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# Optimal scientific production over the life cycle

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## Abstract

The paper develops an optimization model of the career of a scientist. Recognizing that research efforts and networking get more efficient if the scientist is more knowledgeable, history dependent solutions are developed. We give a theoretical underpinning of the four different research patterns detected in [Way et al. \(2017, Proceedings of the National Academy of Sciences\)](#). If the scientist does not bother about his reputation at the end of his career, we show that a sufficient education level is needed for the scientist to develop a typical research pattern where productivity increases in the beginning of his career, while it declines towards retirement. If the education level is not sufficient, a fading research pattern will result where productivity declines over time. On the other hand, when the scientist appreciates to have a good reputation at the end of his career, sufficient education will result in increasing productivity over the career lifetime, preventing a midlife slump.

*Keywords:* Optimal control, research patterns, Skiba, age-dependent scientific production, networking: initial knowledge

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## 1. Introduction

The publications of a scientist over his/her career are usually not evenly spaced in time. The main idea is that productivity patterns quite often show an intuitively plausible time course: scientific creativity tends to rise rapidly to a peak and then gradually declines. There are many studies of career paths of creative people since the famous statistician [Quételet \(1835\)](#) started pertinent research almost 200 years ago. Typical life cycle patterns are not only observed in academia, but also in artistic production, in criminal behavior and other fields.

[Way et al. \(2017\)](#) provide a readable introductory survey into age patterns of scientific production. Using a large data set originating in computer science departments in the U.S. and Canada, these authors show that the conventional narrative of a steeply increasing and then gradually declining production trajectory is not generally true. They identify a much richer diversity of production patterns. One important purpose of the present paper is to explain how such a diversity might come about. While virtually all models dealing with the dynamics of scientific production are descriptive

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(see, e.g., [Rinaldi et al., 2000](#); [Rinaldi and Amigoni, 2000](#)), in what follows we propose a normative approach.

We depart from the assumption that productivity of a scientist depends on the scientist's knowledge and his reputation. A scientist invests in his human capital, or knowledge, by working behind the desk, reading books and papers, developing new ideas, etc. Once a certain stock of knowledge has been built up, the scholar can fruitfully work on his reputation. Investing in the stock of reputation consists of presenting at conferences, contacting colleagues, networking at receptions, etc. These efforts can be summarized as networking investment. The output of a scientist is publishing papers. Necessary for that is the accumulation of a sufficient stock of knowledge. On the other hand, a scientist's productivity also depends on his reputation in the sense that reputation helps to find better coauthors and to get more informative about to which journals one should submit his papers in order to get more successful.

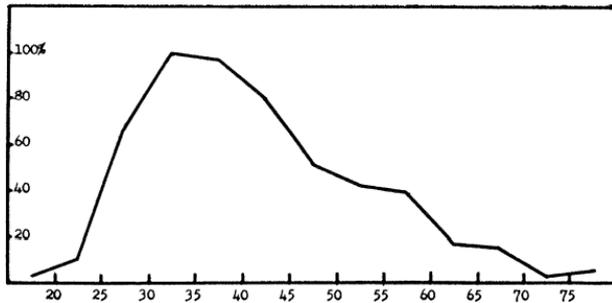
The aim of the scientist is to maximize the discounted stream of publications over time taking into account the costly investments both in knowledge as well as in reputation. The dynamics of the system is given by two ordinary differential equations describing the impact of investment in knowledge and networking on the stocks of knowledge and reputation, respectively. An important feature of the model is that the efficiency of both investment in human capital and networking investment depends on the stock of knowledge. The larger the scientist's knowledge the more effective his investment in research is. Also the networking investment will be more effective if a scientist performs networking while exposing more knowledge about relevant research topics.

Solving the resulting optimal control model by using Pontryagin's maximum principle shows that the shape of the optimal paths depends crucially on the initial situation. If the stock of knowledge is initially too small, it turns out that the researcher's career will not be very productive. If, however, a certain human capital endowment (the so-called Skiba threshold) is exceeded initially, the career will flourish. A large stock of knowledge fosters the investment in networking, making the leverage effect of the stock of reputation work. Besides the case of an infinite planning horizon, the case of a given finite end time is considered. Various terminal conditions (ranging from 'there are no pockets in a shroud' until 'the reputation in the posterity is quite important') will lead to different scientific career patterns.

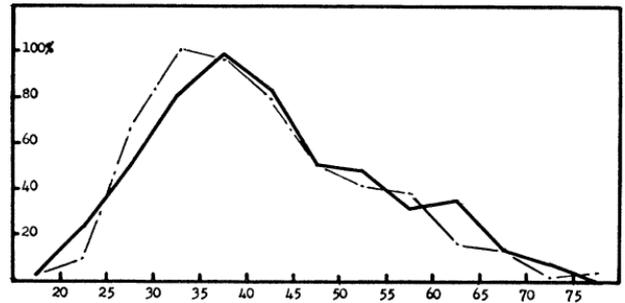
### *Literature review*

In an old, but still readable survey paper on the economics of science, [Stephan \(1996\)](#) referred to the extreme inequality of scientific productivity. Temporal patterns of scientific productivity vary substantially not only across individuals but also for each researcher over time. Some individuals remain productive throughout their career. Others who show early promise become deadwood after some time: Some authors publish papers like a well-oiled machine, while others produce at an erratic rate ([Goodwin and Sauer, 1995](#)).

Typically, an initial rise of creative productivity is followed by an eventual decline. Both the aging process as well as human capital models predict a hump-shaped productivity pattern over the life cycle. There is however, an obvious asymmetry in the distribution of research productivity over the life cycle. While for most of the researchers productivity rises sharply in the early stages of the career, its decline is rather slowly, of which [Figure 1](#) forms a nice illustration. Almost one century ago the famous demographer and mathematician [Lotka \(1926\)](#) stressed the highly skewed nature of scientific publications. Studying nineteenth century physics journals, he observed that approximately six percent of publishing scientists produced half of all papers. Inequality in creative research between scientists at a given time can be explained by different motivation and abilities. There is, however a second kind of inequality in productivity between scientists, namely those of a cohort of scientists over time.



(a) Age versus creative production rate for Russians only, in science and mathematics



(b) Solid line: age versus creative production rate for Englishmen only, in science and mathematics. Broken line, same as in panel (a).

Figure 1: Age vs. creative production rate for Russians only, in science and mathematics. (Source: H.C. Lehman, Men's creative production rate at different ages and in different countries, *The Scientific Monthly*, May 1954, 321-326).

Stephan (1996) mentions a few studies illustrating this dynamic inequality. Besides different capacity and efficiency of scientists, she refers to some form of feedback mechanism denoted by 'the winner takes all'. Past success in research usually acts as leverage for future productivity. Scientists of considerable reputation have a 'cumulative advantage' in the sense that they will accrue greater increments of recognition for their contributions. This peculiar fact has also been denoted as Matthew Effect in science. Essentially it says 'Once a Nobel Laureate, always a Nobel Laureate'. The Gospel According to St. Matthew puts it this way: 'the Matthew effect consists in the accruing of greater increments of recognition for particular scientific contributions to scientists of considerable repute and the withholding of such recognition from scientists who have not yet made their mark' (compare Lehman, 1954; Merton, 1968). Our model accounts for this effect due to the fact that the stock of reputation has a positive effect on the researcher's output.

Faria and McAdam (2015) show that tenure and promotion of academic specialists are characterized by a bang-bang solution in which the scholar shifts from maximum to minimum effort levels and productivity depending on incentives and impatience. Yegorov et al. (2016) explain the differences in payoffs to talent by analyzing the impact of the initial stock of human capital (which is a productive capital) as well as his or her initial market access (or bargaining powers) within an optimal control framework. McDowell (1982) considers varying learning behavior of professors over their career which may be explained as response to the durability of knowledge. An optimal control model is analyzed but also evidence of obsolescence of knowledge by providing literature decay rates is given. Symonds et al. (2006) show discrepancies in the publication rate between women and men appearing early in their careers.

El Ouardighi et al. (2013) study by means of an optimal control model how individual investments into research and teaching skills subject to a fixed time budget can affect academic careers. Seidl et al. (2016) extend their approach by considering the option to leave academia. To do so, they compare, similar to the approach analyzed in Caulkins et al. (2015), solutions with a free end time (i.e. solutions, where it is optimal to leave academia) to infinite time horizon solutions (i.e. solutions, where it is optimal to stay in academia). They find that the optimal long-run career strategy can be history-dependent, i.e. it depends on the initial skills in research and teaching.

Since the early work of Becker (1962), economists have studied the question how behavior varies over the life cycle in occupations where human capital plays an important role. These models typically produce a peaked profile. First, investment in human capital increases, but at some point, due to finiteness of life, it starts declining. While all life-cycle models for scientists deal with efficient allocation of research time, they differ in the objective functionals of the scientists. Usually it is not the income stream alone which is maximized, but a utility function including among other things, the research output. Levin and Stephan (1991) provide an interesting ansatz in that direction, where our

model can be seen as an extension of the Levin-Stephan model. [Simonton \(1989, 1997\)](#) develops a simple three-state model explaining both longitudinal and cross-sectional variations in the scientific output.

Reviewing life-cycle models until mid of the nineties, [Stephan \(1996\)](#) comes to the conclusion that

*“the human capital approach does not provide the cornerstone on which we should model the behavior of scientists.”*

It does not provide an explanation of why the productivity of a cohort of scientists is unequal over time. The author concludes further

*“that the production of scientific knowledge is far more complex than the human capital model assumes and that these complexities have a great deal to say about patterns that evolve over the life cycle.”*

Our model tries just to take this aspect into account. In particular, by explicitly considering the reputation of a scientist we are able to describe unequal age patterns of scientific production.

[Kanazawa \(2003\)](#) refers to the similarity of the age-pattern and scientific creativity and those of crime. While this might be seen as curiosity we mention the fact that many human behavioral aspects exhibit a one-peaked right-skew age-pattern: both the age-specific first marriage intensity as well as fertility provide typical demographic examples. In his Handbook of Genius, [Simonton \(2014\)](#) provides a rich collection of interesting contributions to creativity in a broader sense. Specifically, see [Jones et al. \(2014\)](#).

The paper is organized as follows. In Section 2 the model is presented including some basic analysis by application of Pontryagin’s maximum principle. In Section 3 we show among other things the existence of multiple equilibria whose basins of attraction are separated by Skiba curves ([Grass et al., 2008](#)), i.e. the effect of initial knowledge is considered by developing history-dependent optimal trajectories. Section 4 shows how the four career patterns recently identified by [Way et al. \(2017\)](#) can be generated as optimal trajectories for appropriate parameter values. In particular, the slump of life satisfaction (‘mid-life crisis’) is discussed in this framework (see [Schwandt, 2016a,b](#)). Section 5 concludes the paper.

## 2. Model

For a scientist investing in knowledge,  $K$ , is the major activity. Such knowledge investments,  $I$ , consists of, e.g., working behind the desk, including developing new ideas, making calculations, and reading. In addition, the scientist has the option to establish a reputation,  $R$ , by networking,  $N$ , which one does, e.g., by making presentations at conferences, talking to colleagues, visiting receptions, inviting colleagues, and writing emails.

The output of a scientist is publishing papers,  $P$ . A necessary condition to do so is having built up a stock of knowledge being strictly positive. Building up reputation can work as a leverage with respect to productivity. To model this we introduce the scientific production function

$$P = P(K, R) = K^\alpha(R + 1)^\beta, \tag{1}$$

with  $\alpha$  and  $\beta$  denoting positive constants smaller than one. The functional form reflects that one can be productive without working on reputation.

We consider a representative scientist, who has the aim to maximize the discounted stream of his scientific publications plus the utility arising from investing in knowledge and networking. An important feature of our model is the fact that doing research and networking usually create utility for a scientist as long as it is done ‘to a reasonable extent.’ Only if  $I$  and  $N$  exceed some thresholds,

these activities are connected with disutilities, i.e. they must be seen as costly. In mathematical terms the following optimization model results:

$$\max_{I(\cdot), N(\cdot)} \int_0^T e^{-rt} (c_0 P(K(t), R(t)) - C_1(I(t)) - C_2(N(t))) dt + e^{-rT} (\kappa_1 K(T) + \kappa_2 R(T)) \quad (2a)$$

$$\text{s.t. } \dot{K}(t) = g(K(t))I(t) - \delta_1 K(t), \quad t \in [0, T] \quad (2b)$$

$$\dot{R}(t) = h(K(t))N(t) - \delta_2 R(t), \quad t \in [0, T] \quad (2c)$$

$$K(0) = K_0 \geq 0, R(0) = R_0 \geq 0 \quad (2d)$$

$$I(t) \geq 0, N(t) \geq 0, \quad t \in [0, T]. \quad (2e)$$

with

$$[0, T] := \begin{cases} [0, T] & \text{if } T < \infty \\ [0, \infty) & \text{otherwise.} \end{cases}$$

Eqs. (2b) and (2c) describe the accumulation of knowledge and reputation dependent on research and network activities, respectively. The functions  $g(\cdot), h(\cdot)$  are assumed to be convex-concave (S-shaped) and  $C_{1,2}(\cdot)$  are linear-quadratic. Specifically we choose

$$g(K) := \frac{a(l_1 + K^\theta)}{1 + K^\theta} \quad (2f)$$

$$h(K) := \frac{e(l_2 + K^\sigma)}{1 + K^\sigma} \quad (2g)$$

$$C_1(I) := d_1 I^2 - c_1 I \quad \text{and} \quad C_2(N) := d_2 N^2 - c_2 N, \quad (2h)$$

where  $g(K)$  is an increasing function reflecting that investing in knowledge is more fruitful if one has already built up some knowledge. Also, working on reputation is much more effective if one is knowledgeable, so that  $h(K)$  is an increasing function as well. Note that the functional forms  $C_1(I)$  and  $C_2(N)$  are such that the scientist enjoys positive marginal utility from knowledge investment and networking as long as  $I$  and  $N$  are not too large.

Depending on the parameter  $\kappa_2$ , we either take the salvage value to be equal to zero ( $\kappa_2 = 0$ ), to reflect that “there are no pockets in a shroud”. One could also argue that taking a salvage value into account is reasonable to reflect that the scientist’s reputation at the end of his career should be valued positively. That would mean formulating a salvage value being increasing in  $R$ , which is achieved by assuming  $\kappa_2$  to be positive.

The model as such is interpretable in a more general sense. It does not only hold for the career of a scientist, but also for an artist, an office worker, a football player, and so on. It also holds for a startup like some “Silicon Valley” firm that first develops an idea, then works to extend it ( $I$  and  $K$ ), but then starts the networking process to find venture capitalists, attracting promising people, etc. ( $N$  and  $R$ ).

We solve the model by applying Pontryagin’s maximum principle (see, e.g. [Grass et al., 2008](#)). This method starts out with setting up the current-value Hamiltonian

$$\mathcal{H} = c_0 K^\alpha (R + 1)^\beta - d_1 I^2 + c_1 I - d_2 N^2 + c_2 N + \lambda_1 \left( \frac{a(l_1 + K^\theta)}{1 + K^\theta} I - \delta_1 K \right) + \lambda_2 \left( \frac{e(l_2 + K(t)^\sigma)}{1 + K^\sigma} N - \delta_2 R \right),$$

from which we obtain that

$$\begin{aligned}\partial_K \mathcal{H} &= c_0 \alpha K^{\alpha-1} (R+1)^\beta + \lambda_1 \left( \frac{aK^{\theta-1}(b\theta - l_1 K)}{(1+K^\theta)^2} I - \delta_1 \right) + \lambda_2 \frac{bK^{\sigma-1}(e\sigma - l_2 K)}{(1+K^\sigma)^2} N \\ \partial_R \mathcal{H} &= c_0 \beta K^\alpha (R+1)^{\beta-1} - \lambda_2 \delta_2 \\ \partial_I \mathcal{H} &= c_1 - 2d_1 I + \lambda_1 \frac{a(l_1 + K^\theta)}{1+K^\theta} \\ \partial_N \mathcal{H} &= c_2 - 2d_2 N + \lambda_2 \frac{e(l_2 + K(t)^\sigma)}{1+K^\sigma}.\end{aligned}$$

Application of the maximum principle generates the following necessary optimality conditions:

$$\begin{aligned}I^* &= \frac{1}{2d_1} \left( c_1 + \lambda_1 \frac{a(l_1 + K^\theta)}{1+K^\theta} \right) \\ N^* &= \frac{1}{2d_2} \left( c_2 + \lambda_2 \frac{e(l_2 + K(t)^\sigma)}{1+K^\sigma} \right).\end{aligned}$$

We only consider the interior control solutions since boundary solutions, i.e.  $I = 0$  or  $N = 0$ , will not occur due to the linear-quadratic cost functions Eq. (2h).

$$\begin{aligned}\dot{K}(t) &= \frac{aK(t)^\theta}{1+K(t)^\theta} I^*(t) - \delta_1 K(t) \\ \dot{R}(t) &= \frac{eK(t)^\sigma}{1+K(t)^\sigma} N^*(t) - \delta_2 R(t) \\ \dot{\lambda}_1(t) &= r\lambda_1(t) - \partial_K \mathcal{H}(t) \\ \dot{\lambda}_2(t) &= r\lambda_2(t) - \partial_R \mathcal{H}(t).\end{aligned}$$

### 3. When does the initial knowledge level matter?

As argued in the Introduction, knowledge investments and networking are more effective if the knowledge level of the scientist is high. For this reason his initial knowledge level, which the scientist developed during his studies preceding his career, can be a crucial factor for career development. So, the question in particular is whether it is realistic to pursue a scientific career in case of a too low initial knowledge level. In this light the next proposition provides information, because it defines scenarios under which a scientific career will not be fruitful in the sense that the long run equilibrium prescribes that both the level of knowledge as well as scientific reputation will vanish.

**Proposition 1.** *The equilibrium at the origin exists for all  $\theta > 1$  and  $\sigma \geq 1$ . For  $\theta = 1$  this equilibrium exists if and only if*

$$\delta_1 - \frac{ac_1}{2d_1} > 0. \quad (3)$$

*Proof.* Appendix A. □

We have chosen the parameter values specified in Table 1 as the base case. Specifically we set  $\theta = \sigma = 1$ , and therefore the equilibrium at the origin only exists if expression (3) is satisfied.

$r$	$a$	$e$	$l_1$	$l_2$	$\theta$	$\sigma$	$\delta_1$	$\delta_2$	$\alpha$	$\beta$	$c_1$	$c_2$	$d_1$	$d_2$	$\kappa_1$	$\kappa_2$
0.04	0.5	0.5	0	0	1	1	0.2	0.1	0.8	0.5	1	1	*	1	0	*

Table 1: The specified parameter values for the (Skiba) base case.

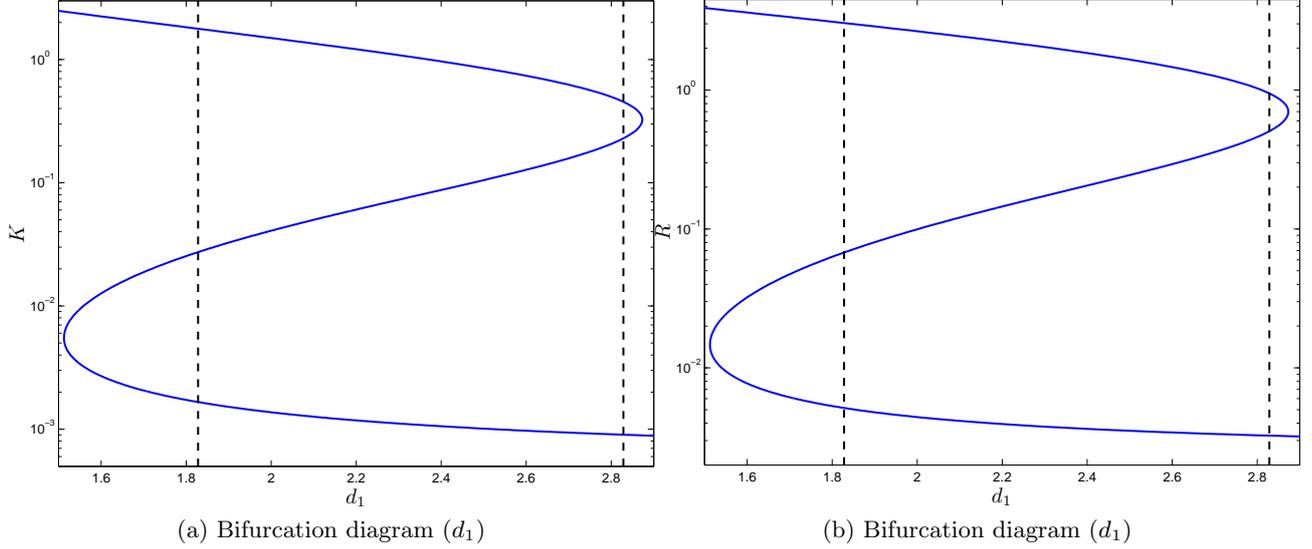


Figure 2: Bifurcation diagram for parameter  $d_1$ . For the region in-between the dashed lines Skiba solutions exist, whereas outside this region the long run equilibrium is unique. For the numerical treatment of the equilibrium at the origin we introduce a small value  $\tau$  (see Appendix A.1). This explains the discrepancy in the positive values shown in the bifurcation diagram and the exact value zero.

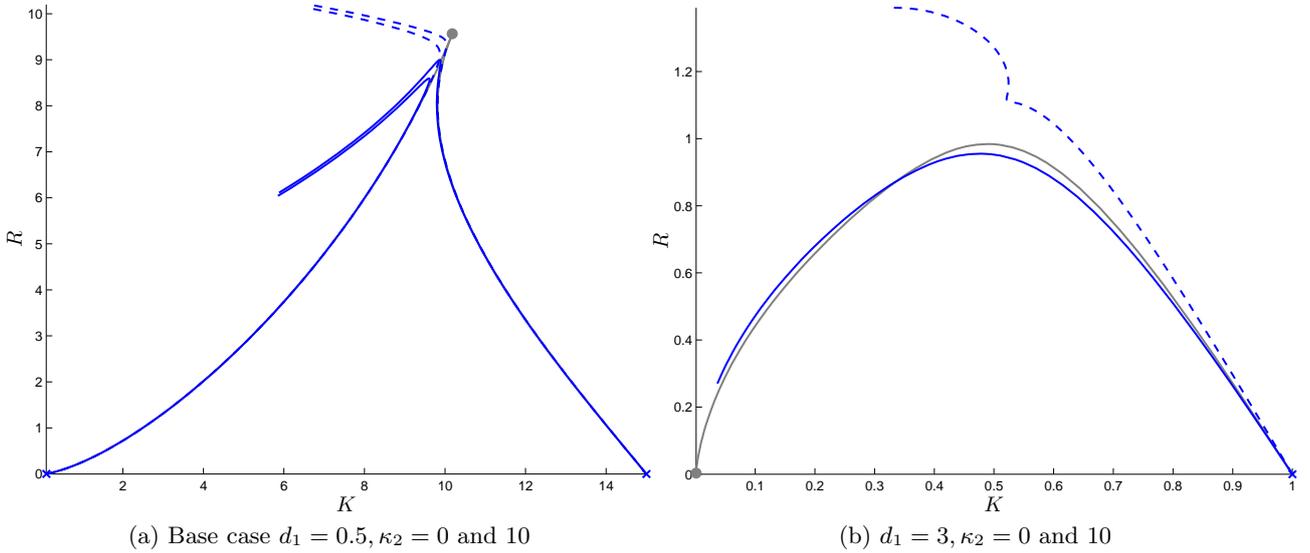


Figure 3: Solution for the base case with different parameter values  $d_1$ . The blue curves correspond to the finite time horizon solutions with  $T = 50$ . The solid curves correspond to  $\kappa_2 = 0$  and the dashed curves to  $\kappa_2 = 10$ . The grey curves depict the infinite time horizon solutions. The crosses denote the initial states of the solution paths. An equilibrium (of the infinite time horizon problem) is depicted by a dot.

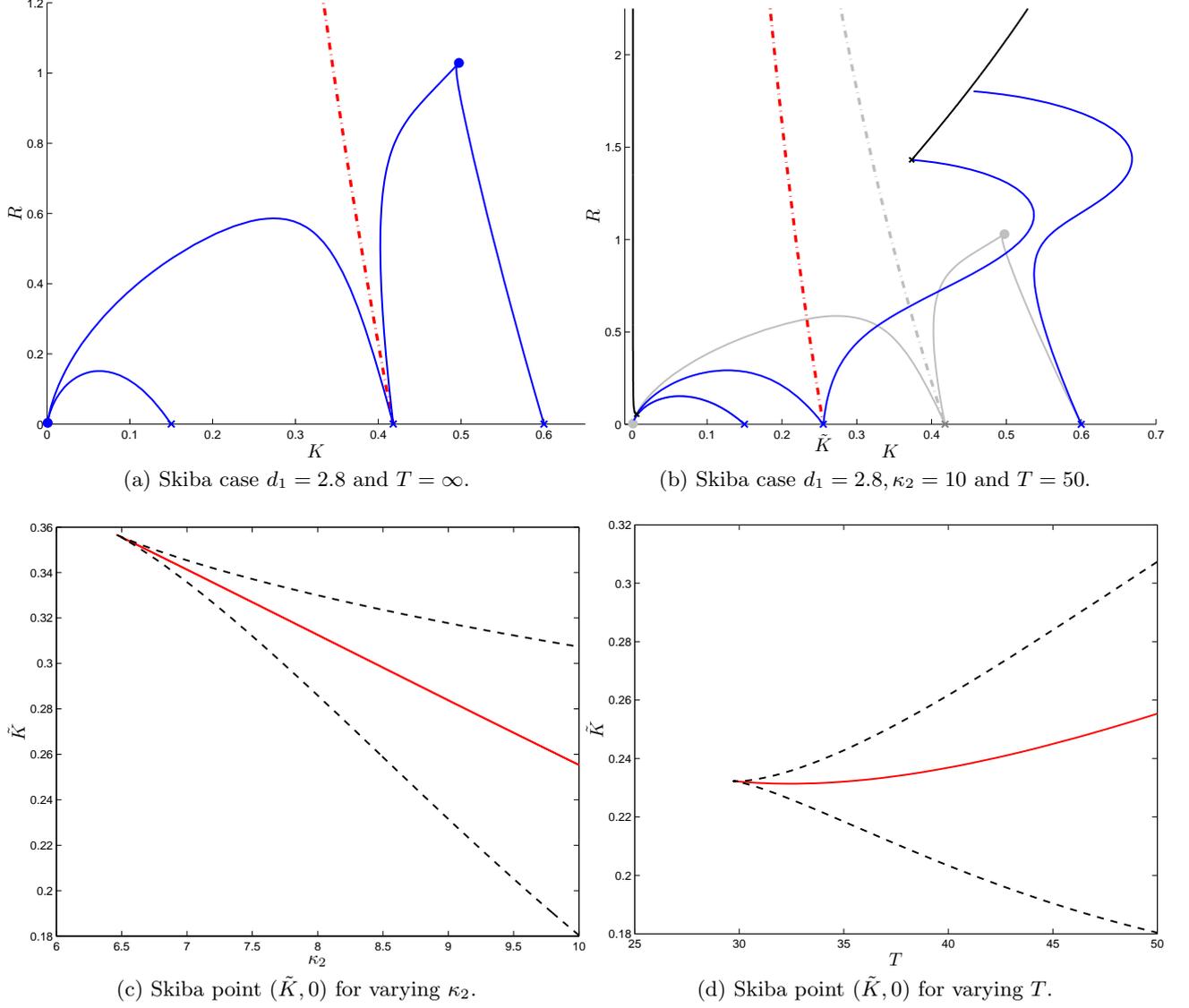


Figure 4: In (a) the Skiba case for the infinite time horizon problem is depicted. The crosses denote the initial states, an equilibrium is represented by a dot. The Skiba curve is given by the red curve. In panel (b) the Skiba solutions for the finite (colored) and infinite time horizon (grey) are depicted. The dashed-dotted curves show the Skiba-curves. Panel (c) shows the change of the Skiba value for  $K$  with fixed  $R = 0$  and varying  $\kappa_2$  (red) curve. The dashed lines enclose the region of initial  $K$  values, where two different solutions exist. Panel (d) shows that a Skiba solution for the finite time horizon only exists for a large enough time horizon, in the actual case  $T \geq 29.73$ . Moreover, the according Skiba point increases for larger values of  $T$ , which is consistent with (b), where the Skiba point of the infinite time horizon problem lies to the right of  $\tilde{K}$ .

Figure 2 shows bifurcation diagrams regarding the possible steady state values of knowledge,  $K$ , and reputation,  $R$ . The bifurcation parameter is  $d_1$ , where an increase of this value raises the cost of investment in knowledge. We first observe that Figure 2 confirms Eq. (3), from which we obtain that for  $d_1 \leq 1.25$  the equilibrium at the origin does not exist, whereas for  $d_1 > 1.25$  this equilibrium exists. In particular, the parameter  $d_1$  specifies how fast marginal utility of the scientist declines, once he decides to invest more in knowledge. A large value of  $d_1$  thus implies it is costly to increase knowledge fast. Therefore, only convergence to the small steady state takes place in such a case, as can be inferred from Figure 2.

Figure 3b shows optimal trajectories for a large value of  $d_1$ , and indeed they end up at the origin. A small  $d_1$  means that the scientist is so clever that it is relatively easy to create knowledge. Figure 3a

depicts trajectories ending up at the larger steady state, where, indeed, the value for  $d_1$  is relatively low. A positive  $\kappa_2$  indicates that at the horizon date reputation is positively valued. The corresponding trajectories therefore end up at larger values of reputation in both Figure 3, where in Figure 3a this happens at the expense of the final value of knowledge that is lower there.

For intermediate levels of  $d_1$  we have a situation with multiple equilibria as Figure 2 shows. Figure 4a depicts the solution of the model if we take an infinite horizon date. Different trajectories are considered starting out from different initial situations. The most reasonable initial situation is such that the reputation of a researcher is low initially, so then  $R(0)$  should be small, i.e. close to zero. Before the career starts, it is very seldom the researcher has build up a network by then. The initial knowledge level  $K(0)$  should also be relatively small, but note that the scientist starts his scientific career after his studies. During his studies the scientist has built up some knowledge, so  $K(0)$  need not be too close to zero.

In Figure 4a a Skiba curve occurs, separating the regions of attraction of the smaller and the larger equilibrium. To the left of the curve we have initial states from which the optimal trajectory ends up at the equilibrium at the origin, and to the right the optimal trajectories converge to the equilibrium with positive knowledge and reputation values. The Skiba phenomenon is caused by the control-state interaction term  $Ig(K)$  in the state equation for knowledge. This term makes that investment in knowledge especially pays off when knowledge is large. Hence, therefore the scientist does not invest enough in knowledge to let it increase when the knowledge level is low.

Since, as argued above, typically  $R(0)$  is close to zero, the point where the Skiba curve intersects with the  $K$ -axis is very relevant, since it determines a threshold value for  $K(0)$ , denoted by  $\tilde{K}$ . The meaning of the threshold value is that for initial knowledge levels above  $\tilde{K}$  the scientist goes for a scientific career that let him end at the upper steady state. If, however, the initial knowledge level is such that it falls below  $\tilde{K}$ , the researcher under consideration will invest more time in other activities than research like, for instance, teaching or administration. The result is that he ends up in the equilibrium at the origin.

Considering the different trajectories we observe that when knowledge  $K$  is low relative to  $R$ , the scientist does not invest too much in networking so that it loses reputation value. On the other hand, the scientist substantially invests in reputation in situations where the knowledge is relatively large. We conclude that  $R$  runs after  $K$ :  $R$  increases (decreases) when it is small (large) compared to  $K$ . The intuition for such behavior is that it especially pays to invest in networking when the knowledge is large, i.e. the reputation investment effectivity function  $h(K)$  is increasing in knowledge  $K$ .

In reality a scientist has no infinite life so it makes most sense to consider a finite time model. Figure 4b shows trajectories where the scientific career encompasses 50 years. The objective function includes a salvage value, which is such that at the end of the scientific life reputation is positively valued. Like with the infinite time solution, also here a Skiba curve can be detected, implying that a sufficiently large initial knowledge level is required to build up a scientific career. Figure 4b shows that the finite time Skiba curve lies to the left of the infinite time one, where having finite time reduces  $\tilde{K}$  from about 0.42 to 0.26. Hence, in more cases it is optimal to build up a scientific career in finite time. This is because of the relatively high valuation of reputation at the end of the scientific lifetime. The latter also causes that the two trajectories to the right of the Skiba curve have in common that at a final time interval reputation increases while knowledge decreases.

The phase portrait in Figure 4b contains two manifolds of end-states that are depicted as black curves. These manifolds are the finite time counterparts to the two equilibria of the infinite time problem in Figure 4a. The manifold corresponding to the low equilibrium is very closely located to the  $R$ -axis. The second manifold has substantially positive knowledge values. The location on such a manifold where a particular trajectory will end, depends on the initial values of knowledge and reputation. The shape of the Skiba curve is such that it intersects the  $K$ -axis, while it approaches the  $R$ -axis asymptotically. This implies that it is not optimal to build up a scientific career if initially the scientist has zero knowledge and very high reputation, which is quite unlikely of course. The reason

is that investments in both knowledge and reputation are not effective with zero knowledge.

Figure 4d shows that the threshold  $\tilde{K}$  increases with the horizon date. This implies that if the scientific career lasts for a longer time, a higher initial knowledge level is needed for building up a fruitful scientific career. The reason is that under this parametrization, the scientist very much appreciates to have a high scientific reputation at the end of his career. The earlier this event takes place the higher the incentive to increase reputation and thus the more attractive it is to invest in a scientific career. The history dependent equilibria separated by the Skiba curve, only exist for values of  $T$  larger than 30 years. For smaller  $T$  building up a scientific career is more attractive, which explains that for  $T$  small enough only convergence to the larger steady state will occur.

That, if at the end of the career reputation is more highly valued, the scientist has a larger incentive to build up a scientific career, is confirmed in Figure 4c. We see that for a larger value of  $\kappa_2$  the threshold is lower, implying that a lower initial knowledge level is needed for starting a trajectory that converges to the larger equilibrium. Moreover, for  $\kappa_2$  small enough only the smaller steady state exists so that the scientist will not invest too much in a scientific career resulting in convergence to the origin, whatever his initial knowledge level is.

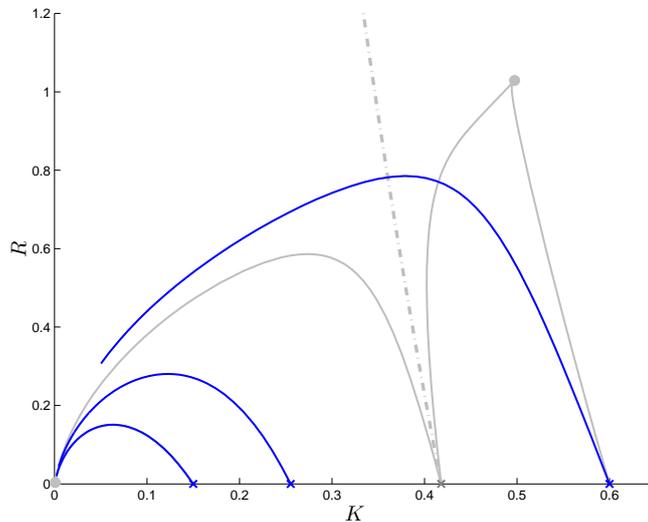


Figure 5: Finite time solution with zero salvage value and  $d_1 = 2.8$ .

Figure 5 shows what happens if the scientist does not care about his reputation at the end of his career, thus when there is no salvage value. Here the scientist is of the opinion that 'there are no pockets in a shroud', so that he assigns no positive value to his reputation at the end of the scientific life. We see that this results in trajectories where the scientist is not interested in building up knowledge, whereas investments in reputation, i.e. networking, will be stopped when the scientist reaches the end of his career.

#### 4. Optimal patterns of scientific production

Using data of computer science departments in North-America, [Way et al. \(2017\)](#) identify four patterns of individuals' scientific productivity, see Figure 6. For simplicity, the authors restrict themselves to two stages in the productivity cycle. They use a piecewise linear model

$$f(t) = \begin{cases} b + m_1 t & 0 \leq t \leq t^* \\ b + m_1 t^* + m_2 (t - t^*) & t > t^* \end{cases} \quad (4)$$

to describe two phases of scientific research. By denoting the slopes of the two phases by  $m_1$  and  $m_2$ , they are able to depict the four cases in the four quadrants  $Q_i$  of the  $(m_1 \ m_2)$ -plane ( $i = 1, \dots, 4$ ).

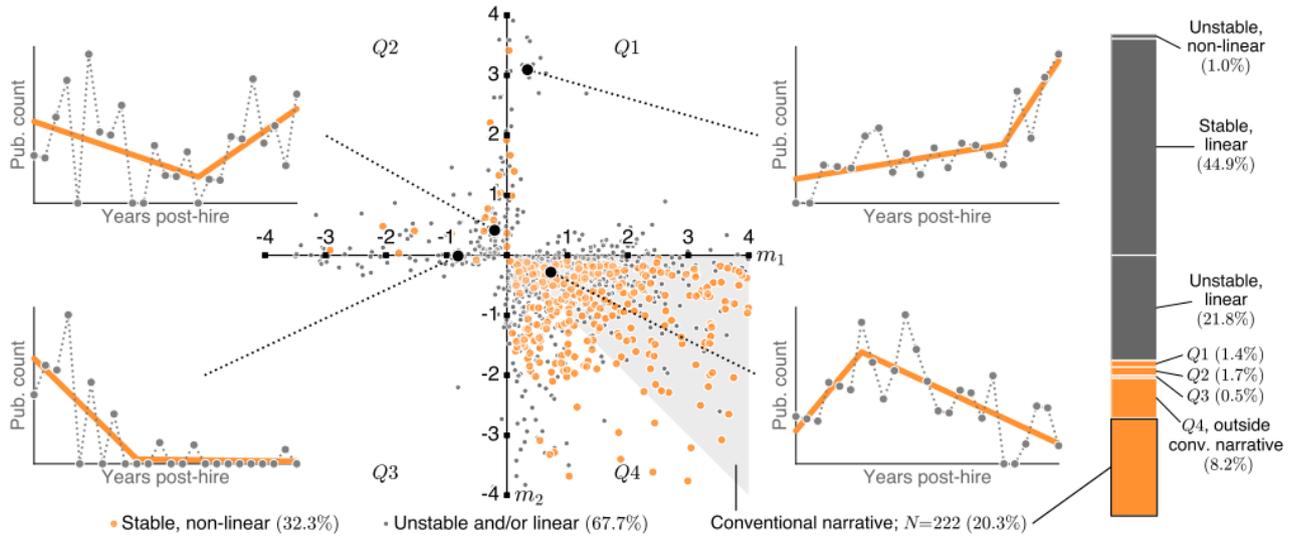


Figure 6: Distribution of individuals' productivity trajectory parameters. Diverse trends in the individual productivity fall into four quadrants based on their slopes  $m_1$  and  $m_2$  in the piecewise linear model Eq. (4). Plots show example publication trajectories to illustrate general characteristics of each quadrant. The shaded triangular region (*Bottom Center*) corresponds to the conventional narrative of early increase followed by gradual decline. Color distinguishes trajectories on two classes: those that are stable and nonlinear (orange) and those that are either unstable or linear (gray). The plot at *Right* describes how researchers are distributed within these two classes. (Source: Way et al., 2017).

Clearly, the most frequent productivity pattern corresponds to  $Q_4$  (positive  $m_1$ , negative  $m_2$ ). However, as Way et al. (2017) claim, rising productivity followed by a declining publication activity is by no means universal. The data of the authors suggest that although this distribution of scientific productivity is indeed the dominant one, two thirds of the population exhibit different productivity patterns. The aim of this section is to illustrate that for each of the four patterns, a parameter scenario can be developed for which our model generates a solution characterizing the associated pattern. All regimes are based on the parameter values depicted in Table 2. To develop the different productivity patterns we vary three different parameters, namely  $d_1$ , indicating the disutility part of knowledge investments,  $\kappa_1$ , being the unit value of knowledge at the end of the scientist's career, and  $\kappa_2$ , being the unit value of reputation at the horizon date. Table 3 gives an overview of their values in the different regimes.

$l_1$	$\theta$	$d_1$	$\kappa_1$	$\kappa_2$
0.25	2	*	*	*

Table 2: The parameter values that changed compared with Table 1.

Regime	$d_1$	$\kappa_1$	$\kappa_2$
$Q_4$ (typical)	3	0	0
$Q_3$ (fading)	6	0	0
$Q_2$ (slump)	5	20	5
$Q_1$ (busy)	5	30	5

Table 3: The values of  $K(0)$ : initial value of the knowledge stock,  $\kappa_1$ : salvage value of  $K(T)$ ,  $\kappa_2$ : salvage value of  $R(T)$ , in the four different regimes.

In what follows we treat the four regimes of Figure 6 in a clockwise manner, starting with the most common life cycle pattern  $Q_4$ . Figure 7 illustrates the case of a first increasing and then decreasing path of scientific productivity, called the *typical* case, see also Figure 1 in the Introduction. In the typical case the scientist starts off investing in his career, resulting in a steady increase of his productivity. Furthermore, a typical scientist is of the opinion that “there are no pockets in a shroud.” This is the reason that at the end of his career he gradually builds off his activities, continuously approaching retirement.

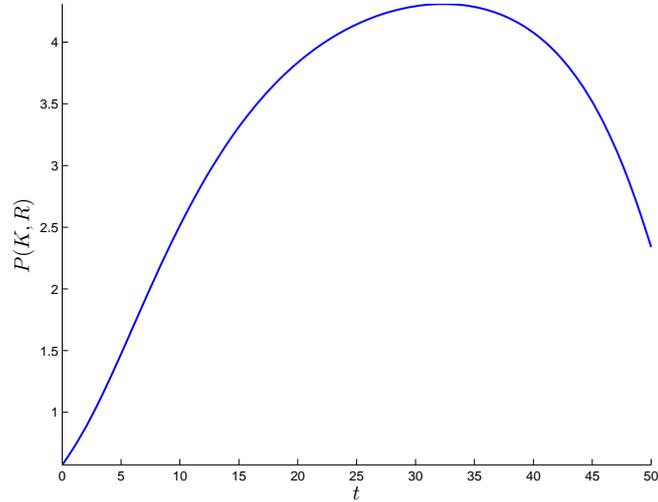


Figure 7: Typical case. Scientific productivity over age. Case of first increasing and then decreasing productivity. The basic parameters are given in Table 2, the varying parameters are specified as  $\kappa_1 = 0$  and  $d_1 = 3$ . The initial state values are  $K(0) = 0.5$  and  $R(0) = 0$ .

The pattern  $Q_3$  depicted in Figure 8 is another distribution which is familiar, called the *fading* case. In the beginning of his career the scientist produces still at a reasonable level, based on the knowledge he obtained during his studies. After getting tenure, the publication rate decreases, converging to zero productivity at the end of his career. Besides being of the opinion of “having no pockets in a shroud,” the fading scientist also experiences heavy disutilities from investing in knowledge ( $d_1$  relatively large), so he gradually reduces his research activities over time.

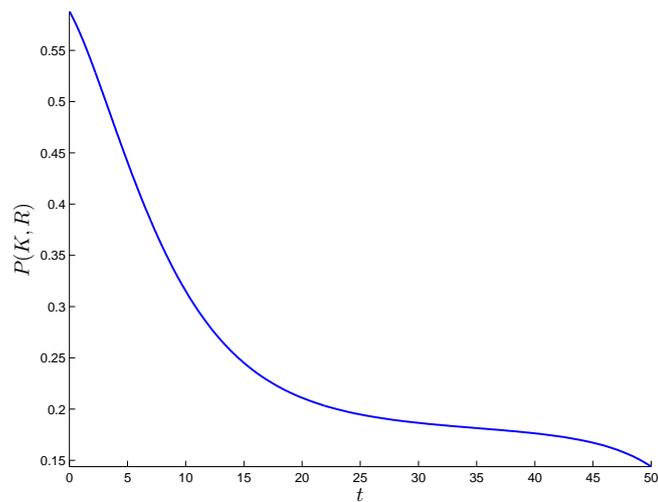


Figure 8: Fading case. Scientific productivity over age. Case of decreasing productivity. The basic parameters are given in Table 2, the varying parameters are specified as  $\kappa_1 = 0$  and  $d_1 = 6$ . The initial state values are  $K(0) = 0.5$  and  $R(0) = 0$ .

The next productivity distribution,  $Q_2$ , shows a *slump* somewhere in the middle of the career (Figure 9). This can typically be caused by a slump of life satisfaction ('midlife crisis'), as discussed in Schwandt (2016a). Towards the end of his career, the "slump" scientist resumes research activities. The reason is that he wants to be remembered because of his knowledge and being a researcher with high reputation. This is reflected in a positive salvage value being increasing in both the final values of knowledge and reputation.

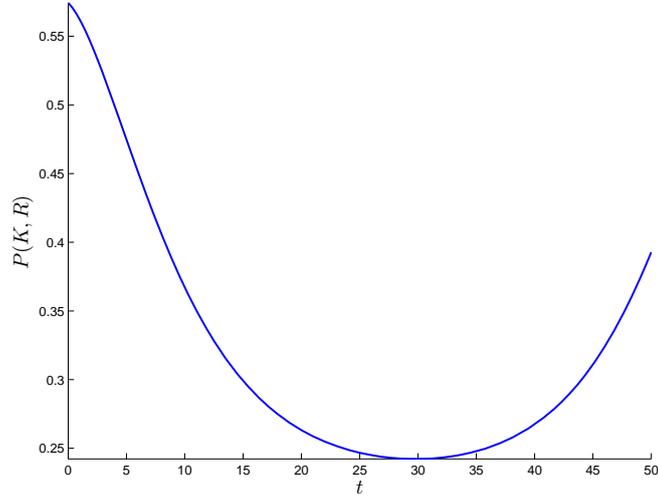


Figure 9: Slump case. Scientific productivity over age. Case of first decreasing and then increasing productivity. The basic parameters are given in Table 2, the varying parameters are specified as  $\kappa_1 = 20$  and  $\kappa_2 = 5$  and  $d_1 = 5$ . The initial state values are  $K(0) = 0.5$  and  $R(0) = 0$ .

The last regime,  $Q_1$ , describes the 'late researcher', called the *busy* case. While at the beginning he produces few papers, he gradually gets his career going, as illustrated in Figure 10. Towards the end he accelerates research intensity because he highly values being knowledgeable and having a high reputation at the end of his career. This is reflected in the relatively high values of  $\kappa_1$  and  $\kappa_2$ .

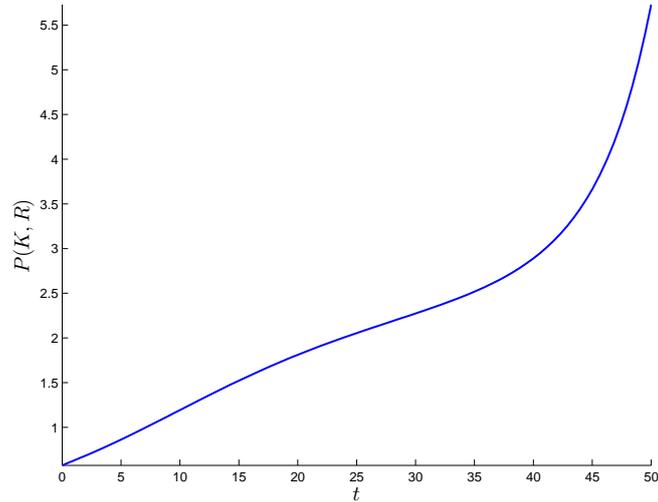


Figure 10: Busy case. Scientific productivity over age. Case of increasing productivity. The basic parameters are given in Table 2, the varying parameters are specified as  $\kappa_1 = 30$  and  $\kappa_2 = 5$  and  $d_1 = 5$ . The initial state values are  $K(0) = 0.5$  and  $R(0) = 0$ .

*The importance of the initial knowledge level regarding research patterns.*

Education matters, especially in how a researcher starts a career. This already became apparent in Section 3 where in several cases a Skiba curve separated career patterns starting out with a high

initial knowledge level resulting in an active career and a relatively low initial knowledge level being not enough to become a productive researcher. At the end of Section 4 we show how the “Skiba approach” sheds light on how a researcher chooses between the different research patterns.

First, consider the two patterns where the researcher does not assign a high value to having a lot of knowledge and reputation at the end of his career, namely the *typical* and the *fading* pattern. Just a little bit more knowledge can result in a *typical* career, rather than a *fading* one, where the *typical* researcher is far more productive. Based on the parameter values of Table 4, Figure 11 illustrates this fact where the two depicted trajectories start out from the threshold value  $\tilde{K}$ . Here the *typical* pattern results for any  $K(0) > \tilde{K}$ , whereas a *fading* career is the result of having  $K(0) < \tilde{K}$ .

$r$	$a$	$e$	$l_1$	$l_2$	$\theta$	$\sigma$	$\delta_1$	$\delta_2$	$\alpha$	$\beta$	$c_1$	$c_2$	$d_1$	$d_2$	$\kappa_1$	$\kappa_2$
0.04	0.5	0.5	0.25	0	2	1	0.2	0.1	0.8	0.5	1	1	4.6198	1	0	0

Table 4: The specified parameter values for Figure 11.

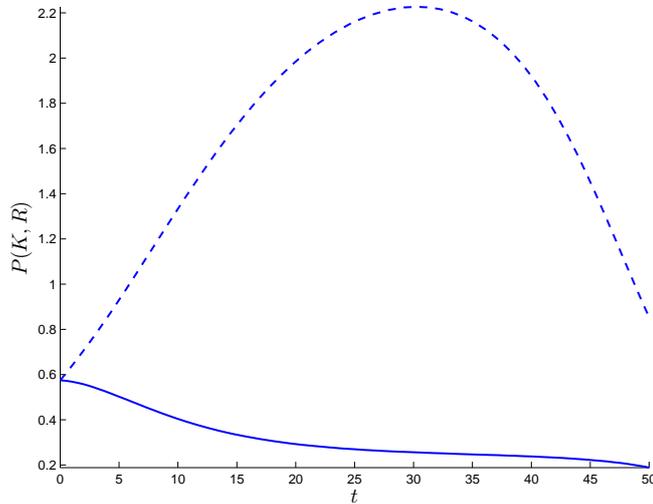


Figure 11: A Skiba solution exhibiting the two regimes  $Q_3$  and  $Q_4$  for  $K(0) = 0.5$  and  $R(0) = 0$ . For the parameter values see Table 4.

Second, we compare the patterns where a high value is assigned to being knowledgeable with high reputation at the end of the career, namely *slump* and *busy*. For such a researcher a *slump* can be avoided if he is well educated. In particular, Figure 12, being based on the parameter scenario of Table 5, again has a threshold value of  $\tilde{K} = 0.5$ , which implies that for  $K(0) > 0.5$  it is optimal to start a *busy* career with increasing research productivity. Knowledge investments and networking are highly effective if the researcher possesses a lot of knowledge, which explains increasing activities during the life of the *busy* researcher. However, in all other cases, thus where education did not result in a high enough initial knowledge level, a *slump* in the career cannot be avoided.

$r$	$a$	$e$	$l_1$	$l_2$	$\theta$	$\sigma$	$\delta_1$	$\delta_2$	$\alpha$	$\beta$	$c_1$	$c_2$	$d_1$	$d_2$	$\kappa_1$	$\kappa_2$
0.04	0.5	0.5	0.25	0	2	1	0.2	0.1	0.8	0.5	1	1	5.254	1	20	5

Table 5: The specified parameter values for Figure 12.

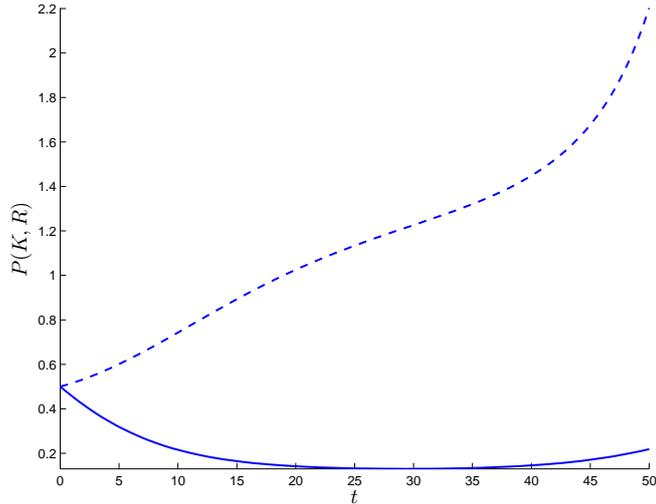


Figure 12: A Skiba solution exhibiting the two regimes  $Q_1$  and  $Q_2$  for  $K(0) = 0.5$  and  $R(0) = 0$ . For the parameter values see Table 5.

## 5. Conclusion

While there exist many papers describing the life cycle behavior of scientific production, in our normative approach we focus on the optimal way in which different age-patterns of publication output comes about. Since the empirical research in [Way et al. \(2017\)](#) we know that four different patterns of scientific productivity can be discerned. In particular, regarding the productivity of the scientist during his scientific life, we can identify patterns called *typical* (increasing and then decreasing productivity), *fading* (decreasing productivity), *slump* (decreasing and then increasing productivity), and *busy* (increasing productivity). The present paper aims to provide a theoretical foundation for this empirical research. To do so, we design an optimization model of the representative scientist. The scientist derives positive utility from publishing papers and from activities like performing research, which we denote by 'investing in knowledge', and networking. However, working too hard causes a disutility, i.e. too large investments in knowledge and networking are costly.

We show that *typical and fading* patterns typically arise in scenarios where the scientist does not assign a too high positive value to being regarded as knowledgeable or having a high reputation at the end of his career. In such a case the scientist will choose for a *typical* pattern if the disutility for hard working is not too high. Here it could help that during his studies the scientist obtained a lot of knowledge. This is because, if the scientist starts his career being knowledgeable, investments in knowledge and networking become more efficient.

If the scientist does assign a substantial positive value to being regarded as knowledgeable with a high reputation at the end of his career, the patterns *slump* and *busy* come into the picture. We show that a *slump* pattern, where the scientist is not very productive halfway his career, can be avoided by high quality education. Again, starting the career with a lot of knowledge make investments in knowledge and networking efficient. This raises productivity along the lifetime, resulting in the *busy* pattern.

Trying to capture some problem in a model by definition means some real life aspects are left out. We did capture that an increase of the scientist's knowledge results in more efficient research activities. What also could have been included is that performing research benefits from the scientist's reputation. If the scientist get stuck somewhere, it is easier to ask a distinguished colleague for help if the scientist himself is of high reputation.

Also, as we have it now the model is autonomous. Another realistic extension would be that the efficiency of accumulating knowledge and reputation explicitly depends on timing, so that effects of aging and learning can be taken into account.

## A. Proof of Proposition 1

For the following analysis we assume  $l_1 = l_2 = 0$ . For  $K = R = 0$  the terms  $K^{\alpha-1}R^\beta$  and  $K^\alpha R^{\beta-1}$  become singular and hence we have to handle diverging costates. Therefore we need to check the transversality conditions

$$\lim_{t \rightarrow \infty} e^{-rt} K(t) \lambda_1(t) = \lim_{t \rightarrow \infty} e^{-rt} R(t) \lambda_2(t) = 0. \quad (\text{A.1})$$

Near the origin and for  $\theta > 1$  and  $\sigma > 1$  the state dynamics is mainly driven by the linear terms and we approximately find

$$K(t) = e^{-\delta_1 t} K(0) \quad (\text{A.2a})$$

$$R(t) = e^{-\delta_2 t} R(0). \quad (\text{A.2b})$$

Plugging these expressions into the costate equations yield

$$\dot{\lambda}_1(t) = (r + \delta_1) \lambda_1(t) - \varphi_1 e^{-(\delta_1(\alpha-1) + \delta_2\beta)t}$$

$$\dot{\lambda}_2(t) = (r + \delta_2) \lambda_2(t) - \varphi_2 e^{-(\delta_2(\beta-1) + \delta_1\alpha)t}$$

with

$$\varphi_1 := \alpha K(0)^{\alpha-1} R(0)^\beta \quad \text{and} \quad \varphi_2 := \beta K(0)^\alpha R(0)^{\beta-1}$$

Solving the ODEs we find

$$\begin{aligned} \lambda_1(t) &= e^{(r+\delta_1)t} \lambda_1(0) - \varphi_1 e^{(r+\delta_1)t} \int_0^t e^{-(r+\delta_1+\delta_1(\alpha-1)+\delta_2\beta)s} ds \\ &= e^{(r+\delta_1)t} \lambda_1(0) + \frac{\varphi_1 e^{(r+\delta_1)t}}{r + \delta_1\alpha + \delta_2\beta} \left( e^{-(r+\delta_1\alpha+\delta_2\beta)t} - 1 \right) \\ \lambda_2(t) &= e^{(r+\delta_2)t} \lambda_2(0) - \varphi_2 e^{(r+\delta_2)t} \int_0^t e^{-(r+\delta_2+\delta_2(\beta-1)+\delta_1\alpha)s} ds \\ &= e^{(r+\delta_2)t} \lambda_2(0) + \frac{\varphi_2 e^{(r+\delta_2)t}}{r + \delta_2\beta + \delta_1\alpha} \left( e^{-(r+\delta_2\beta+\delta_1\alpha)t} - 1 \right). \end{aligned}$$

Hence the transversality conditions are satisfied if the following conditions hold

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-rt} K(t) \lambda_1(t) &= \lim_{t \rightarrow \infty} e^{-rt} e^{-\delta_1 t} e^{(r+\delta_1)t} \left( \lambda_1(0) + \frac{\varphi_1 \left( e^{-(r+\delta_1\alpha+\delta_2\beta)t} - 1 \right)}{r + \delta_1\alpha + \delta_2\beta} \right) \\ &= \lambda_1(0) - \frac{\varphi_1}{r + \delta_1\alpha + \delta_2\beta} = 0 \\ \lim_{t \rightarrow \infty} e^{-rt} R(t) \lambda_2(t) &= \lim_{t \rightarrow \infty} e^{-rt} e^{-\delta_2 t} e^{(r+\delta_2)t} \left( \lambda_2(0) + \frac{\varphi_2 \left( e^{-(r+\delta_2\beta+\delta_1\alpha)t} - 1 \right)}{r + \delta_2\beta + \delta_1\alpha} \right) \\ &= \lambda_2(0) - \frac{\varphi_2}{r + \delta_2\beta + \delta_1\alpha} = 0. \end{aligned}$$

Finally the corresponding time paths are

$$\begin{aligned} \lambda_1(t) &= \frac{\varphi_1 e^{-(\delta_1(\alpha-1) + \delta_2\beta)t}}{r + \delta_1\alpha + \delta_2\beta} \\ \lambda_2(t) &= \frac{\varphi_2 e^{-(\delta_2(\beta-1) + \delta_1\alpha)t}}{r + \delta_2\beta + \delta_1\alpha}. \end{aligned}$$

Plugging the state and costate paths into the expressions for the controls yield

$$I^* = \frac{c_1}{2d_1} \quad \text{and} \quad N^* = \frac{c_2}{2d_2} \quad (\text{A.2c})$$

since

$$\begin{aligned} \lim_{t \rightarrow \infty} K(t)^\theta \lambda_1(t) &= \lim_{t \rightarrow \infty} e^{-(\delta_1 \theta + \delta_1(\alpha-1) + \delta_2 \beta)t} = 0 \\ \lim_{t \rightarrow \infty} R(t)^\sigma \lambda_2(t) &= \lim_{t \rightarrow \infty} e^{-(\delta_2 \sigma + \delta_2(\beta-1) + \delta_1 \alpha)t} = 0. \end{aligned}$$

Finally we consider the case  $\theta = \sigma = 1$ . Again we assume that  $\lim_{t \rightarrow \infty} K(t) = \lim_{t \rightarrow \infty} R(t) = 0$ . We note that the corresponding controls are bounded if  $d_1 > 0$  and  $d_2 > 0$ . Thus  $\lim_{t \rightarrow \infty} N^*(t) = \bar{N}$  and  $\lim_{t \rightarrow \infty} I^*(t) = \bar{I}$ . Then for  $t$  large enough the state dynamics can be written as

$$\dot{K}(t) = \frac{a\bar{I}}{b} K(t) - \delta_1 K(t) = \left( \frac{a\bar{I}}{b} - \delta_1 \right) K(t) = -\xi K(t)$$

with

$$\begin{aligned} \xi &:= \delta_1 - \frac{a\bar{I}}{b} \\ \dot{R}(t) &= \frac{e\bar{N}}{b} K(t) - \delta_2 R(t) = \rho K(t) - \delta_2 R(t) \end{aligned}$$

with

$$\rho := \frac{e\bar{N}}{b}$$

where the higher order terms of the Taylorseries of

$$\frac{1}{b+x} = \frac{1}{b} - \frac{x}{b^2} + \frac{x^2}{2b^3} + \dots$$

were omitted. Subsequently we can assume that  $\xi > 0$ . Then we find

$$\begin{aligned} K(t) &= e^{-\xi t} K(0) \quad (\text{A.3a}) \\ R(t) &= e^{-\delta_2 t} R(0) + e^{-\delta_2 t} \rho K(0) \int_0^t e^{(\delta_2 - \xi)s} ds = e^{-\delta_2 t} R(0) + \frac{\rho K(0) e^{-\xi t}}{\delta_2 - \xi} \end{aligned}$$

We assume  $\delta_2 \neq \xi$ . Therefore either  $\delta_2 > \xi$  or  $\delta_2 < \xi$  is satisfied. First we assume  $\delta_2 < \xi$  then the last equation can be written as

$$R(t) = e^{-\delta_2 t} \left( R(0) + \frac{\rho K(0) e^{(\delta_2 - \xi)t}}{\delta_2 - \xi} \right) \quad (\text{A.3b})$$

then for  $t$  large enough  $R(0) \gg \frac{\rho K(0) e^{(\delta_2 - \xi)t}}{\delta_2 - \xi}$  and

$$R(t) = e^{-\delta_2 t} R(0) \quad (\text{A.3c})$$

For the second case we assume  $\delta_2 > \xi$ . This yields

$$R(t) = e^{-\xi t} \left( R(0) e^{\xi - \delta_2 t} + \frac{\rho K(0)}{\delta_2 - \xi} \right)$$

then for  $t$  large enough  $R(0) \ll \frac{\rho K(0) e^{(\delta_2 - \xi)t}}{\delta_2 - \xi}$  and

$$R(t) = e^{-\xi t} \frac{\rho K(0)}{\delta_2 - \xi}. \quad (\text{A.3d})$$

In both cases Eq. (A.3c) or Eq. (A.3d) the equations together with Eq. (A.3a) are structurally the same as Eqs. (A.2a) and (A.2b) and we can proceed analogously. Moreover from Eq. (A.2c) we find

$$\bar{I} = \frac{c_1}{2d_1} \quad \text{and} \quad \bar{N} = \frac{c_2}{2d_2}.$$

In a last step we consider the case  $\delta_2 = \xi$ . This has only influence on the second costate and therefore yields for Eq. (A.3b)

$$R(t) = e^{-\delta_2 t} R(0) + e^{-\delta_2 t} \rho K(0) t = t e^{-\delta_2 t} \left( \frac{R(0)}{t} + \rho K(0) \right)$$

Thus for  $t$  large enough this equation becomes

$$R(t) = t e^{-\delta_2 t} \rho K(0).$$

Plugging this expression into the second costate equation we find

$$\begin{aligned} \dot{\lambda}_2(t) &= (r + \delta_2) \lambda_2(t) - c_0 \beta K(t)^\alpha R(t)^{\beta-1} \\ &= (r + \delta_2) \lambda_2(t) - c_0 \beta \varphi_1 e^{-\alpha \xi t} t^{\beta-1} e^{-\delta_2(\beta-1)t} \rho K(0) \\ &= (r + \delta_2) \lambda_2(t) - t^{\beta-1} \Psi e^{(\delta_2 - \xi)t} \end{aligned}$$

with

$$\Psi := c_0 \beta \varphi_1 \rho K(0) \quad \text{and} \quad \xi := \alpha \xi + \delta_2 \beta.$$

Therefore the costate is calculated as

$$\lambda_2(t) = e^{(r+\delta_2)t} \lambda_2(0) - e^{(r+\delta_2)t} \Psi \int_0^t s^{\beta-1} e^{-(r+\xi)s} ds \quad (\text{A.4a})$$

To handle the latter integral we resort to the definition of the exponential integral, see e.g. [Temme \(1996\)](#); [Abramowitz and Stegun \(1972\)](#); [Milgram \(1985\)](#). The exponential integral (gamma function)  $\Gamma(\cdot)$  and incomplete gamma function  $\gamma(\cdot, \cdot)$  are defined as

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt \quad \text{and} \quad \gamma(z, t) := \int_0^t s^{z-1} e^{-s} ds$$

Thus the integral in Eq. (A.4a) can be rewritten as

$$\int_0^t s^{\beta-1} e^{-(r+\xi)s} ds = \frac{1}{(r+\xi)^\beta} \int_0^t l^{\beta-1} e^{-l} dl = \frac{\gamma(\beta, t)}{(r+\xi)^\beta}$$

For the transversality condition Eq. (A.1) the following relation has to hold

$$\lim_{t \rightarrow \infty} e^{-rt} R(t) \lambda_2(t) = \lim_{t \rightarrow \infty} t \rho K(0) \left( \lambda_2(0) - \Psi \frac{\gamma(\beta, t)}{(r+\xi)^\beta} \right)$$

Setting  $\lambda_2(0) = \Psi \frac{\Gamma(\beta)}{(r+\xi)^\beta}$  we find

$$\lim_{t \rightarrow \infty} t \rho K(0) \left( \lambda_2(0) - \Psi \frac{\gamma(\beta, t)}{(r+\xi)^\beta} \right) = 0$$

To justify the last identity we apply the rule of de l'Hopital where we use the identity

$$\lim_{t \rightarrow \infty} t^2 t^{\beta-1} e^{-t} = \lim_{t \rightarrow \infty} t^{\beta+1} e^{-t} = 0.$$

Finally we proved that the equilibrium at the origin exists for all  $\theta > 1$  and  $\sigma \geq 1$  and for  $\theta = 1$  this equilibrium exists if and only if the parameters satisfy

$$\xi = \delta_1 - \frac{a\bar{I}}{b} = \delta_1 - \frac{ac_1}{2d_1 b} > 0. \quad (\text{A.5})$$

### A.1. Numerical treatment

For the numerical treatment of the equilibrium at the origin we introduce a parameter  $\tau \ll 1$  in the state dynamics, yielding

$$\dot{K}(t) = \tau + g(K(t))I(t) - \delta_1 K(t), \quad t \in [0, T] \quad (\text{A.6a})$$

$$\dot{R}(t) = \tau + h(K(t))N(t) - \delta_2 R(t), \quad t \in [0, T]. \quad (\text{A.6b})$$

Due to this adaptation the equilibrium at the origin is shifted away from zero and therefore avoids the divergence of the costates.

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