligotrophic area of the lake. This increase might involve extra costs for the rapid adjustment of taxes to higher levels, or may violate the regulator's preferences for intertemporal smoothing of tax payments. The time interval during which the tax increases, reaches the peak, and then goes back to normal in figure 3, is 13.5% of the total time needed to reach a band of 0.1% around the steady state.

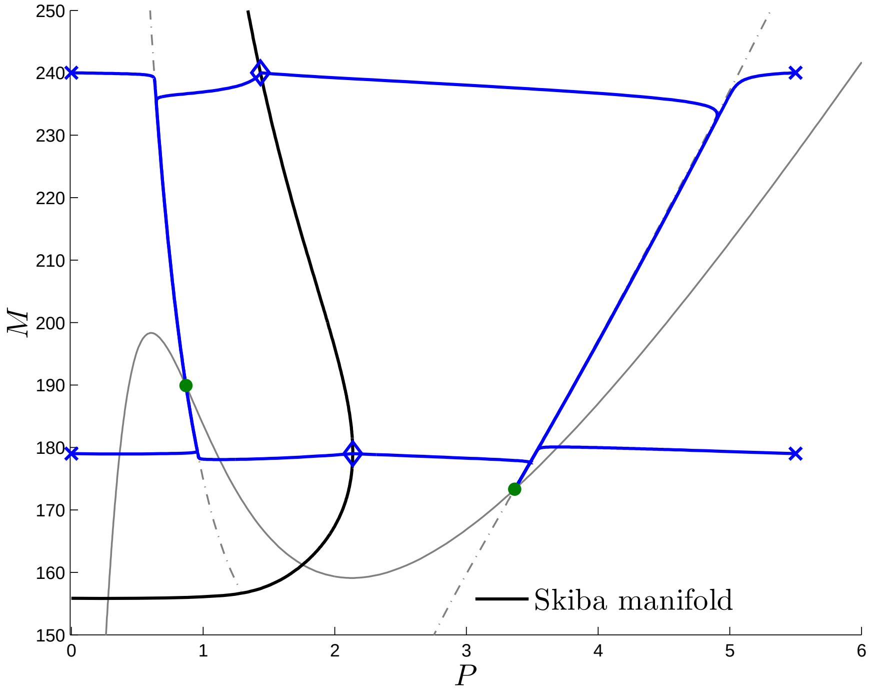


Figure 4: Phase portrait of the full-cooperative outcome for $c_M = 0.0868$ on the (P, M)-plane.

Figure 4 presents the projection of the full-cooperative outcome for the medium-value cost parameter $c_M = 0.0868$ on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M. This case has two possible long-run steady states, namely $(P^C_{M1}, M^C_{M1}) = (0.87017, 189.90)$ and $(P^C_{M2}, M^C_{M2}) = (3.3551, 173.09)$ (the green points), one in the oligotrophic area and one in the eutrophic area of the lake, with a traditional Skiba manifold separating the basins of attraction (the black curve) (the first subscript M refers to c_M). The green points are the intersection points of the isoclines $\dot{P} = 0$ (the dashed grey curves) and $\dot{M} = 0$ (the grey curve) under optimal loading L^*. The phosphorus density in the lake water becomes again somewhat lower than in the one-dimensional analysis of the lake. Starting in $(P_0, M_0) = (0, 179)$, there is first a relatively fast adjustment of P to approximately the oligotrophic steady-state value of P in the one-dimensional analysis of the lake, followed by a slow adjustment of M upwards and P downwards (along the optimal isocline $\dot{P} = 0$). Starting in $(P_0, M_0) = (5.5, 179)$, there is first a relatively fast adjustment of P to approximately the eutrophic steady-state value of P in the one-dimensional analysis of the lake, followed by a slow adjustment of M and P downwards (along the optimal isocline $\dot{P} = 0$). Furthermore, starting in either one of the diamonds on the Skiba manifold, there are two optimal trajectories, one towards the oligotrophic long-run steady state and one towards the eutrophic long-run steady state, with the same total discounted net benefits. Starting in $(P_0, M_0) = (0, 240)$ or in $(P_0, M_0) = (5.5, 240)$ basically yields the same story as for $M_0 = 179$. However, if the initial phosphorus density in the surface sediment M_0 is lower than 155.84, all the optimal trajectories converge to the eutrophic steady state $(P^C_{M2}, M^C_{M2}) = (3.3551, 173.09)$ (but see Figure 6 with explanations below).

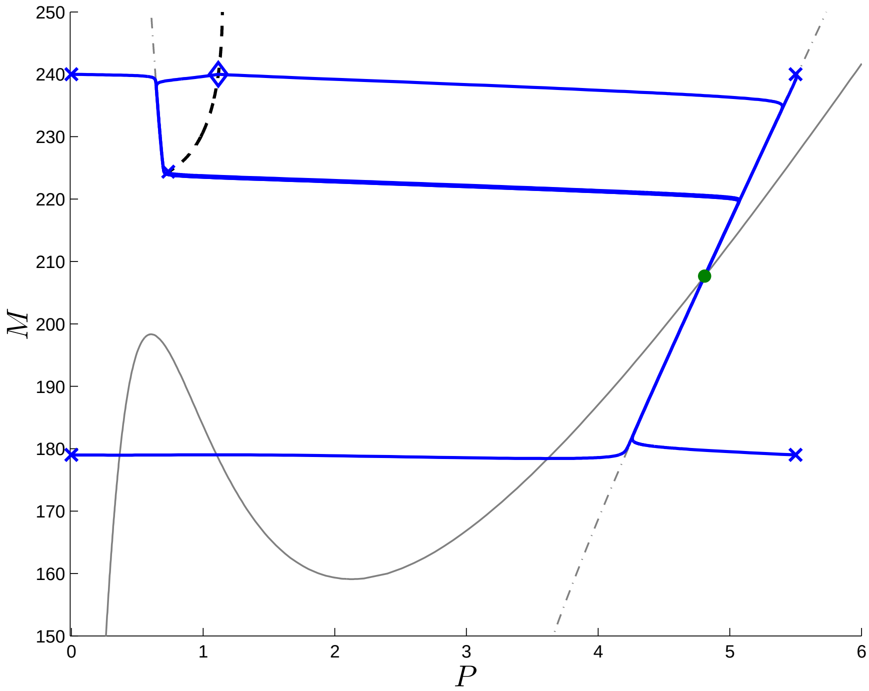


Figure 5: Phase portrait of the full-cooperative outcome for $c_L = 0.057867$ on the (P, M)-plane.

Figure 5 presents the projection of the full-cooperative outcome for the low-value cost parameter $c_L = 0.057867$ on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M. All the optimal trajectories converge to $(P_L^C, M_L^C) = (4.7957, 207.29)$ (the green point) in the eutrophic area of the lake (the subscript L refers to c_L). It is the intersection point of the isoclines $\dot{P} = 0$ (the dashed grey curves) and $\dot{M} = 0$ (the grey curve) under optimal loading L^*. Starting in $(P_0, M_0) = (0, 179)$ or in $(P_0, M_0) = (5.5, 179)$, there is first a relatively fast adjustment of P, followed by a slow adjustment of M upwards and P upwards (along the optimal isocline $\dot{P} = 0$). Starting in higher values of M_0, a manifold of a different type of Skiba points occurs again (the black dashed curve). Those weak Skiba points do not separate different basins of attraction towards different steady states but separate different optimal trajectories towards the same long-run steady state. More specifically, starting in the diamond on that weak Skiba manifold yields two optimal trajectories towards the steady state, with the same total discounted net benefits. The first trajectory moves into the eutrophic area of the lake immediately and then slowly down towards the long-run steady state (along the optimal isocline $\dot{P} = 0$). The other trajectory moves fast into the oligotrophic area of the lake first, then slowly down (along a part of the optimal isocline $\dot{P} = 0$), then fast again into the eutrophic area of the lake, and finally slowly down again towards the long-run steady state (along another part of the optimal isocline $\dot{P} = 0$). Furthermore, starting in $(P_0, M_0) = (0, 240)$ or in $(P_0, M_0) = (5.5, 240)$, the story is basically the same as above. Finally, starting at the end of the black Skiba manifold there is only one optimal trajectory towards the long-run steady state. Actually, starting anywhere below that point, there is again only one optimal trajectory.

The general picture is clear. The parameter c reflects the relative importance of the costs versus the benefits of phosphorus loadings. If c is high, it is optimal to end up in the oligotrophic area of the lake with a high level of ecological services and if c is low, it is optimal to end up in the eutrophic area of the lake with a low level of ecological services. For intermediate values of c, the outcome depends on the initial conditions of the lake, which implies that Skiba points occur where one is indifferent between ending up in the oligotrophic or in the eutrophic area of the lake. Figures 1-5 show three characteristic cases but it is interesting to consider the transition from one case to the other. Thus in figure 6 we describe the emergence of traditional and weak Skiba manifolds as the relative cost parameter changes.

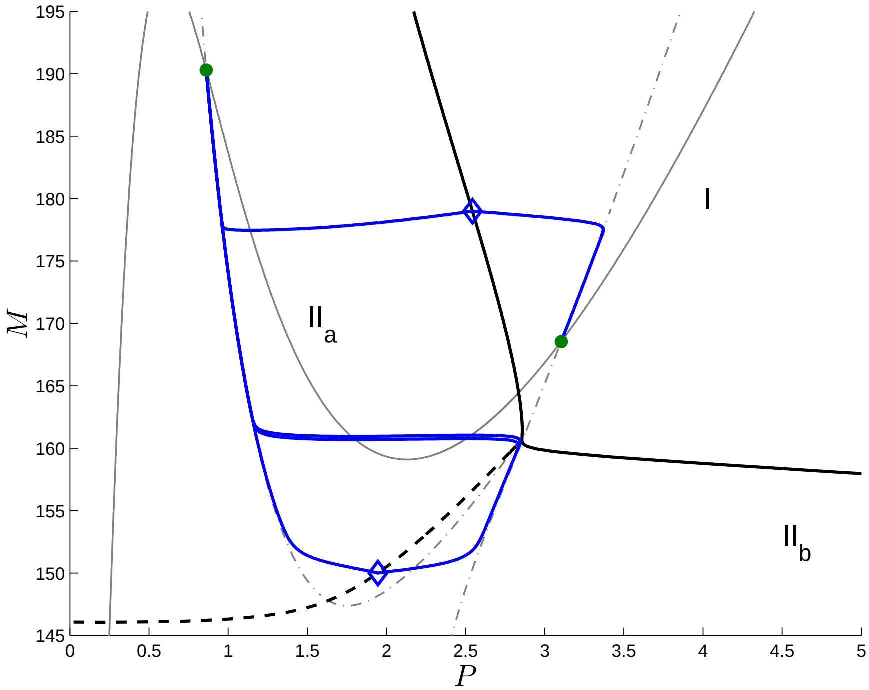


Figure 6: Phase portrait of the full-cooperative outcome for $c = 0.092152$ on the (P, M)-plane.

Figure 6 presents the projection of the full-cooperative outcome for $c = 0.092152$ on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M. This picture arises as follows. When we gradually decrease the value of the relative cost parameter c from $c_H = 0.1736$ to $c_M = 0.0868$, first the second possible long-run equilibrium point in the eutrophic area of the lake appears, with a traditional Skiba manifold separating the basins of attraction. This Skiba manifold (the black curve in Figure 6) bends to the right below, so that all the optimal trajectories for low initial values M_0 still converge to the oligotrophic steady state. Then a second type of Skiba manifold, a weak Skiba manifold, appears (dashed black curve, becomes black curve in Figure 4). These are points from where two equivalent optimal trajectories originate that converge to the same oligotrophic steady state. This leads to the situation depicted in Figure 6. Area I indicates the basin of attraction of the eutrophic long-run steady state and areas IIa and IIb indicate the basins of attraction of the oligotrophic long-run steady state, separated by the weak Skiba manifold. Decreasing the value of the relative cost parameter c just a little bit further, the outcome bifurcates to the situation depicted in Figure 4, with only the traditional Skiba manifold that is separating the basins of attraction. As we have seen, this Skiba manifold now bends to the left below, so that all the optimal trajectories for low initial values M_0 converge to the eutrophic steady state. When we gradually decrease the value of the relative cost parameter c from $c_M = 0.0868$ to $c_L = 0.057867$, at some point the long-run equilibrium in the oligotrophic area of the lake disappears and the outcome becomes as depicted in Figure 5. This is because with low c it is optimal to end up in the eutrophic area of the lake with a low level of ecological services and high benefits from loading.

5 Results for Open-Loop Nash Equilibria

It was shown in Section 3 that for the relative cost parameter $c_H = 0.1736$ and the number of communities $n = 2$ we get the same modified Hamiltonian system for the open-loop Nash equilibrium as for optimal management with $c_M = 0.0868$. This implies that we get the same two possible long-run steady states $(P^N_{H1}, M^N_{H1}) = (P^C_{M1}, M^C_{M1}) = (0.87017, 189.90)$ and $(P^N_{H2}, M^N_{H2}) = (P^C_{M2}, M^C_{M2}) = (3.3551, 173.09)$ as in Figure 2. However, the Nash equilibrium trajectories, driven by the individual welfare indicators (21), differ from the optimal trajectories.

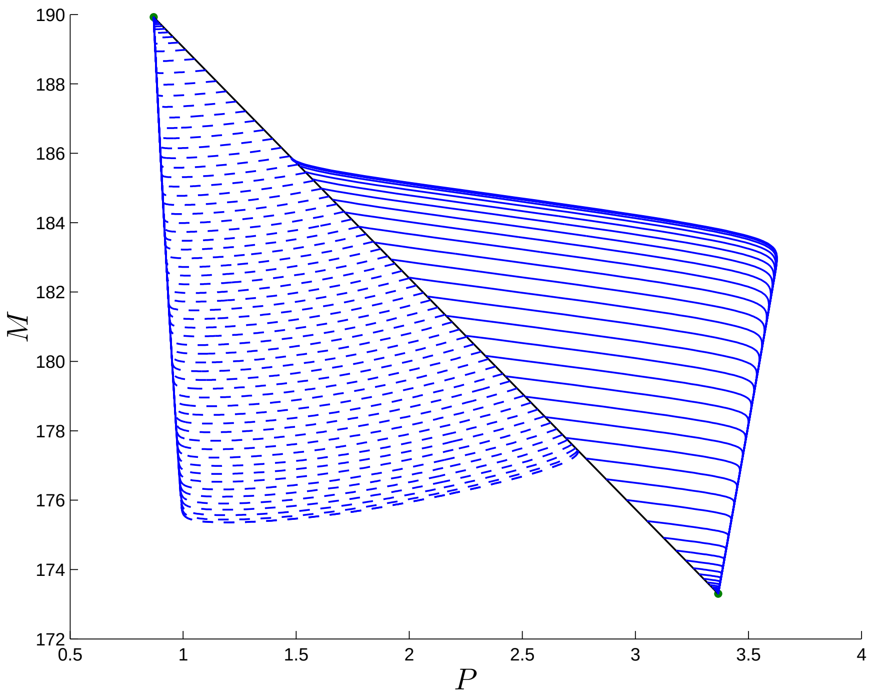


Figure 7: Projection of the Nash equilibrium trajectories on the (P, M)-plane.

Figure 7 presents the projection of the Nash equilibrium trajectories on the (P, M)-plane for the phosphorus densities in the lake water P and in the surface sediment M, starting in initial conditions on a line connecting the two long-run steady states. On the upper part of that line, only one Nash equilibrium exists, leading to the oligotrophic long-run steady-state (the dashed blue lines). On the lower part of that line, also only one Nash equilibrium exists, leading to the eutrophic long-run steady-state (the solid blue lines). However, in the middle part of that line, two Nash equilibria exist, one leading to the oligotrophic steady state and one leading to the eutrophic steady state. The patterns are the same as for the optimal trajectories in Section 4: first a fast adjustment of P, followed by a slow adjustment of M and P.

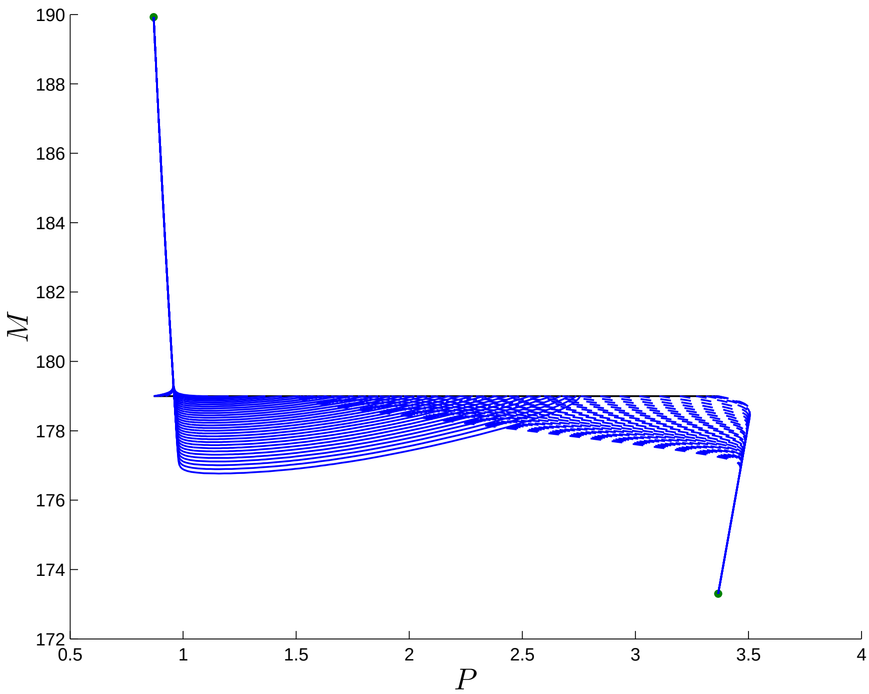


Figure 8: Projection of the Nash equilibrium trajectories on the (P, M)-plane for fixed initial state $M_0 = 179$.

Figure 8 is similar to Figure 7 but now for initial conditions with a fixed $M_0 = 179$. The initial points with two Nash equilibria in both Figure 7 and Figure 8 do not have to be Skiba indifference points. Although we focus on symmetric Nash equilibria, so that the welfare levels for the two communities are the same in each Nash equilibrium, these welfare levels may be different when we compare the two Nash equilibria.

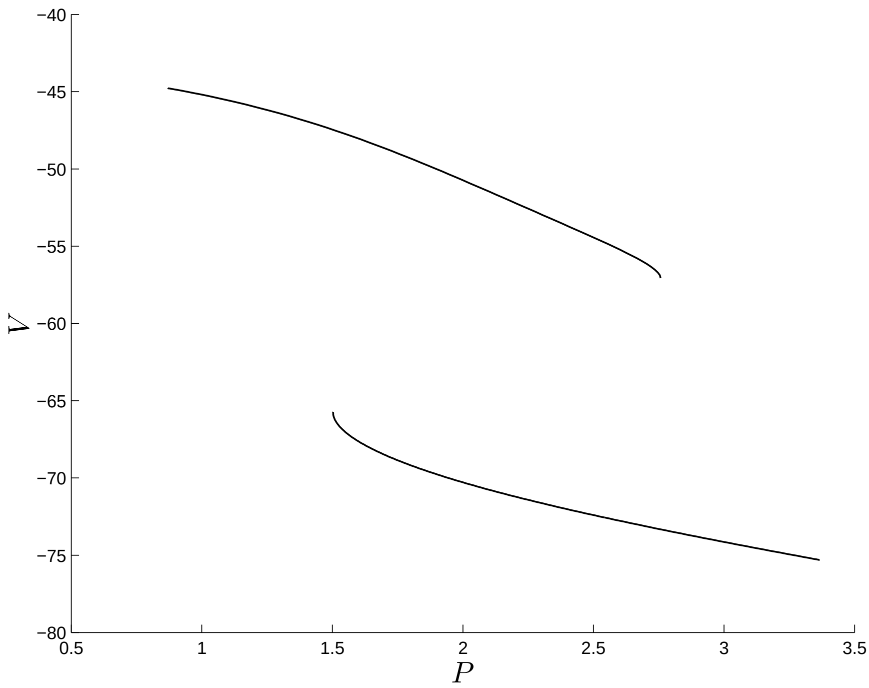
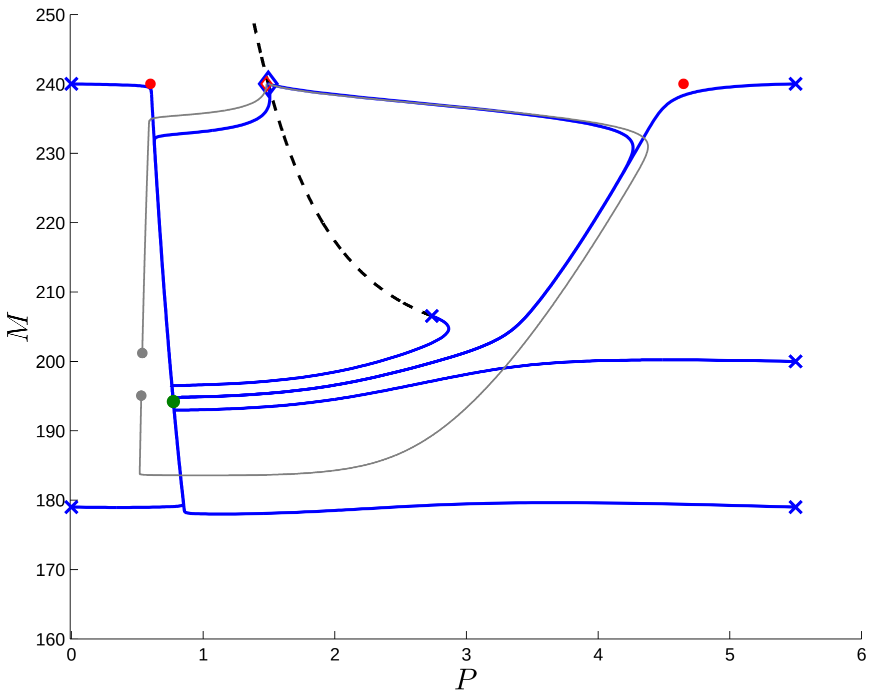


Figure 9: The discounted net benefits (21) for the Nash equilibrium trajectories in Figure 8 as a function of the initial condition P_0.

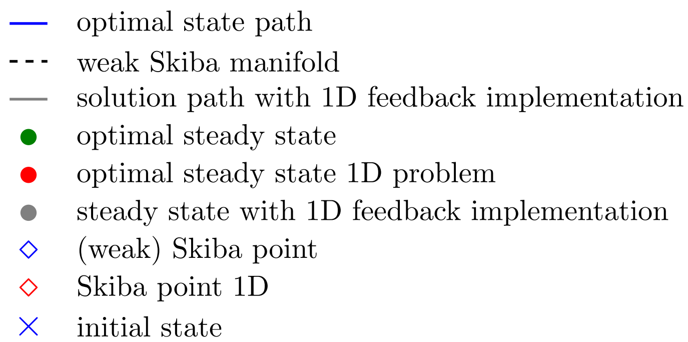
Figure 9 depicts the resulting discounted net benefits (21) for the Nash equilibrium trajectories in Figure 8 as a function of the initial condition P_0. It shows that when two Nash equilibria exist, the one leading to the oligotrophic steady state has higher welfare everywhere. We may argue that the two communities will try to coordinate on the best Nash equilibrium but then an interesting phenomenon occurs. When we increase the initial condition P_0 beyond the point where two Nash equilibria still exist, the communities cannot coordinate on this best Nash equilibrium anymore, because it does not exist anymore. They have to switch to the bad Nash equilibrium, leading to the eutrophic long-run steady state. The drop in welfare at this point is considerable (from -57.06 to -73.32, a 28.5% drop). This point can be viewed as the beginning of a pollution trap. Moving beyond this point implies that the good Nash equilibrium is not reachable any more and that the lake will end up in a eutrophic state with a considerable drop in welfare. This is because, due to open-loop strategies, users commit to their optimal path given the specific initial condition. If the initial condition is such that the oligotrophic steady state is not reachable, then users are committed to a path that traps them in the eutrophic region with a considerable drop in welfare.

6 Implications from Ignoring Slow Dynamics

In this section we return to optimal management but now we consider what happens if the slow dynamics of the phosphorus accumulation in the surface sediment M is ignored. We approach the problem in the following way. We use as a benchmark the solution of the two-dimensional problem under optimal management. Then we derive the optimal management solution when the slow dynamics are ignored and M is fixed at the initial value used for the solution of the two-dimensional model. Finally, we implement the optimal loading that is found with the one-dimensional representation of the lake in the two-dimensional lake system, and compare the result with the optimal management of the two-dimensional lake system. We implement the optimal loading derived from the one-dimensional model in the full model, L_{1D}^{*}, in two ways: (i) in a feedback form $L_{1D}^{*}(P)$, and (ii) in an open-loop form $L_{1D}^{*}(t)$. Our results are presented with the help of figures 10-12.



(a) Phase portrait



(b) Legend for the phase portraits with 1D feedback

Figure 10: Phase portrait of the full-cooperative outcome together with the implementation of the feedback control ignoring the slow variable M, for $c_H = 0.1736$ on the (P, M)-plane.

Figure 10 presents the optimal management of the two-dimensional lake system for relative cost parameter $c_H = 0.1736$. In that case there exists a unique globally stable steady state $(0.7742, 194.19)$ (green dot). The blue diamond indicates a weak Skiba point at $(1.4955, 240)$. Two paths emanate from this point, one which moves into the eutrophic basin and the other one which moves into the oligotrophic basin, but both meet eventually at the unique optimal steady state. Suppose now that a regulator who seeks to manage the lake ecosystem optimally ignores slow dynamics and considers M as a fixed parameter at 240. The optimal solution for this problem is characterized by a traditional Skiba point at $P = 1.4820$, denoted by the red diamond, and two steady states denoted by red dots at $P = 0.6005$, the oligotrophic steady state, and $P = 4.6486$, the eutrophic steady state. For initial $P < 1.4820$, a regulator who ignores slow dynamics is expected to attain the oligotrophic state while for $P > 1.4820$, a regulator who ignores slow dynamics is expected to attain the eutrophic state state. Assume that the regulator implements the optimal loading derived by the one-dimensional model as a feedback rule $L^*_{1D}(P)$, then the actual system that involves slow dynamics will have the following behavior. For initial $P < 1.4820$, the system will follow the gray path to the left of the red diamond and will converge to the steady state $(0.5378, 201.19)$. For initial $P > 1.4820$, the system will follow the gray path to the right of the red diamond and will converge to the steady state $(0.5305, 195.06)$. It is interesting to note that the paths of the full system under the loading obtained by the one-dimensional system are close to the optimal paths (blue paths), and that the corresponding steady states are close to the optimal steady state. Of course the regulator who ignores slow dynamics will be surprised especially if $P > 1.4820$, since the system will converge to a steady state which is in a different basin of attraction than the one expected when regulation was designed at the initial time.

Since the steady states to which the full system converges under the one-dimensional control are different from the optimal steady state, it is appropriate to examine the welfare impact from ignoring slow dynamics. Figure 11 shows the maximum welfare obtained under optimal management of the full model (red line) for different P values and initial $M_0 = 240$. The maximum welfare obtained when the loading from the one-dimensional system is implemented in the two-dimensional system and M is kept fixed at 240 is shown by the blue line for P <1.4820, which is to the left of the traditional Skiba point of the one-dimensional model, by the black line for $1.4820 < P$ <1.4955, which is the interval between the traditional and the weak Skiba points, and by the green line for $P > 1.4955$.

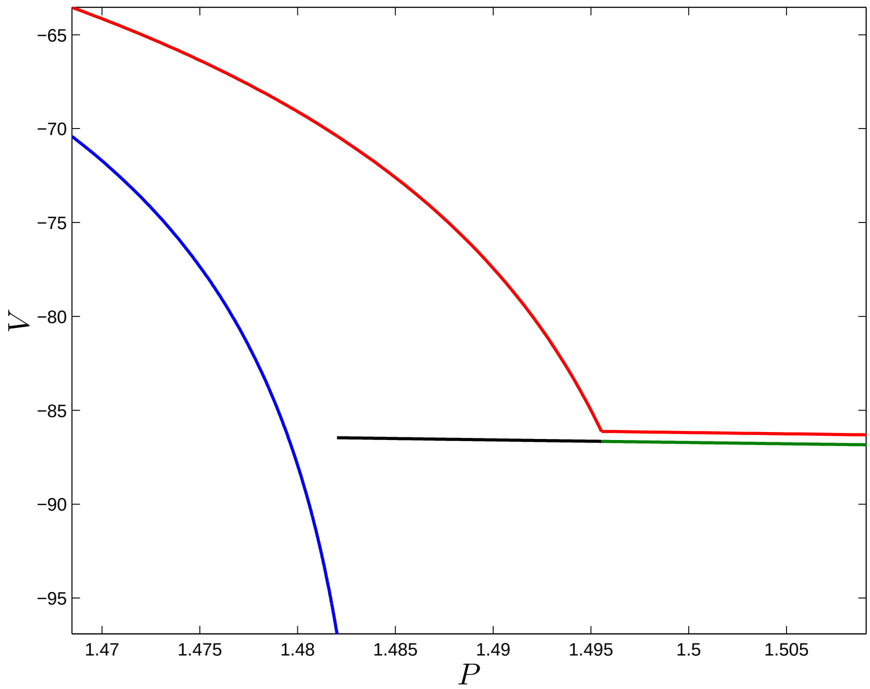


Figure 11: Welfare for the full model and the impact from ignoring slow dynamics.

The largest welfare loss from ignoring slow dynamics occurs when initial P is between the two Skiba points and closer to the traditional one. This is because in this case the one-dimensional control pushes the full system towards the eutrophic state (gray path to the right of the red diamond) before the path bends toward the steady state $(0.5305, 195.06)$, in Figure 10. Thus this path spends a lot of time away from the optimal basin of attraction which is the oligotrophic area. On the other hand, the optimal path for the full system brings that system fast towards the optimal steady state (the blue path to the left of the blue diamond in Figure 10). Thus the largest loss in welfare occurs when the one-dimensional control and the two-dimensional control push the system in different directions before the system eventually converges near the optimal steady state under the two-dimensional control. This happens when initial P is between the two Skiba points. When initial P is on the same side of the two Skiba points, that is, either P <1.4820 or $P > 1.4955$, the optimal path of the two-dimensional system and the path when the one-dimensional control is implemented are close to each other and thus welfare differences are small. We run a number of simulations with different initial values for M and the results are very similar.

When the open loop one-dimensional control, $L^*_{1D}(t)$, is implemented in the full system, the results are similar to the case where the feedback control is implemented. The main difference is that the path under the one-dimensional control that enters the oligotrophic basin converges eventually to a limit cycle near the optimal steady state, while welfare results are similar to those for the feedback control case.

We finally examine system behavior and welfare losses under one-dimensional dynamics, when the cost parameter takes the medium value $c_M = 0.0868$. This is the case where the one- and the two-dimensional systems exhibit traditional Skiba points under optimal control.

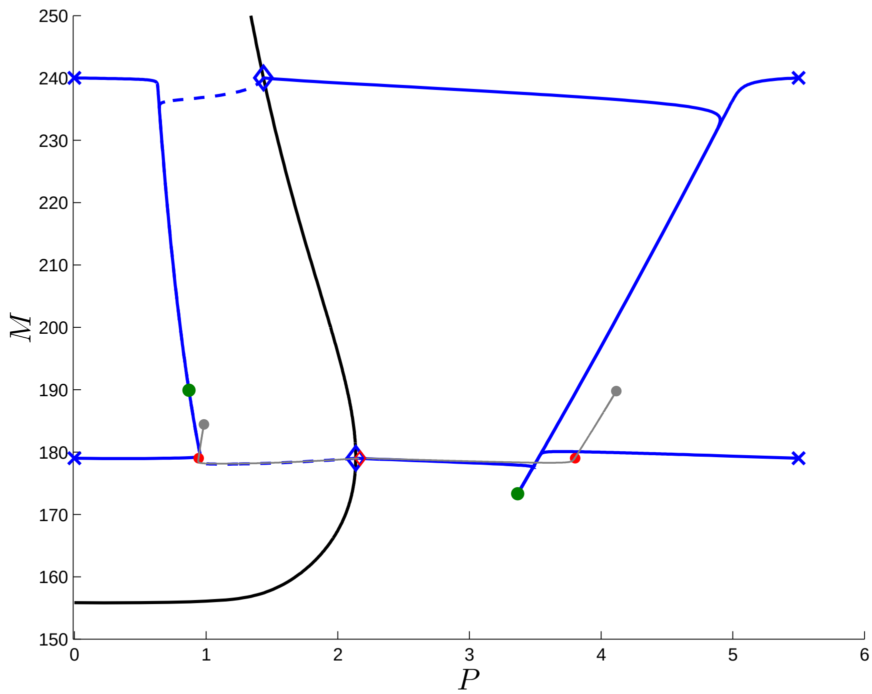


Figure 12: Phase portrait of the full-cooperative outcome together with the implementation of the feedback control ignoring the slow variable M, for $c_M = 0.0868$ on the (P, M)-plane.

Figure 12 presents the optimal management of the two-dimensional lake system for relative cost parameter $c_M = 0.0868$. In that case there exist two locally stable optimal steady states (green dots), the oligotrophic $(0.8697, 189.93)$ and the eutrophic $(3.3663, 173.30)$, which are separated by a traditional Skiba manifold. For example, the paths emanating from either side of one of these Skiba points at the lower blue diamond $(2.1364, 179)$ (blue paths) converge to the corresponding steady states, respectively. When the regulator ignores slow dynamics and considers M as a fixed parameter at 179, the optimal solution is characterized by a traditional Skiba point at $P = 2.1709$, denoted by the red diamond, and two steady states denoted by red dots at $P = 0.9432$, the oligotrophic steady state, and $P = 3.8024$, the eutrophic steady state. For initial P between the two Skiba points, $2.1364 < P < 2.1709$, the regulator who ignores slow dynamics expects to attain the oligotrophic state (red dot), since initial $P < 2.1709$. The full system, however, under feedback control, will converge to the gray dot $(0.9837, 184.45)$ near the oligotrophic optimal steady state. Furthermore, since initial $P > 2.1364$, the optimal steady state for the full system is the eutrophic steady state (green dot at $(3.3663, 173.30)$) and not the oligotrophic steady state at the green dot $(0.8697, 189.93)$.

Thus there may not only be surprises for the regulator who ignores slow dynamics, because the system may converge to a different steady state than the one expected at the beginning of the regulation, but it can also happen that under the one-dimensional control the system ends up in the basin of attraction which is not the optimal one. In our parametrization, welfare differences are not substantial, even when the system ends up in a sub-optimal steady state, because the Skiba points are close to each other. Furthermore, the response pattern of the full system to one-dimensional control is the same when we run simulations with different initial M values. Finally, implementation of the one-dimensional control in the open-loop form, $L_{1D}^{*}(t)$, instead of in the feedback form $L_{1D}^{*}(P)$, usually leads to a limit cycle near the optimal steady state, but the welfare differences are not substantial.

Summarizing, our results suggest that ignoring slow dynamics might produce surprises and even convergence to a sub-optimal steady state, depending on initial conditions and the structure of Skiba points in the one- and two-dimensional systems. For our parametrization, the welfare losses from ignoring slow dynamics are small in most of our simulations. However, surprises and associated welfare losses could be more important in other cases where slow dynamics are ignored. In this respect we think that our approach provides valuable insights into the management of fast/slow systems and the implications from ignoring slow dynamics. These implications are discussed in a deterministic set up. It will be undoubtedly interesting to add stochastics as well, but this is beyond the scope of this paper and is left for further research. However, we have done some sensitivity analysis on the parameters of the slow dynamics. Qualitatively, the results do not change.

7 Conclusion

The lake is an ecological system providing services that can be damaged by phosphorus loadings resulting from profitable human activities. This paper considers optimal management and Nash equilibria in the analysis of the full-lake model and compares the results with the previous literature that has focused on a simplified version of the model. The full model includes, besides the damaging accumulation of phosphorus in the water, the slow accumulation of phosphorus in the surface sediment describing the process of sedimentation of phosphorus and recycling back into the water.

The main pattern of the results stays intact. If a high value is attached to the ecological services as compared to the other profitable activities, optimal management will move the lake towards an oligotrophic state with a high level of ecological services. If a low value is attached to the ecological services, optimal management will move the lake towards a eutrophic state with a low level of ecological services. For intermediate values, traditional Skiba indifference points exist with optimal trajectories to either an oligotrophic or a eutrophic state. However, optimal management of the more complicated two-dimensional non-linear system gives rise to another type of Skiba points, which we have called weak Skiba points. Starting in these points, different optimal trajectories can be chosen, leading to the same long-run steady state, with the same level of welfare. Typically, these trajectories either move to the targeted area of the lake directly or move to the other area first and stay there for some time, before moving to the targeted area. The choice between these different optimal trajectories can be based on the characteristics of the tax trajectories that are required in a decentralized regulatory system.

Another interesting result was found in case of common property, with a number of communities who are damaging the lake but also using the common ecological services. Non-cooperative behavior is characterized with an open-loop Nash equilibrium of this differential game. An area of initial points exist with two possible Nash equilibria, one leading to an oligotrophic steady state and the other one leading to a eutrophic steady state. However, these initial points are not Skiba points because the welfare levels are different in the two Nash equilibria. In fact, the Nash equilibrium moving towards the oligotrophic state always dominates the Nash equilibrium moving towards the eutrophic state. It can be argued that the communities will coordinate on the good Nash equilibrium, but then a potential pollution trap exists where the communities have to switch to the bad Nash equilibrium. The good one does not exist anymore in the sense that it is not reachable from that specific set of initial conditions. The pollution trap implies a substantial drop in welfare.

An important question concerns the implications from implementing the optimal phosphorus loadings found with the simplified one-dimensional version of the lake model. In this case the regulator designs policy by ignoring slow dynamics but the actual ecosystem evolves under the dynamics of the two-dimensional model. Our results suggest that, depending on initial conditions and the structure of the Skiba points in relation to these initial conditions, we have two general conclusions. In the first, the two-dimensional system under one-dimensional regulation ends up near the steady state obtained under optimal two-dimensional regulation, without substantial welfare losses. In the second, one-dimensional regulation produces surprises in the sense that the two-dimensional system ends up in a basin of attraction different from the one expected by the regulator at the outset of the regulatory process, or even converges to a sub-optimal basin of attraction. In this second case, welfare losses can be substantial for certain sets of initial conditions, although in our parametrization this does not frequently occur.

The lake model is a metaphor for many of the environmental problems facing the world today. This paper provides insights and analytical tools for developing optimal management of the lake and characterizing common-property outcomes. These systems are complicated because they are nonlinear and because some parameters are slowly changing. This paper shows how these systems can be analyzed and managed properly.

8 Appendix A

For $b = 0$ the steady state of the system (1)-(3) is given by:

$$(\hat{P}, \hat{M}) = \left(\frac{\hat{L}}{h}, \frac{s\hat{L}}{rh}\left(1 + \frac{(hm)^q}{\hat{L}^q}\right)\right) \tag{34}$$

The Jacobian $\hat{J}$ becomes:

$$\hat{J} = \begin{bmatrix} -(s+h) + r\hat{M}f'(\hat{P}) & rf(\hat{P}) \\ s - r\hat{M}f'(\hat{P}) & -rf(\hat{P}) \end{bmatrix} \tag{35}$$

so that the determinant is given by:

$$\det \hat{J} = hrf(\hat{P}) = \frac{hr}{1 + \tilde{m}^q}, \tilde{m} = \frac{hm}{\hat{L}} \tag{36}$$

For $r = 0$ the determinant of the Jacobian $\hat{J}$ is zero, so that one of the eigenvalues of the Jacobian $\hat{J}$ is zero with eigenvector $[0, 1]^T$. This implies that for small values of r and large values of q one of the eigenvalues of the Jacobian $\hat{J}$ is close to zero with an eigenvector close to $[0, 1]^T$, so that the second variable M is a slow variable. This will still be the case for small values of b. Q.E.D..

9 Appendix B

The OCMat toolbox is especially designed to adequately handle discounted, autonomous, infinite-horizon optimal control problem in Matlab. Therefore information about the possible behavior of the solutions when time goes to infinity is used. This embraces the possibility of convergence to a steady state, limit-cycle, manifold of steady states and even divergent behavior. We explain the basic ideas for the simplest case, following Grass (2012). This is the convergence to a saddle-point steady state.

To guarantee that the solution path $X(\cdot)$ of the canonical system ends near the steady state a large enough time T is chosen. Therefore T is chosen depending on the slowest convergence rate at the steady state. Moreover it is requested that the path ends at the linear stable manifold as an approximation for the (nonlinear) stable manifold, which yields a linear boundary condition on $X(T)$. This condition is called the *asymptotic boundary condi-*

tion (Lentini and Keller, 1980). Together with the initial condition(s) this yields a well-defined boundary value problem.

To start the numerical procedure an approximate function $\tilde{X}(\cdot)$ has to be provided. Usually such a function is not at hand. Therefore the initial point is chosen such that a solution path is known. This can either be a solution from a previous calculation or, at the very beginning, the steady state. Subsequently, the solutions of the boundary value problem (BVP) for the initial state values P_0 and M_0 are computed by stepwise adjusting the initial states towards P_0 and M_0, starting from the known solution.

This procedure is called a continuation process. Different to the native Matlab solvers this continuation step is integrated in the BVP solver used by OCMat. Specifically OCMat uses the so-called *arclength continuation* or *Moore-Penrose continuation* (Kuznetsov, 1998). It is beyond the scope of this paper to explain this algorithm in more detail.

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