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Approximating optimal finite horizon feedback by model predictive control

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Abstract

We consider a finite-horizon continuous-time optimal control problem with nonlinear dynamics, an integral cost, control constraints and a time-varying parameter which represents perturbations or uncertainty. After discretizing the problem we employ a model predictive control (MPC) algorithm for this finite horizon optimal control problem by first solving the problem over the entire time horizon and then applying the first element of the optimal discrete-time control sequence, being a constant in time function, to the continuous-time system over the sampling interval. Then the state at the end of the sampling interval is measured (estimated) with certain error, and the process is repeated at each step over the remaining horizon. As a result, we obtain a piecewise constant function in time as control which can be regarded as an approximation to the optimal feedback control of the continuous-time system. In our main result we derive an estimate of the difference between the MPC-generated solution and the optimal feedback solution, both obtained for the same value of the perturbation parameter, in terms of the step-size of the discretization and the measurement error. Numerical results illustrating our estimates are reported.

Key Words: optimal feedback control, model predictive control, discrete approximations, parameter uncertainty, error estimate.

MSC 2010: 49N35, 49K40, 49M25, 93B52.

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1 Introduction

In this paper we consider an optimal control problem over a finite time interval, which involves an integral cost functional, a control system described by an ordinary differential equation, and control constraints. Both the dynamics and the cost depend on a time-varying parameter which represents perturbations or uncertainty. Only a prediction of the time evolution of this parameter is available in advance which in addition may be inaccurate. On the other hand, it is possible to obtain measurements or estimates of the current state that may be corrupted by errors. In such a situation, using optimal feedback control has certain advantages, when compared with open-loop control.

The Model Predictive Control (MPC) algorithm, applied to a continuous-time system, in our case the one with the predicted parameter, is regarded as a feedback control strategy which provides an approximation of the optimal feedback law. Roughly, MPC involves an initial discretization of the problem at hand and then repeated solving of discrete-time optimal control problems over a shrinking horizon till the final time, for different initial conditions corresponding to measurements of the system states at the beginning of each time interval.

Specifically, at the first stage the MPC algorithm considered involves solving the discretized problem with the predicted parameter over the entire time horizon and then applying the optimal control obtained for the first time-step, being a constant in time function, to the "real" system; that is, the one involving the actual (and unknown) parameter. At the end of the first time-step the state is measured, with some error, and it becomes the initial state for the second stage at which the corresponding discrete-time problem, with the predicted parameter, is solved over the time interval starting at the beginning of the second time-step until the fixed final time. Repeating this procedure for each stage leads to an approximation of the optimal feedback control law. It is the purpose of this paper to derive an estimate for the error of the approximation of the exact optimal feedback by the MPC-generated feedback.

The MPC has been extensively explored in the last decades; see e.g. the books [9] and [13] for a broad coverage of the basic aspects of it, and has found numerous applications in various industries. More common variants of MPC involve receding or moving horizon implementation which leads to a time-invariant feedback law for stabilization and tracking. In other problems which involve spacecraft landing or docking, helicopter landing on ships, missile guidance, way point following, control of chemical batch processes etc., the control is performed over a finite time interval. The application of MPC in this setting leads to shrinking horizon formulations. Shrinking horizon MPC has been considered, for instance, in [14, 15], but compared to the receding horizon MPC, it has been less studied. As for the receding horizon MPC, the shrinking horizon MPC is expected to provide an approximation of the optimal finite horizon feedback control law that can be used to handle systems whose dimension is higher than those to which dynamic programming is applicable. Simultaneously, similarly to close-loop control, shrinking horizon MPC would improve robustness to uncertainty when compared with open-loop finite horizon control. The latter expectation is also supported by the results in this paper. The effect of numerical discretization addressed in the current paper, has been considered for sampled data MPC in receding horizon settings (see e.g., [7, 8, 12]), but not for the finite horizon formulation studied here.

To put the stage, consider the following optimal control problem, which we call problem

$\mathcal{P}_p$:

(1) $$\min \left\{ J_p(u) := g(x(T)) + \int_0^T \varphi(p(t), x(t), u(t))\,\mathrm{d}t \right\},$$

subject to

(2) $$\dot{x}(t) = f(p(t), x(t), u(t)), \quad u(t) \in U \quad \text{for a.e. } t \in [0,T], \quad x(0) = x_0,$$

where the time $t \in [0,T]$, the state x is a vector in $\mathbb{R}^n$, the control u has values $u(t)$ that belong to a convex and closed set U in $\mathbb{R}^m$ for almost every (a.e.) $t \in [0,T]$. The initial state x_0 and the final time $T > 0$ are fixed. The set of feasible control functions u, denoted in the sequel by $\mathcal{U}$, consists of all Lebesgue measurable and bounded functions $u : [0,T] \to U$. Accordingly, the state trajectories x, that are solutions of (2) for feasible controls, are Lipschitz continuous functions of time $t \in [0,T]$.

The functions φ and f depend on a parameter p which is a function of time on $[0,T]$ representing uncertainty. We assume that each representation p is a Lebesgue measurable and bounded function with values in $\mathbb{R}^l$. For a specific choice of p the equation (2) is thought of as representing the "true" dynamics of a "real" system. Although the "true" representation of the parameter p is unknown, some reasonable prediction $\bar{p}$ for it is assumed to be known. In this paper we do not assume that the prediction p is updated throughout the iterations; taking into account such updates however is of considerable interest and will be addressed in further research.

Let us now describe in more detail the MPC algorithm applied to the problem at hand. Let $x[u,p]$ denote the solution of equation (2) for given functions $u \in \mathcal{U}$ and p. If p is the "true" representation of the parameter, then the function $t \mapsto x[u,p](t)$ is interpreted as the trajectory of the "real" system obtained for control u.

Given a natural number N, let $\{t_k\}_0^N$ be a grid on $[0,T]$ with equally spaced nodes t_k and a step-size $h = T/N$. Let $\bar{p}$ be a prediction for the unknown "true" representation p of the parameter. To describe the MPC iteration, fix $k \in \{0, 1, \ldots, N-1\}$ and assume that a control u^N with $u^N(t) \in U$ is already determined on $[0, t_k)$. This control is applied to the real system (that with the parameter p). Assume that its state at time t_k, $x[u^N, p](t_k)$, is measured (or estimated) with error ξ_k, that is, the vector $x_k^0 := x[u^N, p](t_k) + \xi_k$ becomes available at time t_k. The next step is to solve the discrete-time optimal control problem

$$\min \left\{ g(x_N) + h \sum_{i=k}^{N-1} \varphi(\bar{p}(t_i), x_i, u_i) \right\},$$

subject to

(3) $$x_{i+1} = x_i + h f(\bar{p}(t_i), x_i, u_i), \quad u_i \in U, \quad i = k, \ldots, N-1, \quad x_k := x_k^0.$$

Note that this problem is solved for the known (predicted) representation $\bar{p}$ of the parameter. For $k = 0$ we have $x_0^0 = x_0 + \xi_0$. Suppose that a locally optimal control $(\tilde{u}_k, \ldots, \tilde{u}_{N-1})$ is obtained as a solution of this problem. Define the constant in time function

$$u^N(t) = \tilde{u}_k \quad \text{for } t \in [t_k, t_{k+1}),$$

change k to $k+1$ and continue the iterations as long as $k < N$, obtaining at the end a control u^N which we call the *MPC-generated control*. Note that the MPC-generated control u^N may not be uniquely determined, e.g., because the discrete-time optimal control problem appearing at some stage does not have a unique solution. Also note that we keep the final time T fixed, so that the time horizon shrinks at each iteration.

The MPC algorithm so described can be regarded as a method for finding an approximate feedback control. Assume that there exists an (exact) optimal feedback $u^*(t,x)$ for problem $\mathcal{P}_{\bar{p}}$ with the predicted representation $\bar{p}$ of the parameter provided that the state x in $u^*(t,x)$ is measured exactly. In this paper we give an answer to the following question: what is the loss of optimal performance if the MPC-constructed control u^N (possibly in presence of measurement errors) is used in the real system (2) with p instead of the exact optimal feedback u^*. As a quantitative measure of the loss of optimal performance we may use an appropriate norm of the deviation of $x[u^N, p]$, the solution of (2) generated by the MPC-constructed control, from $x[u^*, p]$, the solution of (2) generated by the optimal feedback control u^* for problem $\mathcal{P}_{\bar{p}}$, both solutions obtained for the "real" system, with parameter p. One of the main results of the paper stated as Theorem 2.3 is that, under appropriate conditions, there exists a constant c such that, for all sufficiently large N, for every p sufficiently close to $\bar{p}$, and for every ξ sufficiently close to zero, one has

$$\|u^N - \hat{u}\|_1 + \|x[u^N, p] - x[u^*, p]\|_{W^{1,1}} \le c\Big(h + h\sum_{i=1}^{N-1} |\xi_i|\Big), \tag{4}$$

where $\|\cdot\|_1$ and $\|\cdot\|_{W^{1,1}}$ are the standard norms of, respectively, the space of Lebesgue integrable functions and the space of absolutely continuous functions whose first derivatives are Lebesgue integrable.

Clearly, the first term in the brackets in the right-hand side of (4) comes from the discretization, while the second term is due to the measurement errors. Remarkably, the actual size of the parameter uncertainty, $\|p - \bar{p}\|$, does not appear in the bound (4). That is, a possible change of the parameter p affects in the same way (modulo $O(h)$) both the trajectories generated by the (hypothetical) optimal feedback control and those obtained by applying the MPC method.

A slight modification of estimate (4) also holds for the difference, measured by the l_∞ norm, between the values of the control at the grid points t_k resulting from application of the optimal feedback control in the "true" system and the one generated by the MPC algorithm with measurement errors ξ_i.

The proof of the main result of this paper uses results obtained in two companion papers, [4] and [5], which are currently submitted for publications and available as preprints. In both papers the optimal control problem (1)–(2) is considered under basically the same assumptions. In [4] it is shown that the solution mapping of the discrete-time problem has a Lipschitz continuous single-valued localization with respect to the parameter whose Lipschitz constant and the sizes of the neighborhoods do not depend on the number N of grid points, for all sufficiently large N. In [5] it is proved that there exists an optimal feedback control which is a Lipschitz continuous function of the state and time. We also utilize results from the previous papers [1] and [2].

In this setting, a natural question to ask is whether by using higher order discretization schemes one can obtain a better order of approximation than (4). The answer is, "conditionally yes". Second-order approximations to control-constrained optimal control problems in the form of $\mathcal{P}_p$ (with $p = 0$) by Runge-Kutta discretizations are obtained in [3] under similar conditions as in the present paper. Results for even higher order approximations (essentially for problems without control constraints) are presented in [10]. Using these results, however, will improve only the first term $O(h)$ in the right-hand side of (4). Utilization of higher order schemes is justifiable only if the total l_1-error in the measurements is consistent with the discretization error.

In the following Section 2 we specify the assumptions under which the problem (1)–(2) is analyzed and state the main result of the paper. In Sections 3 we give a proof of the result. In Section 4 we present a computational example which confirms the theoretical findings.

2 Main result

In this paper we use fairly standard notations. The euclidean norm and the scalar product in $\mathbb{R}^n$ (the elements of which are regarded as vector-columns) are denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively. The transpose of a matrix (or vector) E is denoted by $E^\top$. For a function $\psi : \mathbb{R}^p \to \mathbb{R}^r$ of the variable z we denote by $\mathrm{gph}\,(\psi)$ its graph and by $\psi_z(z)$ its derivative (Jacobian), represented by a $(r \times p)$-matrix. If $r = 1$, $\nabla_z \psi(z) = \psi_z(z)^\top$ denotes its gradient (a vector-column of dimension p). Also for $r = 1$, $\psi_{zz}(z)$ denotes the second derivative (Hessian), represented by a $(p \times p)$-matrix. For a function $\psi : \mathbb{R}^{p \times q} \to \mathbb{R}$ of the variables (z, v), $\psi_{zv}(z, v)$ denotes its mixed second derivative, represented by a $(p \times q)$-matrix. The space L^k, with $k = 1, 2$ or $k = \infty$, consists of all (classes of equivalent) Lebesgue measurable vector-functions defined on an interval of real numbers, for which the standard norm $\|\cdot\|_k$ is finite (the dimension and the interval will be clear from the context). As usual, $W^{1,k}$ denotes the space of absolutely continuous functions on a scalar interval for which the first derivative belongs to L^k. In any metric space we denote by $I\!B_a(x)$ the closed ball of radius a centered at x.

We begin with stating the assumptions under which problem $\mathcal{P}_p$ is considered, some of which were mentioned in the introduction. First, the set of possible realizations of the uncertain function p is assumed to be of the form

(5) $$\Pi = \{p : [0, T] \to \mathbb{R}^l \,:\, p \in L_\infty(0, T),\ \|p\|_\infty \le \bar{M},\ \|p - \bar{p}\|_1 \le \delta\},$$

where $\bar{M}$ and δ are positive constants. Clearly, the set of possible realizations of p could be any subset of Π.

Assumption (A1). The set U is closed and convex, the functions $f : \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\varphi : \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are two times continuously differentiable in (x, u) and these functions together with their derivatives in (x, u) up to second order are locally Lipschitz continuous in (p, x, u).

Assumption (A2). The reference (predicted) parameter $\bar{p} \in \Pi$ is Lipschitz continuous and satisfies $\|\bar{p}\|_\infty < \bar{M}$. Problem $\mathcal{P}_{\bar{p}}$ has a locally optimal solution $(\bar{x}, \bar{u})$.

The local optimality is understood in the following (weak) sense: there exists a number $e_0 > 0$ such that for every $u \in \mathcal{U}$ with $\|u - \bar{u}\|_\infty \le e_0$, either the differential equation (2) has no solution on $[0, T]$ for u and $\bar{p}$, or $J_{\bar{p}}(u) \ge J_{\bar{p}}(\bar{u})$.

In terms of the Hamiltonian

$$H(t, x, u, \lambda) = \varphi(\bar{p}(t), x, u) + \lambda^\top f(\bar{p}(t), x, u)$$

for problem $\mathcal{P}_{\bar{p}}$, the Pontryagin maximum (here minimum) principle claims that there exists an absolutely continuous (here Lipschitz) function $\bar{\lambda} : [0, T] \to \mathbb{R}^n$ such that the triple $(\bar{x}, \bar{u}, \bar{\lambda})$ satisfies for a.e. $t \in [0, T]$ the following optimality system:

$$\begin{array}{rcl} 0 &=& -\dot{x}(t) + f(\bar{p}(t), x(t), u(t)), \quad x(0) - x_0 = 0, \\ 0 &=& \dot{\lambda}(t) + \nabla_x H(t, x(t), u(t), \lambda(t)), \\ 0 &=& \lambda(T) - \nabla_x g(x(T)), \\ 0 &\in& \nabla_u H(t, x(t), u(t), \lambda(t)) + N_U(u(t)), \end{array} \tag{6}$$

where the normal cone mapping N_U to the set U is defined as

$$\mathbb{R}^m \ni u \mapsto N_U(u) = \begin{cases} \{y \in \mathbb{R}^n \mid \langle y, v - u\rangle \le 0 \text{ for all } v \in U\} & \text{if } u \in U, \\ \emptyset & \text{otherwise.} \end{cases}$$

To shorten the notations we skip arguments with "bar" in functions, shifting the "bar" on the top of the notation of the function, so that $\bar{H}(t) := H(t, \bar{x}(t), \bar{u}(t), \bar{\lambda}(t))$, $\bar{H}(t, u) := H(t, \bar{x}(t), u, \bar{\lambda}(t))$, $\bar{f}(t) := f(\bar{p}(t), \bar{x}(t), \bar{u}(t))$, etc. Define the matrices
(7)

$$A(t) = \bar{f}_x(t),\ B(t) = \bar{f}_u(t),\ F = g_{xx}(\bar{x}(T)),\ Q(t) = \bar{H}_{xx}(t),\ S(t) = \bar{H}_{xu}(t),\ R(t) = \bar{H}_{uu}(t).$$

Assumption (A3) – Coercivity. There exists a constant $\rho > 0$ such that

$$y(T)^\top F y(T) + \int_0^T \Big(y(t)^\top Q(t) y(t) + w(t)^\top R(t) w(t) + 2y(t)^\top S(t) w(t)\Big)\,\mathrm{d}t \ge \rho \int_0^T |w(t)|^2\,\mathrm{d}t$$

for all $y \in W^{1,2}$ with $y(0) = 0$, and $w \in L^2$ with $w(t) \in U - U$ for a.e. $t \in [0, T]$, such that $\dot{y}(t) = A(t)y(t) + B(t)w(t)$.

Coercivity condition (A3) first appeared in [11] (if not earlier); after the publication of [2] it has been widely used in studies of regularity and approximations for problems like $\mathcal{P}_{\bar{p}}$.

In the companion paper [5] we studied "ideal" optimal feedback control of the continuous-time problem $\mathcal{P}_{\bar{p}}$. In order to rigorously state what we mean by optimal feedback, we need the following notation. For any $\tau \in [0, T)$ and $y \in \mathbb{R}^n$ consider the following problem, denoted by $\mathcal{P}_{\bar{p}}(\tau, y)$:

$$\min_{\mathcal{U}_\tau} \left\{ J_{\bar{p}}(\tau, y; u) := g(x(T)) + \int_\tau^T \varphi(\bar{p}(t), x(t), u(t))\,\mathrm{d}t \right\},$$

where x is the solution of the initial-value problem

$$\dot{x}(t) = f(\bar{p}(t), x(t), u(t)) \quad \text{for a.e. } t \in [\tau, T], \quad x(\tau) = y, \tag{8}$$

and $\mathcal{U}_\tau$ is the set of feasible controls restricted to the interval $[\tau, T]$. The following natural definition of locally optimal feedback control is used in [5].

Definition 2.1. The function $u^* : [0,T] \times \mathbb{R}^n \to U$ is said to be a locally optimal feedback control around the reference optimal solution pair $(\bar{x}, \bar{u})$ of problem $\mathcal{P}_{\bar{p}}$ if there exist positive numbers η and a, and a set $\Gamma \subset [0,T] \times \mathbb{R}^n$ such that

(i) $\mathrm{gph}\,(\bar{x}) + \{0\} \times I\!B_\eta(0) \subset \Gamma$;

(ii) for every $(\tau, y) \in \Gamma$ the equation

$$\dot{x}(t) = f(\bar{p}(t), x(t), u^*(t, x(t))), \quad \text{for a.e. } t \in [\tau, T], \quad x(\tau) = y, \tag{9}$$

has a unique absolutely continuous solution $\bar{x}[\tau, y]$ on $[\tau, T]$ which satisfies $\mathrm{gph}\,(\bar{x}[\tau, y]) \subset \Gamma$;

(iii) for every $(\tau, y) \in \Gamma$ the function $\bar{u}[\tau, y](\cdot) := u^*(\cdot, \bar{x}[\tau, y](\cdot))$ is measurable, bounded, and satisfies

$$\|\bar{u}[\tau, y] - \bar{u}\|_\infty \le a \quad \text{and} \quad J(\tau, y; \bar{u}[\tau, y]) \le J(\tau, y; u),$$

where u is any admissible control on $[\tau, T]$ with $\|u - \bar{u}\|_\infty \le a$, for which the corresponding solution x of (9) exists on $[\tau, T]$ and is such that $\mathrm{gph}\,(x) \subset \Gamma$;

(iv) $u^*(\cdot, \bar{x}(\cdot)) = \bar{u}(\cdot)$.

In particular, property (iv) yields that $\bar{x}$ is a solution of (9) for $\tau = 0$ and $y = x_0$. Then the uniqueness requirement in (ii) implies that $\bar{x}[0, x_0] = \bar{x}$. In the sequel we call the function $t \mapsto \hat{u}(t) := u^*(t, x[u^*, p](t))$ a *realization* of the feedback control u^* when u^* is applied to (2) with value p of the parameter and with exact measurements. We remind that $x[u, p]$ denotes the solution of equation (2), for an open-loop or a feedback control u.

Recall that the admissible controls can be regarded as elements of the space L^∞, that is, every admissible u is actually a class of functions $u : [0,T] \to U$ that differ from each other on a set of zero Lebesgue measure. Any of the members of this class (call it "representative") generates the same trajectory of (2) and the same value of the objective functional (1). Lemma 4.1 in [5] claims that under (A1)–(A3) there exists a "special" representative of the optimal control $\bar{u}$ which satisfies for *all* $t \in [0,T]$ the inclusion (6) in the maximum principle (with $\bar{x}$ and $\bar{\lambda}$ at the place of x and λ) and the pointwise coercivity condition

$$w^\top R(t) w \ge \rho |w|^2 \quad \text{for or every } w \in U - U. \tag{10}$$

The following condition is introduced in [1] and used in [5] to prove existence of a Lipschitz continuous locally optimal feedback control.

Assumption (A4) – Isolatedness. The function $\bar{u}$ (represented as described in the preceding lines) is an isolated solution of the inclusion $\nabla_u \bar{H}(t, u) + N_U(u) \ni 0$ for all $t \in [0,T]$, meaning that there exists a (relatively) open set $\mathcal{O} \subset [0,T] \times \mathbb{R}^m$ such that

$$\{(t, u) \in [0,T] \times \mathbb{R}^m \;:\; \nabla_u \bar{H}(t, u) + N_U(u) \ni 0\} \cap \mathcal{O} = \mathrm{gph}\,(\bar{u}).$$

For example, the isolatedness assumption holds if for every $t \in [0,T]$ the inclusion $\nabla_u \bar{H}(t, u) + N_U(u) \ni 0$ has a unique solution (which has to be $\bar{u}(t)$). In this case, one can verify the isolatedness condition taking any (relatively) open set $\mathcal{O} \subset [0,T] \times \mathbb{R}^m$ containing $\mathrm{gph}\,(\bar{u})$.

In the proof of Theorem 2.3 we use the following result:

Theorem 2.2. *([5, Theorem 5.2]) Let Assumptions (A1)–(A4) be fulfilled. Then there exists a locally optimal feedback control $u^* : [0,T] \times \mathbb{R}^n \to U$ around $(\bar{x}, \bar{u})$ which is Lipschitz continuous on a set Γ appearing (together with the positive numbers η and a) in Definition 2.1.*

Due to Theorem 2.2, if $\|p-\bar{p}\|_1$ is sufficiently small, the feedback control u^* when plugged in (2) generates a unique trajectory on $[0,T]$, denoted, as in the Introduction, by $x[u^*,p]$. A standard proof of this simple fact will be given in the beginning of the proof of the next theorem.

Recall that for any $p \in \Pi$ and measurement errors $\xi_0,\ldots,\xi_{N-1}$, the MPC method, as described in the introduction, generates a control u^N (possibly not uniquely). In order to indicate the dependence of u^N on p and ξ we sometimes use the extended notation $u^N[p,\xi]$.

The main theorem of this paper follows.

Theorem 2.3. *Suppose that assumptions (A1)–(A4) hold with constants $\bar{M}$ and ρ. Then there exist constants N_0, c and $\delta > 0$ such that for every $N \ge N_0$, for every $p \in \Pi$, where the set Π defined in (5) depends on $\bar{M}$ and δ, and for every $\xi = (\xi_0,\ldots\xi_{N-1})$ with $\max_{k=0,\ldots,N-1}|\xi_k| \le \delta$ there exists a control u^N generated by the MPC algorithm for the system (2) with disturbance parameter p and measurement error ξ such that,*

$$|u^N(t_i)-\hat{u}(t_i)| \le c\Big(h+|\xi_i|+h\sum_{k=0}^{N-1}|\xi_k|\Big), \quad i=0,\ldots,N-1. \tag{11}$$

where $\hat{u}(t) := u^*(t,x[u^*,p](t))$ is the realization of the feedback control u^*. Furthermore, if $\hat{x} = x[\hat{u},p] = x[u^*,p]$ is the trajectory of (2) for $\hat{u}$ and p and $x^N := x[u^N,p]$ is the trajectory of (2) for u^N and p, then

$$\|u^N-\hat{u}\|_1 + \|x^N-\hat{x}\|_{W^{1,1}} \le c\Big(h+h\sum_{k=0}^{N-1}|\xi_k|\Big). \tag{12}$$

We complement the statement of the theorem with two remarks.

Remark 2.4. As mentioned in the introduction, the MPC algorithm does not necessarily define a unique control u^N, in general, since the optimal control sequence $(\tilde{u}_k,\ldots,\tilde{u}_{N-1})$ appearing at each stage k of the MPC does not need to be unique. Of course, the MPC-solution u^N is uniquely determined if each of the discrete optimal control problems which is solved at a stage of the MPC algorithm has a unique solution.

Remark 2.5. The important message of Theorem 2.3 is that the MPC-generated control u^N copes with the uncertainty p in the same way as the ideal optimal feedback in the continuous-time problem (modulo the error in the right-hand side of (12)). Clearly, the term h in the estimation (12) results from the discretization, while the term involving $|\xi_k|$ represents the effect of the measurement error in the implementation of the MPC (the implementation of the "ideal" optimal feedback is assumed to be exact).

3 Proof of Theorem 2.3

Before presenting the proof we give some preparatory material. Let Γ, η, a, and the optimal feedback control u^* be as in Theorem 2.2. Without loss of generality we may assume that the set Γ is bounded. Indeed, one can redefine it as

$$\Gamma := \{(t,x)\in\Gamma \,:\, \text{there exists } (\tau,y)\in\Gamma \text{ such that } \tau \le t,\ y \in I\!B_\eta(\bar{x}(\tau)),\ \text{and } x = \bar{x}[\tau,y](t)\}\,.$$

It is straightforward that the requirements of Definition 2.1 are satisfied by this new set Γ (which is bounded) and the same a and η.

Denote by $\hat{L}$ a Lipschitz constant of u^* on Γ. In further lines M and L denote a bound on the values and a Lipschitz constant of f, f_x and φ_x on the following bounded set:

$$\left\{(s,x,u) \in \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^m : \; |s| \le \bar{M},\, x \in P(\Gamma),\, u \in I\!\!B_a(\bar{u}([0,T]))\right\},$$

where $P(\Gamma) := \{x \in \mathbb{R}^n \;:\; (t,x) \in \Gamma \text{ for some } t \in [0,T]\}$ is the projection of Γ on $\mathbb{R}^n$, and we have abbreviated $I\!\!B_a(\bar{u}([0,T])) := \cup\{I\!\!B_a(\bar{u}(t)) \;:\; t \in [0,T]\}$.

Let $\bar{x}[\tau,y]$ and $\bar{u}[\tau,y](t) := u^*(t,\bar{x}[\tau,y](t))$ be as in Definition 2.1. Then $\bar{u}[\tau,y]$ is a locally optimal solution of problem $\mathcal{P}_{\bar{p}}(\tau,y)$. Remark 5.6 in [5] claims that in fact, $\bar{u}[\tau,y]$ is the *unique* locally optimal control in problem $\mathcal{P}_{\bar{p}}(\tau,y)$ in the set $I\!\!B_a(\bar{u}) \subset L^\infty(\tau,T)$. Moreover, the function $t \mapsto \bar{u}[\tau,y](t)$ is Lipschitz continuous, uniformly in $(\tau,y) \in \Gamma$, namely, for any $t,s \in [0,T]$ we have

$$\begin{aligned} |\bar{u}[\tau,y](t) - \bar{u}[\tau,y](s)| &= |u^*(t,\bar{x}[\tau,y](t)) - u^*(s,\bar{x}[\tau,y](s))| \\ &\le \hat{L}(|t-s| + |\bar{x}[\tau,y](t) - \bar{x}[\tau,y](s)|) \le \hat{L}(1+M)|t-s|. \end{aligned} \tag{13}$$

By applying the Grönwall inequality to (9), for any y_1, y_2 with (τ,y_1), $(\tau,y_2) \in \Gamma$ and $t \in [\tau,T]$ we have that

$$|\bar{x}[\tau,y_1](t) - \bar{x}[\tau,y_2](t)| \le e^{L(1+\hat{L})t}|y_1 - y_2|.$$

Hence, for any such y_1, y_2 and t,

$$\begin{aligned} |\bar{u}[\tau,y_1](t) - \bar{u}[\tau,y_2](t)| &= |u^*(t,\bar{x}[\tau,y_1](t)) - u^*(t,\bar{x}[\tau,y_2](t))| \\ &\le \hat{L}|\bar{x}[\tau,y_1](t) - \bar{x}[\tau,y_2](t)| \le \hat{L}e^{L(1+\hat{L})t}|y_1 - y_2|. \end{aligned}$$

In the proof of Theorem 2.3 we also utilize the following theorem, which is similar to [2, Theorem 6], but there are some important differences; namely, it is about a time-dependent problem and, more importantly, it establishes an error estimate which is uniform with respect to the initial state. It concerns the discrete-time problem

$$\min\left\{g(x_N) + h\sum_{i=k}^{N-1}\varphi(\bar{p}(t_i),x_i,u_i)\right\}, \tag{14}$$

subject to

$$x_{i+1} = x_i + hf(\bar{p}(t_i),x_i,u_i), \quad u_i \in U, \quad i = k,\dots,N-1, \quad x_k = y. \tag{15}$$

Theorem 3.1. *Let assumptions (A1)–(A4) be fulfilled. Then there exist numbers N_0 and c_0 such that for every $N \ge N_0$, every $k \in \{0,\dots,N-1\}$ and every $y \in I\!\!B_\eta(\bar{x}(t_k))$, problem (14)–(15) has a locally optimal control $\tilde{u}^N[k,y] = (\tilde{u}^N_k[k,y],\dots,\tilde{u}^N_{N-1}[k,y])$ that satisfies the estimation*

$$\max_{i=k,\dots N-1}|\tilde{u}^N_i[k,y] - \bar{u}[t_k,y](t_i)| \le c_0 h.$$

Proof. The optimality system of first-order necessary optimality conditions for problem (14)–(15) has the form

$$
(16)\qquad \begin{cases} x_{i+1} &= x_i + hf(\bar p(t_i), x_i, u_i), \quad x_k = y, \\ \lambda_{i-1} &= \lambda_i + h\nabla_x H(t_i, x_i, u_i, \lambda_i), \\ \lambda_{N-1} &= \nabla_x g(x_N), \\ 0 &\in \nabla_u H(t_i, x_i, u_i, \lambda_i) + N_U(u_i), \end{cases}
$$

where $i = k, k+1, \ldots, N-1$ in the first and in the last relations, $i = k+1, k+2, \ldots, N-1$ in the second equation. Here λ_i represents the co-state for the discretized problem and H, as before, is the Hamiltonian. In the proof we consider k and y fixed but monitor how the constants involved depend on them. In the sequel, for brevity, we denote $\bar u_i = \bar u[t_k, y](t_i)$, $\bar x_i = \bar x[t_k, y](t_i)$, $\bar\lambda_i = \bar\lambda[t_k, y](t_i)$ skipping the argument $[t_k, y]$.

First, observe that the sequence $\{(\bar u_i, \bar x_i, \bar\lambda_i)\}_i$ satisfies the system

$$
(17)\qquad \begin{cases} 0 &= -x_{i+1} + x_i + hf(\bar p(t_i), x_i, u_i) + b_i, \quad x_k = y, \\ 0 &= -\lambda_{i-1} + \lambda_i + h\nabla_x H(t_i, x_i, u_i, \lambda_i) + d_i \\ 0 &= -\lambda_{N-1} + \nabla_x g(x_N) + d_N, \\ 0 &\in \nabla_u H(t_i, x_i, u_i, \lambda_i) + e_i + N_U(u_i), \end{cases}
$$

with $b_i = \bar b_i(h)$, $d_i = \bar d_i(h)$, $e_i = \bar e_i(h) = 0$, where

$$
\bar b_i(h) = \int_{t_i}^{t_{i+1}} f(\bar p(s), \bar x[t_k, y](s), \bar u[t_k, y](s))\, \mathrm{d}s - hf(\bar p(t_i), \bar x_i, \bar u_i),
$$

$$
\bar d_i(h) = \int_{t_i-1}^{t_i} \nabla_x H(s, \bar x[t_k, y](s), \bar u[t_k, y](s), \bar\lambda[t_k, y](s))\, \mathrm{d}s - h\nabla_x H(t_i, \bar x_i, \bar u_i, \bar\lambda_i), \quad i = k+1, \ldots, N-1,
$$

$$
(18)\qquad \bar d_N(h) = \bar\lambda_{N-1} - \bar\lambda_N = \int_{t_{i-1}}^{t_i} \nabla_x H(t, \bar x(t), \bar u(t), \bar\lambda(t))\, \mathrm{d}t.
$$

Note that $\bar p$ is assumed to be Lipschitz continuous on $[0, T]$. According to (13), $\bar u[t_k, y]$ is Lipschitz continuous with Lipschitz constant $\hat L(1 + M)$. Moreover, $(t, \bar x[t_k, y](t)) \in \Gamma$ and $\bar u[t_k, y](t) \in I\!B_a(\bar u([0, T]))$. Hence, $\bar x[t_k, y]$ is Lipschitz continuous with Lipschitz constant M. Then $\bar\lambda[t_k, y]$ is also Lipschitz continuous with a Lipschitz constant depending only on M. Hence, there exists a constant c_1, independent of k and y and N, such that

$$
(19)\qquad \max_{k \le i \le N-1} |\bar b_i(h)| + \max_{k+1 \le i \le N-1} |\bar d_i(h)|) \le c_1 h.
$$

In fact, the constant c_1, as well as c_2 and c_3 that appear below, depend on the numbers M, L, $\hat L$ and the Lipschitz constant of $\bar p$ only. From (18) we get that

$$
(20)\qquad |\bar d_N(h)| \le c_2 h.
$$

Now we consider the right-hand side of (17) as a mapping acting on $v = \{(x_i, u_i, \lambda_i)\}_i \in \mathbb{R}^K$, where $K = (2n+m)(N-k)$. For the vector $q = (b, d, e)$ we have $b \in \mathbb{R}^{(N-k)n}$, $d \in \mathbb{R}^{(N-k)n}$, $e \in \mathbb{R}^{(N-k)m}$. In both the domain and the range spaces of the optimality mapping we use the l_∞-norm. This means that in this proof all balls and the Lipschitz constants will be with respect to the l_∞-norm.[1]

In the remainder of the proof we show that the solution mapping of the system (17) has a Lipschitz continuous localization around $\bar{v} := \{(\bar{x}_i, \bar{u}_i, \bar{\lambda}_i)\}_i$ for $\bar{q} = (\bar{b}(h), \bar{d}(h), \bar{e}(h))$ which is uniform in k, y and N; that is, there are constants α, β and γ independent of k, y and N such that for each $q := (b, d, e) \in \mathbb{B}_\beta(\bar{q})$ there exists a unique solution $v(q) := \{(x_i, u_i, \lambda_i)\}_i(q)$ of the system (17) in $\mathbb{B}_\alpha(\bar{v})$ and the mapping $\mathbb{B}_\beta(0) \ni q \mapsto v(q)$ is a Lipschitz continuous function with Lipschitz constant γ.

In order to prove the existence of a Lipschitz continuous localization, we use the approach presented in our companion paper [4]. We considered there a continuous-time optimal control problem of the form (1)–(2) on the same assumptions, stated as (A1)–(A4) in the current paper. We also considered there its discrete approximation (14)–(15) with $k = 0$ and $y = x_0$ and with a time-varying functional parameter p around a reference function $\bar{p}$ of $t \in [0, T]$ which is Lipschitz continuous in $[0, T]$. Based on an enhanced version of Robinson's implicit function theorem given below as Theorem 3.2 and results from the paper [2], we established in [4] that there exists a natural number $\bar{N}$ such that if the number N of the grid points is greater than $\bar{N}$, then the solution mapping of the discrete-time problem has a Lipschitz continuous single-valued localization with respect to the parameter whose Lipschitz constant and the sizes of the neighborhoods involved do not depend on N. Here we have a different situation: the reference point is composed of the solution of the continuous-time problem and a time-varying parameter which enters the optimality system in a specific way, as in (17). The approach we use is the same as in [4], but there are few technical differences; in what follows we describe them in detail.

First, observe that system (17) is a special case of the following variational inequality

$$E(v) + q + N_C(v) \ni 0, \tag{21}$$

where $v \in \mathbb{R}^K$, and $C = (\mathbb{R}^n)^{N-k} \times U^{N-k} \times (\mathbb{R}^n)^{N-k}$ and $U^{N-k} \subset (\mathbb{R}^m)^{N-k}$ is the product of $N-k$ copies of the set U. Here $v = (x, u, \lambda) = (x_{k+1}, \ldots, x_N, u_k, \ldots, u_{N-1}, \lambda_k, \ldots \lambda_{N-1})$ is considered as a vector (column) of dimension $n(N-k)+m(N-k)+n(N-k)$. The components of the vector-function E are as follows: the $N-k$ vectors $-x_{i+1} + x_i + hf(\bar{p}(t_i), x_i, u_i)$ each of dimension n, followed by the $(N-k-1)$ vectors $-\lambda_{i-1} + \lambda_i + hH_x(t_i, x_i, u_i, \lambda_i)$ each of dimension n, followed by the n-dimensional vector $\lambda_{N-1} - g_x(x_N)$, followed by the $N-k$ vectors $H_u(t_i, x_i, u_i, \lambda_i)$ each of dimension m.

Note that the function E is continuously differentiable and the derivative $E_v(v)$ has a sparse structure. Namely, the rows from the first group (corresponding to primal equations) contain at most $2n+m$ non-zero elements, the row from the second group (corresponding to adjoint equations) contain at most $3n+m$ nonzero elements, rows from the third group – $2n$, and from the fourth group – $2n+m$. If we consider E_v on the unit ball $\mathbb{B}_1(\bar{v})$ (in the l_∞-norm), then

[1] Although the spaces involved are finite dimensional, the choice of norms is important since the dimension K depends on N and we need some constants (depending on the choice of norms) to be independent of N.

each of the non-zero components of E_v is a Lipschitz function of its variables with a Lipschitz constant L_1, where L_1 depends on the data f, g and φ. It is straightforward to verify that E_v is Lipschitz continuous in $I\!B_1(\bar{s})$ with respect the operator norm of $E_v : \mathbb{R}^K \mapsto \mathbb{R}^K$ and the l_∞-norms in the two spaces. Thanks to the sparseness, the Lipschitz constant, $\bar{L}$, depends on L_1, n and m only, thus, it is independent of N.

We employ the following version of Robinson's implicit function theorem proved in [4], where the assumption about E_v was verified in preceding lines.

Theorem 3.2. *Let $\bar{v}$ be a solution of (21) for $\bar{q}$. Suppose that the derivative E_v exists and is Lipschitz continuous on $I\!B_1(\bar{v})$ with Lipschitz constant $\bar{L}$. Moreover, suppose that the following condition holds: for each $q \in \mathbb{R}^K$ there is a unique solution $w(q) \in \mathbb{R}^K$ of the linear variational inequality*

$$E_v(\bar{v})w + N_C(w) \ni q, \tag{22}$$

and the solution function $q \mapsto w(q)$ is Lipschitz continuous with a Lipschitz constant ℓ. Then there exist positive numbers α, β and γ which depend on ℓ and $\bar{L}$ only, such that for each $q \in I\!B_\beta(\bar{q})$ system (21) has a uniques solution $v \in I\!B_\alpha(\bar{v})$, and the function $q \mapsto v(q)$ is Lipschitz continuous on $I\!B_\beta(\bar{q})$ with Lipschitz constant γ.

Thus, we have to show that the condition involving (22) holds for the particular optimality system (17). The proof of the latter closely follows the proof of Theorem 1.2 in [4] with some adjustments which we will present in some detail.

Let A_i, B_i, Q_i, S_i, R_i denote the matrices in (7) evaluated at $t = t_i$. Also denote $\bar{H}^i_u = H(t_i, \bar{x}_i, \bar{u}_i, \bar{\lambda}_i)$. Then for $v = \{(x_i, u_i, \lambda_i)\}_i$ we have

$$E_v(\bar{v})(v) = \begin{pmatrix} -x_{i+1} + x_i + hA_i x_i + hB_i u_i \\ -\lambda_{i-1} + \lambda_i + hA_i^T \lambda_i + hQ_i x_i + hS_i u_i \\ -\lambda_{N-1} + F x_N \\ \bar{H}^i_u + R_i u_i + S_i^T x_i + B_i^T \lambda_i \end{pmatrix}.$$

Notice that the matrices A, B, Q, S, R are continuous due to continuity of $\bar{u}$, $\bar{x}$ and $\bar{p}$. Then from [2, Lemma 11], we have that there is a natural number N_1 and a positive constant $\rho_1 \le \rho$ such that for all $N > N_1$ the quadratic form

$$\mathcal{B}(y, \nu) = y_N^T F y_N + \sum_{i=0}^{N-1} (y_i^T Q_i y_i + 2y_i^T S_i \nu_i + \nu_i^T R_i \nu_i) \tag{23}$$

satisfies the discretized coercivity condition:

$$\mathcal{B}(y, \nu) \ge \rho_1 \sum_{i=k}^{N-1} |\nu_i|^2 \tag{24}$$

for all (y, ν) from the set

$$\mathcal{C} = \{(y, \nu) \mid y_{i+1} = y_i + hA_i y_i + hB_i \nu_i,\ \nu_i \in U - U,\ i = k, \dots, N-1,\ y_0 = 0\}.$$

According to (10), for every $i = 0, 1, \dots, N-1$ we have

$$\nu^T R_i \nu \geq \rho |\nu|^2 \quad \text{for all } \nu \in U - U. \tag{25}$$

The remainder of the proof of the Lipschitz function $\mathbb{B}_\beta(\bar{q}) \ni q \mapsto v(q) \in \mathbb{B}_\alpha(\bar{v})$ is just a repetition of the corresponding part of the proof of Theorem 1.2 in [4].

Finally, from (19) and (20) we have that

$$\|\bar{q}\|_\infty \leq c_3 h.$$

By taking N sufficiently large we can ensure that $c_3 h \leq \beta$, this $0 \in \mathbb{B}_\beta(\bar{q})$, thus the value $v(0)$ exists and

$$\|v(0) - \bar{v}\|_\infty = \|v(0) - v(\bar{q})\|_\infty \leq \gamma \|\bar{q}\|_\infty \leq \gamma c_3 h.$$

This implies the claim of the theorem with $c_0 := \gamma c_3$.

To complete the proof it remains to observe that, for the state-control pair $(\bar{x}^N, \bar{u}^N)$ obtained from solving the optimality system (16), the coercivity condition (24) is a sufficient condition for a (strict) local minimum, see [2, Appendix 1]. □

Proof of Theorem 2.3. We continue using the notation we have used/introduced from the beginning of this section.

Choose δ to satisfy

$$0 < \delta < \frac{\eta}{4} e^{-L(1+\hat{L})T} \tag{26}$$

and define Π as in (5). We will prove next that for every $p \in \Pi$ the solution $\hat{x} := x[u^*, p]$ of (2) for u^* and p exists on $[0, T]$ and $\hat{x}(t) \in \mathbb{B}_{\eta/4}(\bar{x}(t))$, $t \in [0, T]$.

Since $\hat{x}(0) = \bar{x}(0)$, the smoothness of f in assumption (A1) implies that $\hat{x}$ exists locally and can be extended to the maximal interval $[0, \theta] \subset [0, T]$ in which $\hat{x}(t) \in \mathbb{B}_{\eta/4}(\bar{x}(t))$ holds. Note that $|\hat{x}(\theta) - \bar{x}(\theta)| = \eta/4$ if $\theta < T$. Then $\Delta(t) := |\hat{x}(t) - \bar{x}(t)|$, $t \in [0, \theta]$, satisfies

$$\begin{aligned}
\Delta(t) &\leq \int_0^t |f(p(s), \hat{x}(s), \hat{u}(s)) - f(\bar{p}(s), \bar{x}(s), \bar{u}(s))| \,\mathrm{d}s \\
&= \int_0^t |f(p(s), \hat{x}(s), u^*(s, \hat{x}(s))) - f(\bar{p}(s), \bar{x}(s), u^*(s, \bar{x}(s)))| \,\mathrm{d}s \\
&\leq \int_0^t L\left[|p(s) - \bar{p}(s)| + \Delta(s) + \hat{L}\Delta(s)\right] \mathrm{d}s,
\end{aligned}$$

where we use the identities $\bar{u}(s) = u^*(s, \bar{x}(s))$ (see property (iv) in Definition 2.1) and $\hat{u}(s) = u^*(s, x[u^*, p](s)) = u^*(s, x[\hat{u}, p](s)) = u^*(s, \hat{x}(s))$. Applying the Grönwall inequality and taking into account (26) we obtain that

$$|\Delta(t)| \leq e^{L(1+\hat{L})t} \|p - \bar{p}\|_1 \leq e^{L(1+\hat{L})t} \delta < \eta/4, \quad t \in [0, \theta]. \tag{27}$$

Since the last inequality is strict, we obtain that $\theta = T$ and that $x[\hat{u}, p](t) \in \mathbb{B}_{\eta/4}(\bar{x}(t))$, $t \in [0, T]$.

Define recursively the sequence

$$d_{k+1} = e^{L(1+\hat{L})h} \left[d_k + L(\hat{L}(1+M) + c_0)h^2 + L\hat{L}|\xi_k|h \right], \tag{28}$$

$k = 0, \dots, N-1$, starting with $d_0 = 0$. By using induction we obtain the explicit representation

$$d_k = \sum_{i=0}^{k-1} e^{L(1+\hat{L})(k-i)h} L \left[(\hat{L}(1+M) + c_0)h^2 + \hat{L}|\xi_i|h \right],$$

which implies the estimation

$$\begin{aligned} d_k &\le e^{L(1+\hat{L})T} L \left(T(\hat{L}(1+M) + c_0)h + \hat{L}h \sum_{i=0}^{N-1} |\xi_i| \right) \\ &\le e^{L(1+\hat{L})T} L \left(T(\hat{L}(1+M) + c_0)h + \hat{L}T\delta \right) =: \bar{d}(\delta, h). \end{aligned} \tag{29}$$

Make $\delta > 0$ and $h = T/N$ smaller if necessary so that

$$\bar{d}(\delta, h) < \frac{\eta}{4} \tag{30}$$

and adjust Π according to (5). We will prove next that the MPC algorithm (applied for disturbance p and vector ξ of measurement errors) generates a control $u^N := u^N[p, \xi]$ with the properties claimed in the statement of the theorem. To do that, we use induction in k.

For $k = 0$ we have $|x^N(0) - \hat{x}(0)| = 0 = d_0$. Assume that $0 \le k < N$ and that the MPC-generated control u^N and trajectory x^N are defined on $[t_0, t_k]$ so that the inequality $|x^N(t) - \hat{x}(t)| \le d_k$ holds for all $t \in [0, t_k]$ and, moreover,

$$|x^N(t_k) - \bar{x}(t_k)| \le \eta/2. \tag{31}$$

Then, from (26) and (31), $y := x^N(t_k) + \xi_k$ satisfies the inequality

$$|y - \bar{x}(t_k)| \le |x^N(t_k) - \bar{x}(t_k)| + |\xi_k| \le \frac{\eta}{2} + \delta < \frac{\eta}{2} + \frac{\eta}{4} < \eta.$$

Let $\tilde{u} := \bar{u}[t_k, y]$ be the (unique) locally optimal control in problem $\mathcal{P}_{\bar{p}}(t_k, y)$ in the set $\mathbb{B}_a(\bar{u})$ (whose existence follows from Theorem 2.2 and the condition $y \in \mathbb{B}_\eta(\bar{x}(\tau))$). Let $\tilde{x}$ be the corresponding trajectory of (2) for $p = \bar{p}$ starting from y at t_k. According to Theorem 3.1, there exists a locally optimal control $\tilde{u}^N[k, y] = (\tilde{u}_k^N[k, y], \dots, \tilde{u}_{N-1}^N[k, y])$ for problem (14)–(15), which satisfies the inequality

$$|\tilde{u}_k^N[k, y] - \tilde{u}(t_k)| \le c_0 h. \tag{32}$$

Clearly, the constant function $u^N(t) = \tilde{u}_k^N[k, y]$ for all $t \in [t_k, t_{k+1})$ is generated by the MPC algorithm and satisfies $|u^N(t) - \tilde{u}(t_k)| \le c_0 h$ for $t \in [t_k, t_{k+1})$.

Let $s \in [t_k, t_{k+1})$; then, utilizing (32), Theorem 2.2, and the definition of $\hat{x}$, we have

$$\begin{aligned} |u^N(s) - \hat{u}(s)| &\le |u^N(s) - \tilde{u}(t_k)| + |\tilde{u}(t_k) - \hat{u}(s)| \\ &\le |\tilde{u}_k^N[k, y] - \tilde{u}(t_k)| + |\tilde{u}(t_k) - \hat{u}(s)| \\ &\le c_0 h + |u^*(t_k, y) - u^*(s, \hat{x}(s)| \\ &= c_0 h + \hat{L}\left(|t_k - s| + |y - \hat{x}(s)|\right) \\ &\le c_0 h + \hat{L}\left(h + |x^N(t_k) - \hat{x}(s)| + |\xi_k|\right) \\ &\le c_0 h + \hat{L}\left(|x^N(s) - \hat{x}(s)| + |\xi_k| + (1+M)h\right). \end{aligned} \tag{33}$$

Furthermore, using the latter estimation in the state equation, for $t \in [t_k, t_{k+1}]$ we get that

$$
\begin{aligned}
|x^N(t) - \hat{x}(t)| &\le |x^N(t_k) - \hat{x}(t_k)| + \int_{t_k}^{t} \big| f(p(s), x^N(s), u^N(s)) - f(p(s), \hat{x}(s), \hat{u}(s)) \big| \, \mathrm{d}s \\
&\le d_k + L \int_{t_k}^{t} \big(|x^N(s) - \hat{x}(s)| + |u^N(s) - \hat{u}(s)| \big) \, \mathrm{d}s \\
&\le d_k + L \int_{t_k}^{t} \Big((1+\hat{L})|x^N(s) - \hat{x}(s)| + \hat{L}|\xi_k| + \hat{L}(1+M)h + c_0 h \Big) \, \mathrm{d}s.
\end{aligned}
$$

The Grönwall inequality yields

$$
|x^N(t) - \hat{x}(t)| \le e^{L(1+\hat{L})h} \Big(d_k + L\hat{L}|\xi_k|h + (L\hat{L}(1+M) + Lc_0)h^2 \Big) = d_{k+1}. \tag{34}
$$

Moreover, using (27) and (30) we obtain that for all $t \in [t_k, t_{k+1}]$ one has

$$
|x^N(t) - \bar{x}(t)| \le |x^N(t) - \hat{x}(t)| + |\hat{x}(t) - \bar{x}(t)| \le d_{k+1} + \frac{\eta}{4} < \frac{\eta}{2}.
$$

The induction step is complete. As a result, from (34) combined with (29), we obtain the existence of a constant c_4 such that the MPC-generated control u^N and the corresponding trajectory x^N satisfy

$$
\|x^N - \hat{x}\|_{C[0,T]} \le c_4 \Big(h + h \sum_{k=0}^{N-1} |\xi_k| \Big). \tag{35}
$$

This last estimate, combined with (33), gives us the estimate (11) in the statement of the theorem with a constant c which depends on L, $\hat{L}$, M, c_0 and c_4 only. By integrating in (33) and utilizing (35), we get

$$
\begin{aligned}
\|u^N - \hat{u}\|_1 &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} |u^N(t) - \hat{u}(t)| dt \\
&\le \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \Big(c_0 h + \hat{L}(|x^N(t) - \hat{x}(t)| + |\xi_k| + (1+M)h \Big) \, \mathrm{d}t \\
&\le \sum_{k=0}^{N-1} \left(c_0 h^2 + \hat{L}h|\xi_k| + \hat{L}(1+M)h^2 + \hat{L} \int_{t_k}^{t_{k+1}} c_4 \left(h + h \sum_{k=0}^{N-1} |\xi_k| \right) dt \right) \\
&\le T(c_0 + \hat{L}(1+M))h + \hat{L}h \sum_{k=0}^{N-1} |\xi_k| + \hat{L}Tc_4 \left(h + h \sum_{k=0}^{N-1} |\xi_k| \right) \\
&\le c \left(h + h \sum_{k=0}^{N-1} |\xi_k| \right),
\end{aligned}
$$

[VV: V gornite formuli sqm mahnal edin red i sqm smenil dt **s** $\mathrm{d}t$**.]** where c depends on L, $\hat{L}$, M, c_0 and c_4 only. Using this last estimate and the smoothness of f, and taking the constant c larger if needed, we obtain (12). This completes the proof. □

4 Numerical examples

In this section we illustrate the result obtained in Theorem 2.3 by considering a problem of axisymmetric spacecraft spin stabilization (nutation damping) from [16] (p. 353). The transversal angular velocity components ω_1 and ω_2 of the spacecraft satisfy

$$\dot{\omega}_1 = \lambda\omega_2 + \frac{M_d}{J_t},$$
$$\dot{\omega}_2 = -\lambda\omega_1 + \frac{M_c}{J_t},$$

where $\lambda = \frac{J_t - J_3}{J_t} n$, J_t is the spacecraft transversal moment of inertia, J_3 is the spacecraft moment of inertia about the spin axis, n is the spin rate, M_d is the disturbance torque which can for instance be caused by thruster misalignment, and M_c is the control moment. Rescaling the time ($t \to \lambda t$), and letting $x_1 = \omega_1$, $x_2 = \omega_2$, $p = \frac{M_d}{J_t}$, $u = \frac{M_c}{J_t}$, we consider the following optimal control problem:

$$\min \left\{ |x(T)|^2 + \frac{\alpha}{2} \int_0^T |u(t)|^2 \, \mathrm{d}t \right\},$$

subject to

$$\dot{x}_1 = x_2 + p,$$
$$\dot{x}_2 = -x_1 + u,$$
$$x_1(0) = x_2(0) = 1, \quad |u| \le a,$$

where $p(\cdot)$ is an uncertain time-dependent parameter. We consider two examples, in both of which we use the specifications $T = 4\pi$, $\alpha = 0.5$, $a = 0.2$ and choose a reference (predicted) functional parameter $\bar{p}(t) \equiv 0$. In the simulations, the measurement error ξ_k is sampled randomly from an uniform distribution, with $|\xi_k| \le 0.01$. The difference between the two examples below is in the choice of the uncertain function $p(\cdot)$.

Example 1. Here $p(\cdot)$ is a piecewise constant function on the uniform mesh with 2560 points in $[0, T]$. The values of $p(t)$ in every subinterval are chosen randomly in the interval $[-\bar{M}, \bar{M}] = [-\delta/T, \delta/T]$ with $\delta = 0.1$. Then any such p belongs to the set Π defined with the above values of $\bar{M}$ and δ.

In the notation of Theorem 2.3, the following two quantities,

$$RE_x := \frac{\|x^N - \hat{x}\|_{W^{1,1}}}{h + h\sum_{k=1}^{N-1} |\xi_k|} \qquad \text{and} \qquad RE_{ux} := \frac{\|u^N - \hat{u}\|_1 + \|x^N - \hat{x}\|_{W^{1,1}}}{h + h\sum_{k=1}^{N-1} |\xi_k|},$$

represent the relative errors with respect to x and with respect to u and x. According to the estimate (12) in Theorem 2.3 these quantities are bounded. The numerical results obtained confirm this result. Indeed, Table 4 presents the numerically obtained values of these ratios for several values of N. The results are consistent with the theoretical estimate (12).

Observe that, despite the presence of perturbations, the performance of the open loop optimal control is not significantly worse than that of the optimal feedback control low. Indeed, denoting by $\hat{J}^{\mathrm{ol}}(p, \xi)$ the value of the objective functional when the optimal open-loop control (for the reference problem with $\bar{p} = 0$) is implemented, by $\hat{J}^{\mathrm{fb}}(p, \xi)$ the objective value when

Table 1: Relative error for perturbation p generated randomly

N	80	160	240	320	400	480	560	640
RE_x	5.140	1.484	0.914	0.653	0.575	0.442	0.448	0.354
RE_{ux}	23.609	40.765	47.809	48.210	47.519	46.353	45.162	43.448

the "exact" feedback u^* is implemented, and by $\hat{J}_N^{\text{mpc}}(p,\xi)$ the objective value when the MPC-generated control with mesh size N is implemented, we obtain

$$\hat{J}^{\text{ol}}(p,\xi) \approx \hat{J}^{\text{fb}}(p,\xi) = 0.0766, \quad \hat{J}_{160}^{\text{mpc}}(p,\xi) = 0.1617, \quad \hat{J}_{640}^{\text{mpc}}(p,\xi) = 0.0858,$$

The reason for this effect is that the high-frequency random (uniformly distributed around the predicted value) perturbations "neutralize" each other, therefore the measurements in the feedback case do not bring a considerable advantage.

Example 2. Here we assume that p is a "systematic" error caused by a model imperfection, namely, $p(t) = 0.1x_2(t)$.

Table 2: Relative error estimation for perturbation $p(t) = 0.1x_2^*(t)$

N	80	160	240	320	400	480	560	640
RE_x	5.667	4.221	4.616	4.995	5.265	5.470	5.624	5.743
RE_{ux}	23.826	43.325	55.730	61.211	55.996	55.170	54.211	52.822

This time the performance of the optimal open-loop control, on one hand, and those of the optimal feedback control and the MPC-generated controls, on the other hand, differ substantially:

$$\hat{J}^{\text{ol}}(p,\xi) = 0.1929, \quad \hat{J}^{\text{fb}}(p,\xi) = 0.0898, \quad \hat{J}_{160}^{\text{mpc}}(p,\xi) = 0.1879, \quad \hat{J}_{640}^{\text{mpc}}(p,\xi) = 0.1002.$$

Figure 1 shows the controls in Example 2 generated by the "true" feedback control law u^* and by the MPC algorithm with $N = 160$ and $N = 640$. The controls in Example 1 look similarly.

Fig. 2 shows the trajectories in Example 2 generated by the "true" feedback control law u^* and by the MPC algorithm with $N = 160$ and $N = 640$. As seen, the MPC algorithm with $N = 640$ discretization points stabilizes the system to the origin almost as well as the "theoretical" optimal feedback control low.

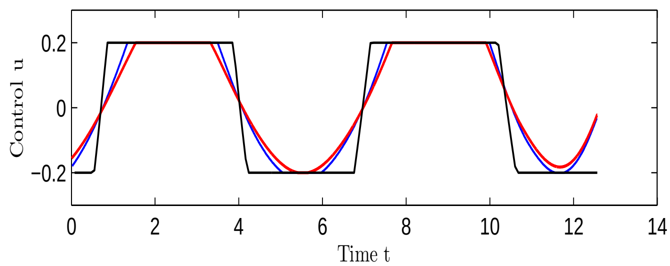

Figure 1: Example 2: Control functions generated by the optimal feedback u^* (red line, with lowest steepness) and by the MPC algorithm with $N = 160$ (black line, the steepest one) and with $N = 640$ (blue line).

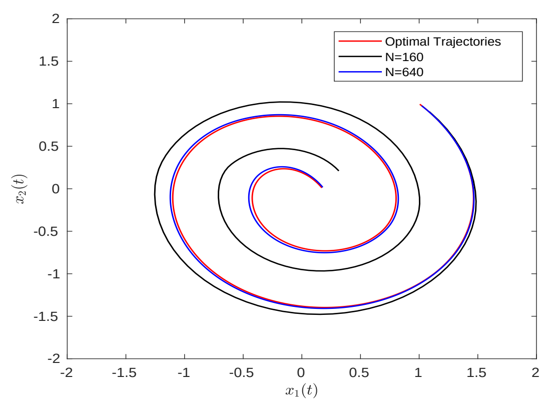

Figure 2: Example 2: Trajectories generated by the optimal feedback u^* (red line) and by the MPC algorithm with $N = 160$ (black line) and with $N = 640$ (blue line).

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