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On the Weak Convergence of the Extragradient Method for Solving Pseudo-Monotone Variational Inequalities*

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Abstract

In infinite-dimensional Hilbert spaces, we prove that the iterative sequence generated by the extragradient method for solving pseudo-monotone variational inequalities converges weakly to a solution. A class of pseudo-monotone variational inequalities is considered to illustrate the convergent behaviour. The result obtained in this note extends some recent results in the literature; especially, it gives a positive answer to a question raised in *Acta. Math. Vietnam, **41**, 251–263 (2016).*

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1 Introduction

Variational inequalities serve as a powerful mathematical model, which unifies important concepts in applied mathematics like systems of nonlinear equations, necessary optimality conditions for optimization problems, complementarity problems, obstacle problems, or network equilibrium problems [1]. Therefore, this model has numerous applications in the fields of engineering, mathematical programming, network economics, transportation research, game theory, and regional sciences [2].

Several techniques for the solution of a variational inequality (VI) in finite-dimensional spaces have been suggested such as projection method, extragradient method, Tikhonov regularization method and proximal point method; see, e.g., [1]. Typically, for guaranteeing

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the convergence to a solution of the VI, some kinds of monotonicity of the assigned mapping is required. In case of gradient maps, generalized monotonicity characterizes generalized convexity of the underlying function [3]. The well known gradient projection method can be successfully applied for solving strongly monotone VIs and inverse strongly monotone VIs [1, 4]. In practice, these assumptions are rather strong. The Tikhonov regularization and proximal point methods can serve as an efficient solution method for solving monotone VIs. For pseudo-monotone VIs, however, it may happen that every regularized problem generated by the Tikhonov regularization (resp. every problem generated by the proximal point method) is not pseudo-monotone [5]. This implies that the regularization procedures performed in Tikhonov regularization and proximal point methods may destroy completely the given pseudo-monotone structure of the original problem and can make auxiliary problems more difficult to solve than the original one.

To overcome this drawback, Korpelevich introduced the extragradient method [6]. In the original paper, this method was applied for solving monotone VIs in finite-dimensional spaces. It is a known fact [1, Theorem 12.2.11] that the extragradient method can be successfully applied for solving pseudo-monotone VIs. Because of its importance, extragradient-type methods have been widely studied and generalized [1].

Recently, the extragradient method has been considered for solving VIs in infinite-dimensional Hilbert spaces [7, 8, 9]. Providing that the VI has solutions and the assigned mapping is monotone and Lipschitz continuous, it is proved that the iterative sequence generated by the extragradient method converges weakly to a solution. However, as stated in [9, Section 6, Q2], it is not clear if the weak convergence is still available when monotonicity is replaced by pseudo-monotonicity. The aim of this paper is to give a positive answer to this question. As a consequence, the scope of the related optimization problems can be enlarged from convex optimization problems to pseudoconvex optimization problems. This guarantees the advantage of extragradient method in comparing with the other solution methods.

The paper is organized as follows: We first recall some basic definitions and results in Sec. 2. The weak convergence of the extragradient method for solving pseudo-monotone, Lipschitz continuous VIs is discussed in Sec. 3. An example is presented in Sec. 4 to illustrate the behavior of the extragradient method. We conclude the note with some final remarks in Sec. 5.

2 Preliminaries

Let H be real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|.\|$ and K be a nonempty, closed and convex subset of H. For each $u \in H$, there exists a unique point in K (see [2, p. 8]), denoted by $P_K(u)$, such that

$$\|u - P_K(u)\| \leq \|u - v\| \quad \forall v \in K.$$

It is well known [2, 10] that the projection operator can be characterized by

$$\langle u - P_K(u), v - P_K(u) \rangle \leq 0 \quad \forall v \in K. \tag{2.1}$$

Let $F : H \to H$ be a mapping. The variational inequality $\mathrm{VI}(K, F)$ defined by K and F consists in finding a point $u^* \in K$ such that

$$\langle F(u^*), u - u^* \rangle \geq 0 \quad \forall u \in K. \tag{2.2}$$

The solution set of $\mathrm{VI}(K, F)$ is abbreviated to $\mathrm{Sol}(K, F)$.

Remark 2.1 *$u^* \in \mathrm{Sol}(K, F)$ if and only if $u^* = P_K(u^* - \lambda F(u^*))$ for all $\lambda > 0$.*

We recall some concepts which are useful in the sequel.

Definition 2.2 *The mapping $F : H \to H$ is said to be*

(a) *pseudo-monotone if*

$$\langle F(u), v - u \rangle \geq 0 \;\Rightarrow\; \langle F(v), v - u \rangle \geq 0 \quad \forall u, v \in H;$$

(b) *monotone if*

$$\langle F(u) - F(v), u - v \rangle \geq 0 \quad \forall u, v \in H;$$

(c) *Lipschitz continuous if there exists $L > 0$ such that*

$$\|F(u) - F(v)\| \leq L\|u - v\| \quad \forall u, v \in H;$$

(d) *sequentially weakly continuous if for each sequence $\{u^n\}$ we have: $\{u^n\}$ converges weakly to u implies $\{F(u^n)\}$ converges weakly to $F(u)$.*

Remark 2.3 *It is clear that monotonicity implies pseudo-monotonicity. However, the converse does not hold. For example the mapping $F :]0, +\infty[\to]0, +\infty[$, defined by $F(u) = \frac{a}{a+u}$ with $a > 0$ is pseudo-monotone but not monotone.*

We recall a result which is called Minty lemma [11, Lemma 2.1].

Proposition 2.4 *Consider the problem* $\mathrm{VI}(K,F)$ *with* K *being a nonempty, closed, convex subset of a real Hilbert space* H *and* $F : K \to H$ *being pseudo-monotone and continuous. Then,* u^* *is a solution of* $\mathrm{VI}(K,F)$ *if and only if*

$$\langle F(u), u - u^* \rangle \geq 0 \quad \forall u \in K.$$

3 Weak Convergence of the Extragradient Method

In this section, we consider the problem $\mathrm{VI}(K,F)$ with K being nonempty, closed, convex and F being pseudo-monotone on H and Lipschitz continuous with modulus $L > 0$ on K. We also assume that the solution set $\mathrm{Sol}(K,F)$ is nonempty.

Extragradient Algorithm

Data: $u^0 \in K$ and $\{\lambda_k\} \in [a,b]$, where $0 < a \leq b < 1/L$.

Step 0: Set $k = 0$.

Step 1: If $u^k = P_K(u^k - \lambda_k F(u^k))$ then stop.

Step 2: Otherwise, set

$$\bar{u}^k = P_K(u^k - \lambda_k F(u^k)),$$
$$u^{k+1} = P_K(u^k - \lambda_k F(\bar{u}^k)).$$

Replace k by $k+1$; go to **Step 1**.

Remark 3.1 *If at some iteration we have* $F(u^k) = 0$*, then* $u^k = P_K(u^k - \lambda_k F(u^k))$ *and the Extragradient Algorithm terminates at step* k *with a solution* u^k*. From now on, we assume that* $F(u^k) \neq 0$ *for all* k *and the Extragradient Algorithm generates an infinite sequence.*

We recall an important property of the iterative sequence $\{u^k\}$ generated by the Extragradient Algorithm; see, e.g., [9, 6].

Proposition 3.2 *Assume that* F *is pseudo-monotone and* L*-Lipschitz continuous on* K *and Sol(*K,F*) is nonempty. Let* u^* *be a solution of* $\mathrm{VI}(K,F)$*. Then, for every* $k \in \mathbb{N}$*, we have*

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \left(1 - \lambda_k^2 L^2\right) \|u^k - \bar{u}^k\|^2. \tag{3.1}$$

We are now in the position to establish the main result of this note. The following theorem states that the sequence $\{u^k\}$ converges weakly to a solution of $\mathrm{VI}(K,F)$. This result extends the Extragradient Algorithm for solving monotone VIs [7, 9] to pseudo-monotone VIs.

Theorem 3.3 *Assume that F is pseudo-monotone on H, sequentially weakly continuous and L-Lipschitz continuous on K. Assume also that Sol(K,F) is nonempty. Then the sequence $\{u^k\}$ generated by the Extragradient Algorithm converges weakly to a solution of* $\mathrm{VI}(K,F)$.

Proof Since $0 < a \le \lambda_k \le b < 1/L$, it holds that

$$0 < 1 - b^2L^2 \le 1 - \lambda_k^2 L^2 \le 1 - a^2L^2 < 1.$$

Therefore, from Proposition 3.2, the sequence $\{u^k\}$ is bounded and

$$\lim_{k\to\infty} \|u^k - \bar{u}^k\| = 0.$$

Since F is Lipschitz continuous on K we have

$$\|F(u^k) - F(\bar{u}^k)\| \le L\|u^k - \bar{u}^k\|.$$

Hence

$$\lim_{k\to\infty} \|F(u^k) - F(\bar{u}^k)\| = 0.$$

As $\{u^k\}$ is a bounded sequence in a Hilbert space, there exists a subsequence $\{u^{k_i}\}$ of $\{u^k\}$ converging weakly to an element $\hat{u} \in K$. Since $\lim_{k\to\infty} \|u^k - \bar{u}^k\| = 0$, $\{\bar{u}^{k_i}\}$ also converges weakly to $\hat{u}$. We will prove that $\hat{u} \in$ Sol(K,F). Indeed, since

$$\bar{u}^k = P_K(u^k - \lambda_k F(u^k)),$$

by the projection characterization (2.1), it holds

$$\left\langle u^{k_i} - \lambda_{k_i} F(u^{k_i}) - \bar{u}^{k_i}, v - \bar{u}^{k_i} \right\rangle \le 0 \quad \forall v \in K,$$

or equivalently,

$$\frac{1}{\lambda_{k_i}} \left\langle u^{k_i} - \bar{u}^{k_i}, v - \bar{u}^{k_i} \right\rangle \le \left\langle F(u^{k_i}), v - \bar{u}^{k_i} \right\rangle \quad \forall v \in K.$$

This implies that

$$\frac{1}{\lambda_{k_i}} \left\langle u^{k_i} - \bar{u}^{k_i}, v - \bar{u}^{k_i} \right\rangle + \left\langle F(u^{k_i}), \bar{u}^{k_i} - u^{k_i} \right\rangle \le \left\langle F(u^{k_i}), v - u^{k_i} \right\rangle \quad \forall v \in K. \tag{3.2}$$

Fixing $v \in K$ and letting $i \to +\infty$ in the last inequality, remembering that $\lim_{k\to\infty} \|u^k - \bar{u}^k\| = 0$ and $\lambda_k \in [a, b] \subset]0, 1/L[$ for all k, we have

$$\liminf_{i\to\infty} \left\langle F(u^{k_i}), v - u^{k_i} \right\rangle \geq 0. \tag{3.3}$$

Now we choose a sequence $\{\epsilon_i\}_i$ of positive numbers decreasing and tending to 0. For each ϵ_i, we denote by n_i the smallest positive integer such that

$$\left\langle F(u^{k_j}), v - u^{k_j} \right\rangle + \epsilon_i \geq 0 \quad \forall j \geq n_i, \tag{3.4}$$

where the existence of n_i follows from (3.3). Since $\{\epsilon_i\}$ is decreasing, it is easy to see that the sequence $\{n_i\}$ is increasing. Furthermore, for each i, $F(u^{k_{n_i}}) \neq 0$ and, setting

$$v^{k_{n_i}} = \frac{F(u^{k_{n_i}})}{\|F(u^{k_{n_i}})\|^2},$$

we have $\left\langle F(u^{k_{n_i}}), v^{k_{n_i}} \right\rangle = 1$ for each i. Now we can deduce from (3.4) that for each i

$$\left\langle F(u^{k_{n_i}}), v + \epsilon_i v^{k_{n_i}} - u^{k_{n_i}} \right\rangle \geq 0,$$

and, since F is pseudo-monotone, that

$$\left\langle F(v + \epsilon_i v^{k_{n_i}}), v + \epsilon_i v^{k_{n_i}} - u^{k_{n_i}} \right\rangle \geq 0. \tag{3.5}$$

On the other hand, we have that $\left\{u^{k_i}\right\}$ converges weakly to $\hat{u}$ when $i \to \infty$. Since F is sequentially weakly continuous on K, $\left\{F(u^{k_i})\right\}$ converges weakly to $F(\hat{u})$. We can suppose that $F(\hat{u}) \neq 0$ (otherwise, $\hat{u}$ is a solution). Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$\|F(\hat{u})\| \leq \liminf_{i\to\infty} \|F(u^{k_i})\|.$$

Since $\left\{u^{k_{n_i}}\right\} \subset \left\{u^{k_i}\right\}$ and $\epsilon_i \to 0$ as $i \to 0$, we obtain

$$0 \leq \lim_{i\to\infty} \|\epsilon_i v^{k_{n_i}}\| = \lim_{i\to\infty} \frac{\epsilon_i}{\|F(u^{k_{n_i}})\|} \leq \frac{0}{\|F(\hat{u})\|} = 0.$$

Hence, taking the limit as $i \to \infty$ in (3.5), we obtain

$$\langle F(v), v - \hat{u} \rangle \geq 0.$$

It follows from Proposition 2.4 that $\hat{u} \in$ Sol(K,F).

Finally, we prove that the sequence $\{u^k\}$ converges weakly to $\hat{u}$. To do this, it is sufficient to show that $\{u^k\}$ cannot have two distinct weak sequential cluster points in Sol(K, F). Let

$\{u^{k_j}\}$ be another subsequence of $\{u^k\}$ converging weakly to $\bar{u}$. We have to prove that $\hat{u} = \bar{u}$. As it has been proven above, $\bar{u} \in$ Sol(K,F). From Proposition 3.2, the sequences $\{\|u^k - \hat{u}\|\}$ and $\{\|u^k - \bar{u}\|\}$ are monotonically decreasing and therefore converge. Since for all $k \in \mathbb{N}$,

$$2\left\langle u^k, \bar{u} - \hat{u}\right\rangle = \|u^k - \hat{u}\|^2 - \|u^k - \bar{u}\|^2 + \|\bar{u}\|^2 - \|\hat{u}\|^2,$$

we deduce that the sequence $\{\left\langle u^k, \bar{u} - \hat{u}\right\rangle\}$ also converges. Setting

$$l = \lim_{k\to\infty} \left\langle u^k, \bar{u} - \hat{u}\right\rangle,$$

and passing to the limit along $\{u^{k_i}\}$ and $\{u^{k_j}\}$ yields, respectively,

$$l = \langle \hat{u}, \bar{u} - \hat{u}\rangle = \langle \bar{u}, \bar{u} - \hat{u}\rangle.$$

This implies that $\|\hat{u} - \bar{u}\|^2 = 0$ and therefore $\hat{u} = \bar{u}$.

Remark 3.4 *The author in [13] studied the extragradient method for solving strongly pseudo-monotone variational inequalities with the following choice of stepsizes:*

$$\sum_{k=0}^{\infty} \lambda_k = \infty, \quad \lim_{k\to\infty} \lambda_k = 0.$$

It was proved that the iterative sequence generated by the extragradient method converges strongly to a solution. By considering an example [13, Example 4.2], the author stated that the condition $\lim_{k\to\infty} \lambda_k = 0$ *cannot be omitted. We have shown that if this condition is violated then the strong convergence reduces to the weak convergence.*

It is also worth stressing that, the basic extragradient method can serve as an adequate solution method for solving pseudo-monotone VIs, which was not guaranteed by the method studied in [13].

Remark 3.5 *If we replace the Lipschitz continuity of* F *on* K *by its Lipschitz continuity on the whole space* H*, then the conclusion in Theorem 3.3 still holds for the subgradient extragradient method [7]. Indeed, a careful reviewing shows that Lemma 5.2 in [7] is also guaranteed for pseudo-monotone mappings instead of monotone ones (see also [12]). The conclusion can be obtained by using a similar technique as in Theorem 3.3.*

Remark 3.6 *When the function* F *is monotone, it is not necessary to impose the sequential weak continuity on* F*. Indeed, in that case, it follows from* (3.2) *and the monotonicity of* F *that*

$$\begin{aligned} \frac{1}{\lambda_{k_i}}\left\langle u^{k_i} - \bar{u}^{k_i}, v - \bar{u}^{k_i}\right\rangle + \left\langle F(u^{k_i}), \bar{u}^{k_i} - u^{k_i}\right\rangle &\le \left\langle F(u^{k_i}), v - u^{k_i}\right\rangle \\ &\le \left\langle F(v), v - u^{k_i}\right\rangle \quad \forall v \in K. \end{aligned}$$

Letting $i \to +\infty$ *in the last inequality, remembering that* $\lim_{k\to\infty} \|u^k - \bar{u}^k\| = 0$ *and* $\lambda_k \in [a,b] \subset]0, 1/L[$ *for all* k*, we have*

$$\langle F(v), v - \hat{u}\rangle \ge 0 \quad \forall v \in K.$$

4 An Illustrative Example

In this section, we present an example to illustrate the main results obtained in Section 3. Another example can be found in [9, Example 5.2], where the mapping F is monotone and Lipchitz continuous. The following example is considered in [13], where the mapping F is pseudo-monotone but not monotone.

Let $H = \ell_2$, the real Hilbert space, whose elements are the square-summable sequences of real numbers, i.e., $H = \{u = (u_1, u_2, \ldots, u_i, \ldots) : \sum_{i=1}^{\infty} |u_i|^2 < +\infty\}$. The inner product and the norm on H are given by setting

$$\langle u, v\rangle = \sum_{i=1}^{\infty} u_i v_i \quad \text{and} \quad \|u\| = \sqrt{\langle u, u\rangle}$$

for any $u = (u_1, u_2, \ldots, u_i, \ldots), v = (v_1, v_2, \ldots, v_i, \ldots) \in H$.
Let $\alpha, \beta \in \mathbb{R}$ be such that $\beta > \alpha > \dfrac{\beta}{2} > 0$. Put

$$K_\alpha = \{u \in H : \|u\| \le \alpha\}, \quad F_\beta(u) = (\beta - \|u\|)\, u,$$

where α and β are parameters. It is easy to verify that F_β is sequentially weakly continuous on H and $\mathrm{Sol}(K_\alpha, F_\beta) = \{0\}$. Note that F_β is Lipschitz continuous and pseudo-monotone on K_α. Indeed, for any $u, v \in K_\alpha$,

$$\begin{aligned}
\|F_\beta(u) - F_\beta(v)\| &= \|(\beta - \|u\|)u - (\beta - \|v\|)v\| \\
&= \|\beta(u - v) - \|u\|(u - v) - (\|u\| - \|v\|)v\| \\
&\le \beta\|u - v\| + \|u\|\|u - v\| + |\|u\| - \|v\||\|v\| \\
&\le \beta\|u - v\| + \alpha\|u - v\| + \|u - v\|\alpha \\
&= (\beta + 2\alpha)\|u - v\|.
\end{aligned}$$

Hence, F_β is Lipschitz continuous on K_α with the Lipschitz constant $L := \beta + 2\alpha$. Let $u, v \in K_\alpha$ be such that $\langle F_\beta(u), v - u\rangle \ge 0$. Then

$$(\beta - \|u\|)\langle u, v - u\rangle \ge 0.$$

Since $\|u\| \le \alpha < \beta$, we have $\langle u, v - u\rangle \ge 0$. Hence,

$$\begin{aligned}
\langle F_\beta(v), v - u\rangle &= (\beta - \|v\|)\langle v, v - u\rangle \\
&\ge (\beta - \|v\|)(\langle v, v - u\rangle - \langle u, v - u\rangle) \\
&\ge (\beta - \alpha)\|u - v\|^2 \ge 0.
\end{aligned}$$

We have thus shown that F_β is pseudo-monotone on K_α. It is worthy to stress that F_β is not monotone on K_α. To see this, it suffices to choose $u = (\frac{\beta}{2}, 0, \ldots, 0, \ldots), v = (\alpha, 0, \ldots, 0, \ldots) \in K_\alpha$ and note that

$$\langle F_\beta(u) - F_\beta(v), u - v\rangle = \left(\frac{\beta}{2} - \alpha\right)^3 < 0.$$

We now apply the extragradient algorithm for solving the variational inequality VI(K_α, F_β). We choose $\lambda_k = \lambda \in \left]0, \frac{1}{L}\right[= \left]0, \frac{1}{\beta+2\alpha}\right[$ and we take starting point as any $u^0 \in K_\alpha$. The projection onto K_α is explicitly calculated as

$$P_{K_\alpha} u = \begin{cases} u, & \text{if} \quad \|u\| \le \alpha, \\ \dfrac{\alpha u}{\|u\|}, & \text{otherwise}. \end{cases} \tag{4.1}$$

Since for all k,

$$0 < \lambda < \frac{1}{\beta + 2\alpha} < \frac{1}{\beta - \|u^k\|},$$

then we have

$$\|u^k - \lambda F_\beta(u^k)\| = \left(1 - \lambda\left(\beta - \|u^k\|\right)\right)\|u^k\| \le \|u^k\| \le \alpha.$$

Therefore,

$$\bar{u}^k = P_{K_\alpha}(u^k - \lambda_k F_\beta(u^k)) = \left(1 - \lambda\left(\beta - \|u^k\|\right)\right) u^k.$$

Similarly, we can deduce that

$$\|u^k - \lambda_k F_\beta(\bar{u}^k)\| \le \alpha.$$

Indeed, we have

$$u^k - \lambda_k F_\beta(\bar{u}^k) = u^k - \lambda\left(\beta - \|\bar{u}^k\|\right)\left(1 - \lambda\left(\beta - \|u^k\|\right)\right) u^k.$$

Since

$$\begin{aligned} 1 - \lambda\left(\beta - \|\bar{u}^k\|\right)\left(1 - \lambda\left(\beta - \|u^k\|\right)\right) &= 1 - \lambda\left(\beta - \|\bar{u}^k\|\right) + \lambda^2\left(\beta - \|\bar{u}^k\|\right)\left(\beta - \|u^k\|\right) \\ &\ge 1 - \lambda\left(\beta - \|\bar{u}^k\|\right) > 0, \end{aligned} \tag{4.2}$$

we can write

$$\|u^k - \lambda_k F_\beta(\bar{u}^k)\| = \left[1 - \lambda\left(\beta - \|\bar{u}^k\|\right)\left(1 - \lambda\left(\beta - \|u^k\|\right)\right)\right]\|u^k\| \le \|u^k\| \le \alpha.$$

This and (4.2) imply that

$$\begin{aligned} \|u^{k+1}\| &= \|P_{K_\alpha}(u^k - \lambda_k F_\beta(\bar{u}^k))\| \\ &= \|u^k - \lambda\left(\beta - \|\bar{u}^k\|\right)\bar{u}^k\| \\ &= \left[1 - \lambda\left(\beta - \|\bar{u}^k\|\right)\left(1 - \lambda\left(\beta - \|u^k\|\right)\right)\right]\|u^k\|. \end{aligned} \tag{4.3}$$

We have

$$\begin{aligned}
\lambda\left(\beta-\|\bar{u}^k\|\right)\left(1-\lambda\left(\beta-\|u^k\|\right)\right) &= \lambda\left(\beta-\|\bar{u}^k\|\right)\left(1-\lambda\beta+\lambda\|u^k\|\right) \\
&\geq \lambda\left(\beta-\|\bar{u}^k\|\right)(1-\lambda\beta) \qquad (4.4)\\
&= \lambda\left(\beta-(1-\lambda\beta)\|u^k\|-\lambda\|u^k\|^2\right)(1-\lambda\beta)\,.
\end{aligned}$$

Considering the function $f(x) := \beta-(1-\lambda\beta)\,x-\lambda x^2$ with $x\in[0,\alpha]$, it is easy to see that f is decreasing on $[0,\alpha]$. Therefore, the minimal value of f is

$$\beta-(1-\lambda\beta)\,\alpha-\lambda\alpha^2,$$

which is attained at $x=\alpha$. Combining this with (4.4) and (4.3) yields

$$\begin{aligned}
\|u^{k+1}\| &\leq \left(1-\lambda\left(\beta-(1-\lambda\beta)\,\alpha-\lambda\alpha^2\right)(1-\lambda\beta)\right)\|u^k\| \\
&= \left(1-\left(\lambda\beta-\lambda\alpha+\lambda^2\alpha\beta-\lambda^2\alpha^2\right)(1-\lambda\beta)\right)\|u^k\| \\
&= \left[1-(\beta-\alpha)\,\lambda\,(1+\alpha\lambda)\,(1-\lambda\beta)\right]\|u^k\|. \qquad (4.5)
\end{aligned}$$

We claim that

$$q := (\beta-\alpha)\,\lambda\,(1+\alpha\lambda)\,(1-\lambda\beta)\in]0,1[.$$

Indeed, since $\alpha<\beta$ and $0<\lambda<\frac{1}{\beta+2\alpha}$, we have $q>0$. To verify that $q<1$, it is sufficient to show that $(\beta-\alpha)\,\lambda\,(1+\alpha\lambda)<1$. Since $\beta/2<\alpha<\beta$ and $0<\lambda<\frac{1}{\beta+2\alpha}$ we have

$$\begin{aligned}
(\beta-\alpha)\,\lambda\,(1+\alpha\lambda) &< (\beta-\alpha)\,\frac{1}{\beta+2\alpha}\left(1+\frac{\alpha}{\beta+2\alpha}\right) \\
&< \frac{\beta}{2}\frac{1}{\beta+\beta}\left(1+\frac{\beta}{\beta+\beta}\right)=\frac{3}{8}.
\end{aligned}$$

This implies that $q\in]0,1[$ and we can deduce from (4.5) that

$$\|u^k\|\leq(1-q)^k\,\|u^0\|,$$

for all $k\in\mathbb{N}$. This means that the sequence $\{u^k\}$ converges strongly to 0, the unique solution of $\mathrm{VI}(K_\alpha,F_\beta)$.

5 Conclusions

We have considered the extragradient method for solving infinite-dimensional variational inequalities with a pseudo-monotone and Lipschitz continuous mapping. We have shown that the iterative sequence generated by the extragradient method converges weakly to a solution of the considered variational inequality, provided that such a solution exists. The strong convergence of the iteration sequence is still an open question that could be an interesting topic for a future research.

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