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Growth and collapse of empires: a dynamic optimization model

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Abstract

This paper addresses the spatial evolution of countries accounting for economics, geography and (military) force. Economic activity is spatially distributed following the AK model with the output being split into consumption, investment, transport costs and military (for defense and expansion). The emperor controls the military force subject to the constraints imposed by the economy but also the geography (transport costs, border length) and the necessity to satisfy the needs of the population. The border changes depending on how much pressure the emperor can muster to counter the pressure of neighboring countries. The resulting dynamic process determines a country's size over time. The model leads to multiple steady states, large empires and small countries being separated by a threshold, and collapse. The resulting patterns can be linked to historical observations.

Keywords: dynamic optimization, growth model, empire, geography, defense.
JEL Classification: B52, C61, E02, N0, N40, N90.

1 Introduction

Most of the economic literature and in particular studies on economic growth assume a country of fixed size or territory[1] and (if at all) interacting with the rest of the world by trading goods. However, many countries have a history of evolution, of ups and downs, in their territory as well as in economic performance and population. In particular, some countries (we call them empires) deliberately expanded their territories by conquering less powerful neighboring countries.

While little is known about the economics of different countries and empires in the distant past, we have much more information about the evolution of their territory. In spite of the complex shape and change of borders, there are certain regularities in the evolution of empires and grand theories about them,e.g. Kennedy (1987) on the rise and decline of the United States, Taylor and Flint (2000) who also suggest the temporal evolution of empires, and Spengler (1918, 1922) on the rise and in particular the decline of

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[1]Territory is, however, considered in regional science, starting from the German school (work of von Thünen (1826), followed by Christaller (1933), and recently also in economic geography, see e.g., Fujita, Krugman and Venables (1999) and Puu (1997).

the West and its culture. Historical examples of empires range from the short-lived albeit vast empires amassed by Alexander the Great and Genghis Khan, to long-lasting ones like the kingdoms of ancient Egypt, the Roman empire, the Austro-Hungarian monarchy, and Russia. In contrast to these empires, many countries have stayed and still stay small within their borders, neither expanding nor ending up in another empire, cases in point ranging from the ancient Greek cities to today's Switzerland.

The objective of this paper is to explain this sort of stylized historical evolutions and patterns by accounting for economics, geography and military force ('war') as additional means of politics (according to von Clausewitz (1832): "War is but a continuation of politics by other means")[2] in order to expand or defend an empire. For this purpose we use a simple model of economics, or more precisely, of a stagnant (non-growing) economy with constant population density, which covers even for around 10,000 years of the developed world, from the agricultural to the industrial revolution[3] (Clark 2009), of geography (accounting for transport costs depending on size) and of the politico-economic objectives of a ruler that leads to a dynamic optimization model. Our assumptions about (geographical) space and the economy extend Yegorov's preliminary paper (2018) in which expansion or retreat are determined by the residual expenditure on defense after accounting for subsistence consumption and capital depreciation. To the best of our knowledge, there is no paper in the economics literature that attempts to endogenize the size and evolution (i.e., expansion as well as decline) of empires. And only few papers are loosely related. Grossman and Mendoza (2001) compare an empire's strategies of annexation (coerced or uncoerced) and conquest but lacks the dimensions of space and dynamics, which are crucial in our model; however, we do not differentiate between annexation and conquest. Turchin et al. (2013) use computer simulation to model the historical growth of empires based on agriculture and the ruggedness of the terrain. This model is able to replicate the locations of historical empires. Our goal is complementary as we aim to explain the historical sequence of empire sizes by economic reasoning as well as their coexistence with smaller countries and their temporal evolution. In this respect we contribute to the existing literature by integrating different aspects from different fields such as (i) socio-physics in the derivation of the core equations, (ii) economics determining output and budget constraints, (iii) political economics by solving for an emperor's optimal intertemporal strategy (i.e., solving an optimal control problem), and (iv) history by interpreting our results in terms of historical patterns and episodes. Our dynamic model introduces to the economics of a stationary economy ('agriculture') military and geographical aspects, represented in analytical form in the tradition of the German school of regional science, cf. Christaller (1993). Despite the model's simplicity, it allows for rather complex mathematical outcomes that can be matched with history.

[2]"Der Krieg ist eine bloße Fortsetzung der Politik mit anderen Mitteln" (Clausewitz 1832)
[3]Therefore, colonialism does not fit into our explanation.

2 Model formulation

2.1 Geography

Any empire is characterized by its situation on the map. Therefore, a model without geographic considerations makes no sense. Empires can be, and typically are, represented by complex geometries that are further complicated by geographic specifics such as rivers and mountains at the borders. However, we simplify the geography to a two-dimensional space, more precisely to a square of length R. Thus the territory is $S = R^2$, which is uniformly populated and the initial size of the empire is given, $S(0) = S_0$. The length of the border is given by the perimeter of the square, $L = 4R$.

2.2 Economy

Production is dispersed in a structured space, like in Yegorov (2005, 2009, 2011). An empire produces a composite good, $Y = \tilde{A}K$ with capital, K, as the only production factor. This output can be used for consumption (C), investment (I), transport costs (T), and military expenditures W (for 'war') to expand or defend the existing territory,

$$Y = C + I + W + T. \tag{1}$$

While the huge body of literature on economic growth—classical as well as endogenous growth (compare Barro and Sala-i-Martin (1995))—focuses on the accumulation of capital (physical and/or human), we consider a stagnant economy, which was characteristic for the world economy prior to the industrial revolution. Therefore, investment must replace the depreciation of the existing capital stock, $I = \delta K$, and in the following we work with the net productivity of capital,

$$A := \tilde{A} - \delta. \tag{2}$$

2.3 Border dynamics

Military activity takes place at the (possibly shifting) borders of a country, at least until the 20$^{\text{th}}$ century, when airplanes entered wars, not to mention long-range missiles, nuclear weapons and meanwhile also unmanned aerial vehicles (drones). The outward pressure on each kilometer of the border is $P = W/(4R)$. We assume that the speed of expansion (V) is determined by the difference in military strength, or more precisely, that it is proportional to the difference in pressure on either side of the border,

$$V = \alpha(P - P_0),$$

where α is a constant coefficient and P_0 is the external and exogenous pressure exercised by the empire's neighbors. Therefore, the gained territory per unit of time is $dS = LVdt$

and the equation of territorial expansion becomes[4]:

$$\frac{dS}{dt} = \alpha W - \alpha P_0 L. \tag{3}$$

2.4 Transport costs

Assuming a sufficiently strong economy and an emperor who is only constrained by providing subsistence consumption (C_0) to his population, the military expenditures, $W = AK - C_0$, would allow for exerting pressure on the borders, thus theoretically leading to unbounded growth. The reason for this is that the military expenditures are quadratic in R, $W = wR^2$ where w denotes defense expenditures per unit of territory, while the opposing pressure at the border is proportional to $4R$. However, all empires in history have faced their limits, even the mighty Romans could not proceed further north in Europe against the counterpressure of the Germanic tribes. A crucial limiting aspect is that size increases transport costs as a large empire defending/expanding its border must keep shipping goods to it. Since the average distance is proportional to R, the volume of transported resources is proportional to total output and thus to R^2, the total transport costs (T) are proportional to R^3. Therefore,

$$T = \beta R^3,$$

in the split-up of the aggregate output according to (1)[5]. Aside from these geometric motivations, 'transport' costs may include other costs related to size such as an increasingly heterogeneous population as the empire expands. For example, annexing areas populated by people of different ethnicities will decrease the likelihood of loyalty to the emperor, in particular with respect to defending the borders. This was the case with the Austro-Hungarian empire when the rise of Panslawism in the 19^{th} century eroded the loyalty of the Slavic people to the House of Habsburg, culminating in the assassination of archduke Franz Ferdinand by a panslawist (Gavrilo Princip in Sarajevo).

2.5 Dynamic evolution of empires

Accounting for output and expenditures for investment, consumption and transport with the residual going to the military, the pressure exercised per unit length of the border

[4]There exist mathematical models of military actions: for example, Lanchester's (1916) laws about relative military strength and Washburn and Kress (2009). The law of von Clausewitz (1832) about the triple advantage for a successful attack is tactically correct, but here we want to model the long-term average territorial gain via the difference in military potentials on the border. Formally our law is similar to the physical law of oil moving in a pipeline; the speed of the viscous fluid is proportional to the difference in pressures.

[5]A cubic relationship also follows when computing the transport costs from all uniformly distributed points of the square to the center (the origin in $\Re^2$ or respectively from the center to each point of the square) assuming the Euclidean distance,

$$\int_{-R/2}^{R/2}\int_{-R/2}^{R/2} \sqrt{x^2+y^2}\,dxdy = const\ R^3.$$

$(4R)$ is

$$P = W/4R = (AK - C - T)/4R.$$

The expansion (without differentiating between annexing and conquering) is determined by the pressure difference at the border according to (3). Differentiating, accounting for $\dot{S} = 2R\dot{R}$, for (3), and substituting the above expression for P yields,

$$\dot{R} = \frac{\alpha\,(AK - C - T)}{2R} - 2\alpha P_0. \tag{4}$$

Since the military expenditure is the residual in the accounting in (1), substituting the (relative) military expenditure (w),

$$AkR^2 - cR^2 - \beta R^3 = wR^2 \Longrightarrow Ak - c - \beta R = w, \tag{5}$$

yields

$$\dot{R} = \frac{\alpha}{2}Rw - 2\alpha P_0 \leq \frac{\alpha}{2}\Big(R\,(Ak - \beta R) - 4P_0\Big). \tag{6}$$

This equation implies that neither small countries nor large empires can be sustained. Small countries cannot muster the resources to counter the external pressure whereas large empires have to spend too many resources on transport and thus lack the money for the army necessary to defend the large border. The right-hand side in (6) is the upper bound on expansion based on the presumption that all output except for replacing capital and transport is given to military. Therefore, the quadratic equation on the right-hand side determines the maximum feasible rate of expansion, which must be positive for some values of R in order to sustain at least empires of limited size. This requires a sufficiently strong economy facing external pressure, P_0. If this condition (of a positive discriminant in the root below) is satisfied, then two positive roots,

$$R_{1,2} = \frac{Ak \pm \sqrt{(Ak)^2 - 16\beta P_0}}{2\beta}, \tag{7}$$

exist and any sustainable interior, $R > 0$, steady state (equilibrium) must be between these bounds. Therefore, any empire starting at $R_0 < R_1$ must end in its own dissolution, i.e., wind up as a part of its neighbor(s); since the above criterion is sufficient, the domain of unsustainable empires could and presumably will include $R_0 < R_1$. All large empires with $R_0 > R_2$ must collapse to smaller ones since not even their transport costs can be covered; this can also be in a discontinuous manner as in the case of Russia selling Alaska to the United States.

2.6 An emperor's objective

Suppose an emperor cares not only about the territory (and thus population size), but also about the utility of consumption of each citizen; or alternatively viewed in a less benevolent interpretation: a starving population is a threat to any ruler and only offering *panem et circenses* will keep people quiet, so the more the safer for the emperor. In short, we assume that the emperor maximizes the intertemporal utility

$$\max \int_0^\infty e^{-rt}\,U\,(c, S)\,dt, \tag{8}$$

i.e., trades off between per-capita consumption (given by c) and the size of the empire, $S = R^2$.

A crucial and robust feature of the model is the existence of multiple interior steady states within the critical bounds (R_1, R_2), irrespective of the detailed specification of the preferences. However, this specification matters for the comparison of (i) the limiting case of $R \to 0$, in which case the emperor must, or finds it optimal, to surrender, with (ii) the alternative of keeping a small country. For example, when considering an additive utility in per-capita consumption and size,

$$U = \ln(c) + \sigma S, \tag{9}$$

the boundary solution, $R \to 0$, will always dominate the alternative and feasible strategy of defending a small country. The reason is that a small country yields only little in terms of grandness but lowers the per-capita consumption in order to cover defense and transport costs. Imposing Inada conditions for size, e.g., either separable $U = \ln(c) + \sigma \ln(S)$ or multiplicative/Cobb Douglas, $U = c^{\theta} S^{1-\theta}$, has desirable limit properties and allows for multiple steady states too. Therefore, the specification of the preferences is not crucial for the existence of interior steady states, but the limiting property of U for $R \to 0$ determines either whether an emperor finds a small country worth defending or prefers to surrender or whether there exists another threshold level of R at which to choose between those two possibilities. For example, one may specify the value at $R = 0$ exogenously, ranging from minus infinity (if losing a war means getting killed—Saddam Hussein and Muammar Gaddafi being recent examples) to a finite number, presumably even positive, for an emperor like the Dalai Lama who cares about his people even when in exile. The specification of this value determines then the region of attraction for $R = 0$.

As a consequence, the following discussion is restricted to interior steady states with the understanding that the limiting steady state $R = 0$ is also a possible outcome. However, this possibility is not further explored, because its existence, the range of its attraction and the speed of convergence (in finite or infinite time) are fully determined by the specification of the preferences and their properties for $R \to 0$, while the study of the interior steady states is robust across different specifications.

3 An emperor's optimal strategy

Small letters denote the relative magnitudes (per size and capita). Therefore $k_0 = K_0/S$ is the capital density per unit of land occupied by one person so that $c = C/S$ is the per-capita consumption and equal to,

$$c = Ak_0 - w - \beta R, \tag{10}$$

in which $w = W/S$ is the defense density (also per unit of land and thus per capita), which of course must be non-negative. This allows to eliminate c from the objective so that the emperor has to solve the following optimal control problem for the specification[6]

[6]The convex term in the objective function has been known since Skiba (1978) to be crucial (but neither necessary nor sufficient, compare Hartl et al. (2004)) for thresholds and multiple long-run outcomes. However, the precise specification of the emperor's preference with respect to size is not crucial

in (9):

$$\max_{w(t)\geq 0} \quad \int_0^\infty e^{-rt}\Big[\ln\Big(Ak_0 - w(t) - \beta R(t)\Big) + \sigma R(t)^2\Big]dt \tag{11}$$

$$\text{s.t.:} \quad \dot{R}(t) = \frac{\alpha}{2}w(t)R(t) - 2\alpha P_0 \tag{12}$$

3.1 Necessary optimality conditions

Defining the current-value Hamiltonian for the above model,

$$\mathcal{H}\big(R(t), w(t), \lambda(t)\big) = \ln\Big(Ak_0 - w(t) - \beta R(t)\Big) + \sigma R(t)^2 + \\ + \lambda(t)\Big(\frac{\alpha}{2}w(t)R(t) - 2\alpha P_0\Big),$$

and focusing on the interior solutions, i.e., $w > 0$ and $R \in (R_1, R_2)$[7], the first order optimality conditions are: the Hamiltionian maximizing condition, $\partial H/\partial w = 0$ since $\partial^2 H/\partial w^2 < 0$, implies for the control (omitting from now on the argument t),

$$w = Ak_0 - \beta R - \frac{2}{\lambda \alpha R}, \tag{13}$$

and the costate (λ) must satisfy the following differential equation,

$$\dot{\lambda} = r\lambda + \frac{\beta}{Ak_0 - w - \beta R} - 2\sigma R - \frac{\alpha}{2}w\lambda. \tag{14}$$

Combining (12), (13) and (14) yields the following canonical equations system with the corresponding limiting transversality condition,

$$\dot{R} = \frac{\alpha}{2}Ak_0 R - 2\alpha P_0 - \frac{\alpha\beta}{2}R^2 - \frac{1}{\lambda}, \; R(0) = R_0, \tag{15}$$

$$\dot{\lambda} = \Big(r - \frac{\alpha}{2}Ak_0\Big)\lambda + \alpha\beta\lambda R - 2\sigma R + \frac{1}{R}, \quad \lim_{t\to\infty} e^{-rt}\lambda(t)R(t) = 0. \tag{16}$$

3.2 Steady states

Solving $\dot{R} = 0$ for the costate,

$$\lambda = \frac{2}{\alpha Ak_0 R - \alpha\beta R^2 - 4\alpha P_0},$$

and substituting this into (16) yields the following fourth-order polynomial

$$\alpha\beta\sigma R^4 - \alpha\sigma Ak_0 R^3 + \Big(\frac{\alpha\beta}{2} + 4\alpha\sigma P_0\Big)R^2 + rR - 2\alpha P_0 \overset{!}{=} 0. \tag{17}$$

(as argued in the previous section). Indeed, the same (and even quantitatively similar) results hold, e.g., for a square root appreciation of the size that obeys the law of diminishing returns and implies a term linear in R; details are reported in the Appendix.

[7] Therefore, the state constraint, $R \geq 0$ corresponding to the non-negativity of the distance, can be ignored.

The roots from the interval (R_1, R_2) determine the steady states of R.

The Jacobian matrix of the above canonical system,

$$\mathcal{J} = \begin{pmatrix} \frac{\partial \dot{R}}{\partial R} & \frac{\partial \dot{R}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial R} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} A k_0 - \alpha\beta R & \frac{1}{\lambda^2} \\ \alpha\beta\lambda - 2\sigma - \frac{1}{R^2} & r - \frac{\alpha}{2} A k_0 + \alpha\beta R \end{pmatrix}, \tag{18}$$

has the eigenvalues,

$$\zeta_{1,2} = \frac{r \pm \sqrt{r^2 - 4\Delta}}{2},$$

since $\mathrm{tr}(\mathcal{J}) = r$ and in which Δ is the determinant. Therefore, one eigenvalue is for sure positive and both will be positive (or have positive real parts) iff $\Delta > 0$, while $\Delta < 0$ implies (saddlepoint) stability.

Remark 1. Although our focus is on interior equilibria as argued in 2.6, i.e., the roots of the 4^{th} order polynomial in (17) from (R_1, R_2), there is another possible long-run outcome, namely paths that converge asymptotically to the left-hand boundary, $R \to R_1$ with $c \to 0$ and $\lambda \to -\infty$ as the $\dot{\lambda} = 0$ isocline has a pole for $R \to R_1$, see (18). In this case, $R \to R_1$, as well as in the ignored case $R \to 0$, the interpretation is that this empire is bound to vanish from the map because it must collapse for any small push to $R_1 - \varepsilon$ (e.g., due to a local revolt or some skirmish at the border) as the budget constraints cannot be matched by non-negative components. This is nevertheless a highly relevant outcome, because history is littered with examples of countries and empires removed from the map by ending up in other empires (annexed or conquered).

4 A base case

Our unit of time is one year, the unit of distance is $R = 1$ =1000 km and the (gross) output per unit of territory (1 million km^2) is normalized, $\tilde{A}k_0 = 1$ and $k_0 = 1$. Agricultural production dominated before the industrial revolution and continued to do so in vast areas on the globe until the 20^{th} century. Geography determines crop variety and affects productivity. Depreciation accounts for the share of the harvest that is needed for seeds and the fraction of domestic animals (cows, sheep, pigs, etc.) that cannot be consumed because they must be kept for reproduction. Knowing that this parameter can vary across regions and technologies (e.g., higher for livestock, smaller for producing grain), we assume $\delta = 0.2$. Hence, $A = 0.8$.

Due to our normalization of per-capita GDP, w denotes the share of military expenditures. Assuming that the external pressure must be balanced along the border, then $4RP_0 = wR^2$ implies $w = 4P_0$ for $R = 1$, i.e., four times the external pressure on the border. Choosing $P_0 = 0.03$ implies that 12% of the GDP are needed to defend a country with $R = 1$.

Starting again from the unit square, $R = 1$, then $\beta = 0.2$ implies that 20% of the gross output must be spent on transport and this share grows with size up to the point at which transport costs equal the net output, rendering such large empires unsustainable (and in fact infeasible to reach, i.e., choosing an even higher initial condition, $R_0 > R_2$, requires either more favorable parameters, e.g., the power and charisma of leaders such

A	k_0	P_0	r	α	β	σ
0.8	1	0.03	0.03	0.1	0.2	1

Table 1: Parameters of the base case.

as Alexander the Great, Genghis Khan, and Napoleon, or a temporary lack of opposing pressure, or simply luck).

The parameter α determines the speed of expansion (in thousand kilometers per annum) if $P - P_0 = 1$. We set $\alpha = 0.1$, which requires a military budget of 40% of the GDP in order to defend a country of size $R = 1$. Such a country will expand at the speed of 100 km per year if facing no defense from neighbors ($P_0 = 0$), which is compatible with historical examples: between 1500 and 1550 the Spanish empire started to conquer Mexico and Peru (an expansion by more than 2000 km on each of the two fronts) and Russia expanded into Siberia by about 3000 km between 1590 and 1650, or 30 km per year at the price of high shares of public expenditure going to the military.

In summary, a (calibrated) parametrization of the base case can be found in Table 1.

5 Properties of steady states and bifurcation analyses

5.1 Optimal vector field

In the remainder of the paper, the following notation is used: R_L, R^H and $\tilde{R}$ denote the low, high and unstable steady states, respectively, R_S stands for the (Skiba) threshold, R_1 and R_2 symbolize the left and right boundaries of sustainable empires given by equation (7) and R_S^L, R_S^H represent the low and high boundaries of the Skiba region. In the following, we introduce the *optimal vector field*; see Kiseleva and Wagener (2010, 2015) for a strict mathematical formulation.

Optimal solutions, i.e., $(R^*(\cdot), w^*(\cdot)))$, can be found among the solutions of the canonical system (15) and (16) satisfying the initial and the limiting transversality conditions. The corresponding solution paths converge to the steady states $R^H, R_L, \tilde{R}$ and the boundary steady state at R_1, or they stay put if initially starting at a steady state. If multiple solution paths exist, the optimal path must be determined by comparing their objective values (11).

Figure 1 shows the solution for the base case. This solution exhibits a so-called Skiba point R_S, cf. Skiba (1978), following the seminal works of Skiba (1978), Dechert and Nishimura (1983), Sethi (1977) and Sethi (1979), cf. Grass et al. (2008) and Grass (2012)), or mathematically more precise, an indifference threshold point, cf. Kiseleva and Wagener (2010, 2015). Solutions starting at this point have the same objective value for both paths converging to the low steady state R_L and the high steady state R^H.

This example is used for the motivation of the optimal vector field. For each state $R \neq R_S$, there exists a unique optimal control value $\omega^*(R)$, called the *optimal policy rule.* For R_S the optimal policy rule is not unique, $\omega^*(R_S) = \{w_1, w_2\}$ with the two controls w_1, w_2 lying on the two solution paths. Plugging the optimal policy rule into the state dynamics (12) defines an autonomous, but possibly multi-valued ordinary differential equation (ODE),

$$\dot{R}(t) = \frac{\alpha}{2}\omega^*(R(t))R(t) - 2\alpha P_0. \tag{19}$$

This ODE is called the *optimal vector field*. An optimal saddle of the canonical system corresponds to a (locally) stable equilibrium of ODE (19) and an optimal unstable node corresponds to an unstable steady state. At a Skiba point, the optimal vector field is multi-valued.

Distinguishing between the canonical system and the optimal vector field should help to better understand possible differences in the bifurcations of both ODEs. For example, a saddle-node bifurcation in the canonical system, with a new saddle and node appearing, may have no influence on the optimal vector field, since the according paths converging to this new saddle may be inferior to the solution converging to the other saddle. Conversely, the appearance of a Skiba point in the optimal vector field does not entail a bifurcation in the canonical system. Last but not least, for ODE (19) the usual terminology of stability can be applied and there is no confusion about a stable saddle.

Curves/lines that concern the optimal vector field are represented as solid lines, whereas curves related only to the steady states of the canonical system, but not appearing in the optimal vector field, are plotted as dashed lines.

5.2 Tipping behavior

What are the policy implications? Consider a large empire R^H and a small country R_L, both locally stable, i.e., starting with $R(0)$ in the corresponding neighborhood, the optimal policy is to converge to the nearby steady state. At the Skiba point, $R_S = 0.61$ (for the base case), the emperor is indifferent between choosing either the route to the small country or the large empire. However, this choice requires very different policies in terms of military expenditures, zero for R_L but around 35% for R^H and more as the empire grows. Since the threshold level is relatively close to the low stable steady state (R_L), the large empire has a much larger region of attraction.

It is interesting to compare the shares (= per capita/area since GDP per capita is independent of the size) of consumption, defense, and transport costs across the different steady-state levels: for the small country, $R_L = 0.19$, $c = 0.13, w = 0.65$ and 0.02 for transport due to its small size; for the large empire, $R^H = 3.58$, $c = 0.05, w = 0.05$, and 0.39 for transport. In this example, consumption is much greater in the small country (which, incidentally, applies even today for Luxembourg, Switzerland, and also to Qatar, the Emirates, and Kuwait, in their case due to the availability of hydrocarbon resources), and although defense expenditures are substantial (again Switzerland in the past—and maybe even today—comes to mind), the cost for transport is insignificant. Large empires, on the other hand, have to spend higher shares on transport than on defense, because T/Y grows with respect to size R, while W/Y declines. However, the impression that small countries actually enjoy higher consumption is not true because the share going to consumption is in many examples non-monotonic with respect to size so that the population of larger empires can also enjoy higher consumption. E.g., setting $\beta = 0.3$ yields a qualitatively similar behavior ($R^H = 1.985$ and $R_L = 0.185$) in which the people of the large empire enjoy a higher consumption.

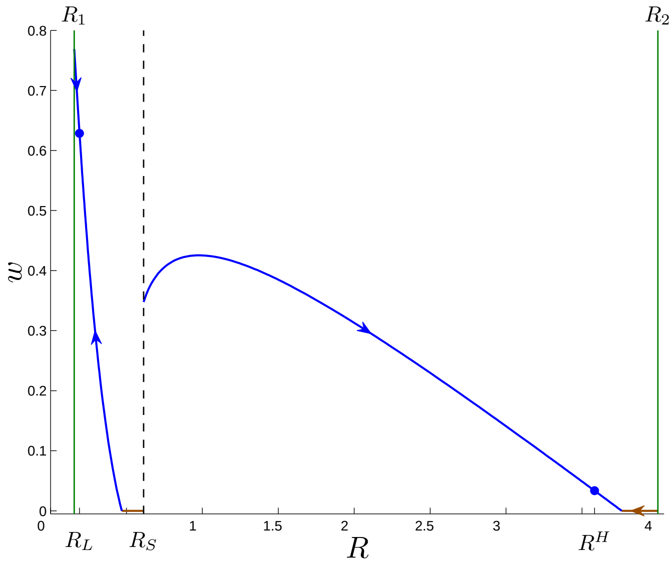


Figure 1: Optimal strategies (blue) and corresponding (Skiba) threshold in the phase space (of state and control) separating the low (R_L) from the high (R^H) equilibrium for the base case parametrization from Table 1. The vertical lines correspond to $R = R_1$ and $R = R_2$. The dashed vertical line is the Skiba threshold R_S.

The qualitative behavior remains the same in the neighborhood of the parameter values specified in Table 1. Figures 6 and 4 numerically specify the neighborhood for the parameter values r, σ, P_0 and β. The Skiba regions are enclosed between the two straight lines denoting a heteroclinic and saddle-node bifurcation for the canonical system. Both bifurcations are explained below.

5.3 From a large (small) country to the coexistence of an empire and a small country

Almost all applied papers following the route of Skiba (1978) are confined to qualitative sketches of phase diagrams in order to show[8]: (i) that multiple steady states exist, (ii) of which the interior is unstable (and implicitly assumed to be spiral, which need not be the case), and (iii) that this combination must lead to a threshold. From there, all policy conclusions are drawn highlighting the sensitivity with respect to initial conditions and the associated jump in the control. This reasoning is insufficient, because one branch can be, and in many cases (also in our model) is, the only optimal one. Instead of the local and geometric analysis, a global one is necessary, and Wagener (2003), motivated by the shallow-lake example, seems to have been the first to check explicitly for the optimality of different paths. The purely geometric argument has at least two additional shortcomings. First, it ignores that one must start with a system that leads to a unique steady state (of the optimal vector field), because given the existence of a (Skiba) threshold it is impossible to reach the high steady state when starting from natural initial conditions, here those of a small country. Only a subsequent change of model parameters can introduce a threshold and the corresponding sensitivity with respect to initial conditions. Therefore,

[8]e.g., Brock (1983), Brock and Dechert (1985), Dechert (1984) are early, and indeed the first, representations of the shallow-lake model, e.g., Mäler (2000), Mäler et al. (2003), are later examples.

understanding the transition from a globally stable large empire (or small country, see 5.3.3) to the coexistence of an empire and a small country is very important.

5.3.1 Continuous policy function

The second fault of an analysis restricted to qualitative diagrams is that they ignore the possibility of continuous policy functions that pass through the unstable steady states serving simultaneously as thresholds (compare Hartl et al. (2004) in a convex-concave dynamic optimization problem, and Wirl and Feichtinger (2005) in various concave cases). Hence, the unstable steady states can be part of the optimal policy function (and indeed are an, albeit unstable, steady state, e.g., for $\beta = 0.14$ and $P_0 = 0.11$, for further details see the online Appendix) with the consequence that the military expenditures w are continuous and differentiable at the threshold level, i.e., in contrast to Figure 1 no abrupt policy change exists between the two saddle point paths towards either the high or low long-run outcome.

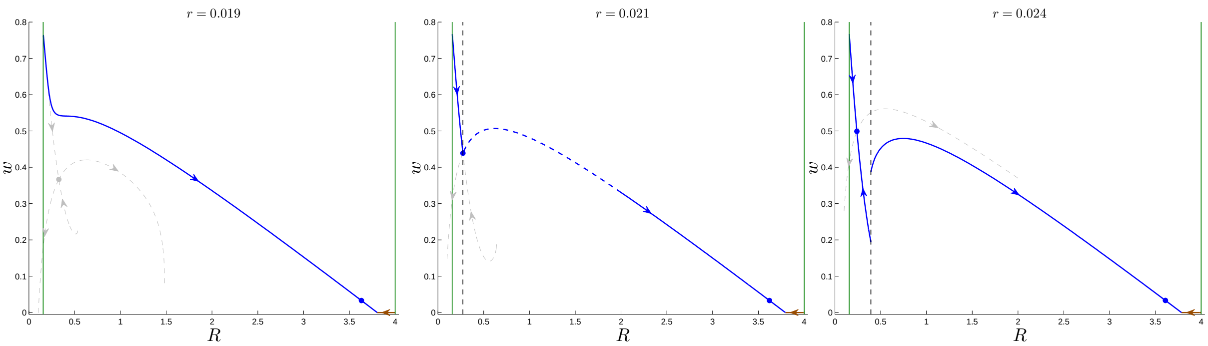
In order to have a closer look at the transition from a globally stable steady state to the existence of two locally stable steady states, we have to distinguish between the two cases,

- a discontinuous policy and a Skiba point,
- a continuous policy and an unstable steady state (weak Skiba point),

that separate the regions of attraction.

5.3.2 Transition to the Skiba case

Mathematically, this transition is characterized by the appearance of a heteroclinic connection. For a better understanding, Figures 2a, 2b, 2c depict the situation before, at, and after this bifurcation for a varying interest rate r. For r smaller than the bifurcation value $r_h = 0.021\ldots$ (Figure 2a for $r = 0.019$) a unique globally stable steady state R^H (blue dot) results. Thus, for every initial country size the optimal policy (blue curve) lets us end up in this empire. Figure 2a indicates a second saddle point R_L (gray dot) in the state-control space with a stable (solid gray) and unstable path (dashed gray). However, the corresponding policy is inferior to the policy of the empire and hence this solution does not appear in the optimal vector field (Equation (19)).



(a) Only R^H is optimal. **(b)** Heteroclinic bifurcation. **(c)** Skiba phenomenon.

Figure 2: Phase portrait for increasing values of the discount rate r for base case parameter values as given in Table 1.

Increasing r, the unstable path (dashed gray) of the small country and the stable path (blue) of the empire come closer until they coincide in a heteroclinic connection, see Figure 2b. This means that starting on this connecting line we approach the empire or, if we reverse time, we approach the small country. This is the first occurrence of a small country as a long-run solution. Thus, starting left from and even at R_L it is optimal to move to, and stay at, the small country. For any initial state larger than R_L, the optimal policy leads to the empire. Increasing r further, the heteroclinic connection breaks up, cf. Figure 2c, and a Skiba point appears which separates the regions of attraction between the empire and the small country.

5.3.3 Transition to the weak Skiba case

What happens in Figure 4b in the neighborhood of $\beta_0 = 0.33\ldots$? For parameter values larger than β_0 a unique small country exists, R_L. Thus, for every initial size of a country the optimal policy leads to the small steady state. If β is reduced, the bifurcation diagram displayed in Figure 4b shows the appearance of a saddle and an unstable node (in the canonical system). A numerical analysis shows that the unstable node is an unstable steady state (weak Skiba) of the optimal vector field for β slightly lower than β_0 and that the high and low saddles are locally stable steady states of the optimal vector field.

What happens exactly at β_0? This is the point where node and saddle coincide, hence this bifurcation is called a saddle-node bifurcation. Concerning the optimal policy at β_0: for countries that are initially larger than (or equal to) the newly emerged empire, the optimal policy is to converge to the empire; when starting with a smaller country, it is optimal to converge to the small country.

5.4 Bifurcation analysis in the (P_0, β) space

Figure 3 shows a complex pattern of outcomes in the (P_0, β) space:

Region 1: There are two locally stable steady states, an empire and a small country, separated by a (weak) Skiba point:

- **a:** The small country is feasible.
- **b:** The small country becomes infeasible and has to be replaced by the boundary steady state (i.e., the country vanishes from the map).

Region 2: There is a unique globally stable small country.

- **a:** The small country is feasible.
- **b:** The boundary steady state, so that a small country becomes infeasible.

Region 3: There is a unique globally stable empire.

Region 4: There is no solution.

Of course, high costs rule out the existence of any country or empire (Region 4). With $P_0 = const$ and a low value of β, there can only be empires , while for a large value ($\beta > 0.37$ and a very low P_0) only small countries can exist. Increasing the pressure along

$\beta = \text{const}$ (the case $\beta = 0.2$ is considered in Figure 4a) leads to a transition from Region 1a (R^H and R_L) to Region 1b (R^H and R_1), and then to Region 2b (R_1).

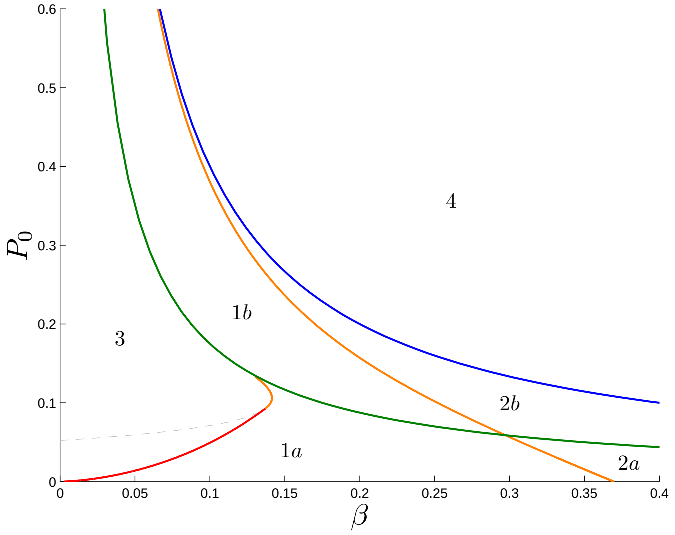
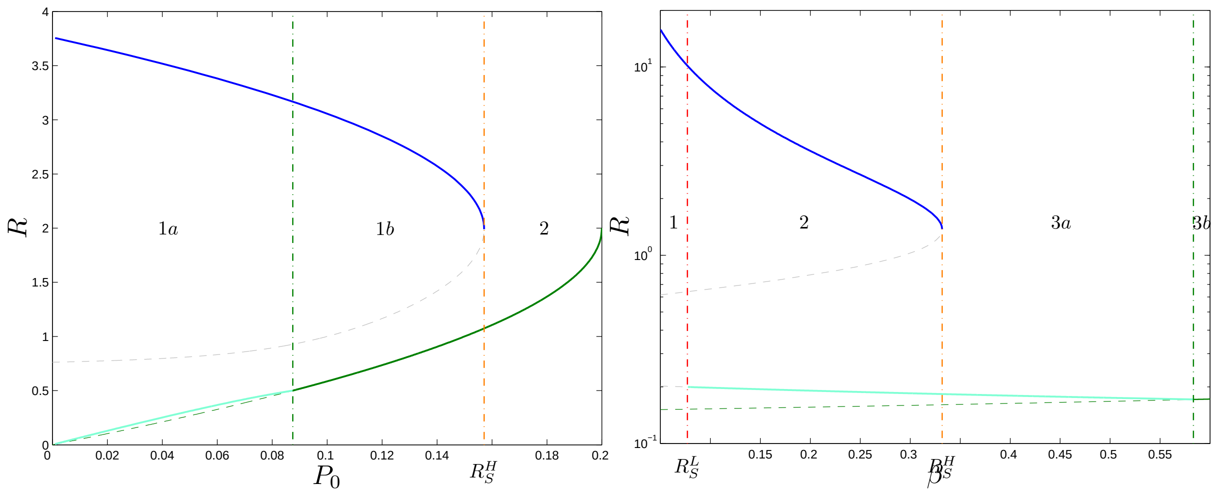


Figure 3: Two-dimensional bifurcation diagram with respect to the base case parametrization from Table 1. 1: multiple optimal solutions; a: (weak) Skiba; b: weak Skiba (R_1); 2: unique low equilibrium; a: R_L only; b: R_1 only; 3: unique high equilibrium; 4: no solution; red: heteroclinic bifurcation; orange: saddle-node bifurcation (R_L/R^H (dis)appears); green: R_1 becomes optimal; gray: saddle-node bifurcation that does not play a role for the optimal vector field; blue: $(Ak_0)^2 - 16\beta P_0 = 0$ (see the discriminant from equation (7)).



(a) External pressure P_0.

(b) Transport costs β.

Figure 4: P_0 bifurcation for $\beta = 0.2$ (a) and β bifurcation for $P_0 = 0.03$ (b), the other parameters as seen in Table 1. Blue: high equilibrium R^H; cyan: low equilibrium R_L; dashed green: boundary R_1; solid green: optimal boundary equilibrium R_1; dashed gray: non-optimal steady states;. (a) 1: Skiba area where R^H and R_L exist. 2: weak Skiba and coexistence of R^H and R_1. 3: only R_1 exists. (b) 1: only R^H. 2: Skiba between R_H and R_L. 3: only R_L. 4: only R_1.

5.4.1 Bifurcation diagram in the external pressure P_0

Figure 4a shows the one-dimensional P_0 bifurcation diagram along $\beta = 0.2$ that allows to distinguish between two main regions, where region 1 is subdivided to facilitate interpretation.

Region 1: There are two locally stable steady states, an empire and a small country, separated by a (weak) Skiba point:

a: The small country is feasible.

b: The small country becomes infeasible and has to be replaced by the boundary steady state (vanishing country).

$\mathbf{R_S^H}$**:** A saddle-node bifurcation separating Regions 1 and 2.

Region 2: There is only the vanishing country.

R^H (empire) exists in a wider range, $0 < P_0 < 0.16$, but a small country also has a substantial ecological niche surviving pressure up to $P_0 < 0.08$. Hence, small countries and large empires can coexist in a wide range of pressures. At larger pressures, $0.08 < P_0 < 0.16$, small countries R_L transform into unstable (in a political-historical sense) boundary steady states. This is intuitive because small countries cannot resist strong pressure and will be annexed or conquered.

5.4.2 Bifurcation diagram in the transport costs β

Figure 4b varies the transport cost parameter β, which is useful for tracking technological evolution both in transport as well as in ruling and governing a country. This allows to distinguish between three main regions of which Region 3 is subdivided due to interpretational reasons

Region 1: A unique globally stable empire.

$\mathbf{R_S^L}$**:** A heteroclinic bifurcation separating Regions 1 and 2.

Region 2: There are two locally stable steady states, an empire and a small country, separated by a (weak) Skiba point.

$\mathbf{R_S^H}$**:** A saddle-node bifurcation separating Regions 2 and 3.

Region 3: There is a unique small country.

a: The small country is feasible.

b: The small country becomes infeasible and has to be replaced by the boundary steady state (vanishing country).

One explanation for the lacking existence of empires of a size $R > 1$ (see the discussion in the next section) before three to four thousand years ago is that transport costs (including the costs of governing, establishment of a bureaucracy, etc.) were simply prohibitive. Given a high value, $0.08 < \beta < 0.58$, the ecological niche for small countries, is wider than for large empires, $0 < \beta < 0.33$ and there is a large overlap in which both coexist, which is consistent with historical observations.

5.5 Bifurcation analysis in the (r, σ) space

The (r, σ) space consists of different regions, which are separated by three curves, cf. Figure 5: the red curve corresponds to the parameter values in which a heteroclinic connection exists; a saddle-node bifurcation takes place at the black curve; the green line separates the regions in which either a small country or only the boundary steady state exists that corresponds to a country that cannot be sustained.

Therefore, we find the following regions of globally stable outcomes:

Region 1: There is a unique globally stable empire.

Region 2: There are two locally stable steady states, an empire and a small country, separated by a (weak) Skiba point.

- **a:** The small country is feasible (Skiba).
- **b:** The small country becomes infeasible and has to be replaced by the boundary steady state (vanishing country) (weak Skiba).

Region 3: There is a unique small country.

- **a:** The small country is feasible.
- **b:** There is only the country at the boundary R_1 which will vanish from the map (for the reason given in Remark 1).

Regions 1 and 3 are not strictly separated. For small values of σ and r the size of empires reduces to that of countries and vice versa. The case of both small r and σ is indeed interesting, because a low r only allows for large empires, while a low value of σ does so only for small countries (see Figures 6a and 6b).

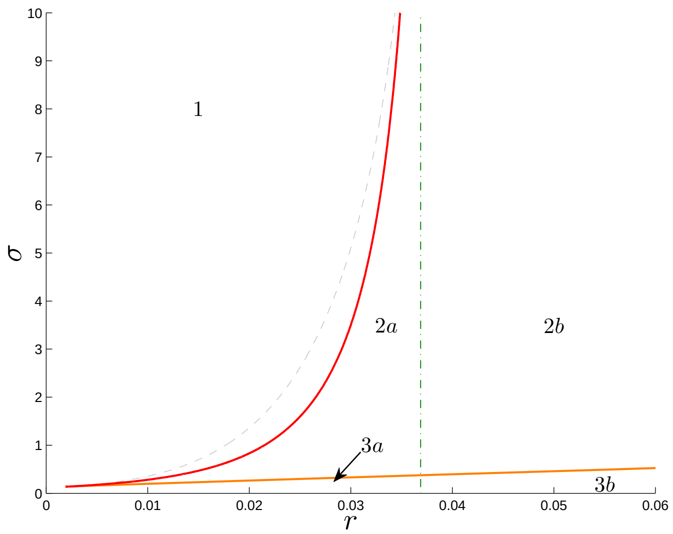


Figure 5: Comparing the different outcomes using a two-dimensional bifurcation diagram with respect to the emperor's preference parameters (r, σ) for the base case parametrization from Table 1. 1: only R^H, 2a: Skiba between R^H and R_L, 2b: Skiba between R^H and R_1, 3a: only R_L, 3b: only R_1. Red: heteroclinic connection; orange: limit point bifurcation, part of the optimal vector field (R^H vanishes/appears); gray: limit point bifurcation, not a part of the optimal vector field (R_L vanishes/appears); green: R_L, R_2 may exist.

5.5.1 Bifurcation diagram in the discount rate r

Figure 6a provides a cut along r of the bifurcation diagram in Figure 5. It shows how the steady state values of R depend on r. Three steady states (in the canonical system) exist over a broad range. The dashed lines correspond to non-optimal steady states. There are three main regions, of which Region 2 is subdivided:

Region 1: There is a unique globally stable empire.

$\mathbf{R_S^L}$: A heteroclinic bifurcation separating regions 1 and 2.

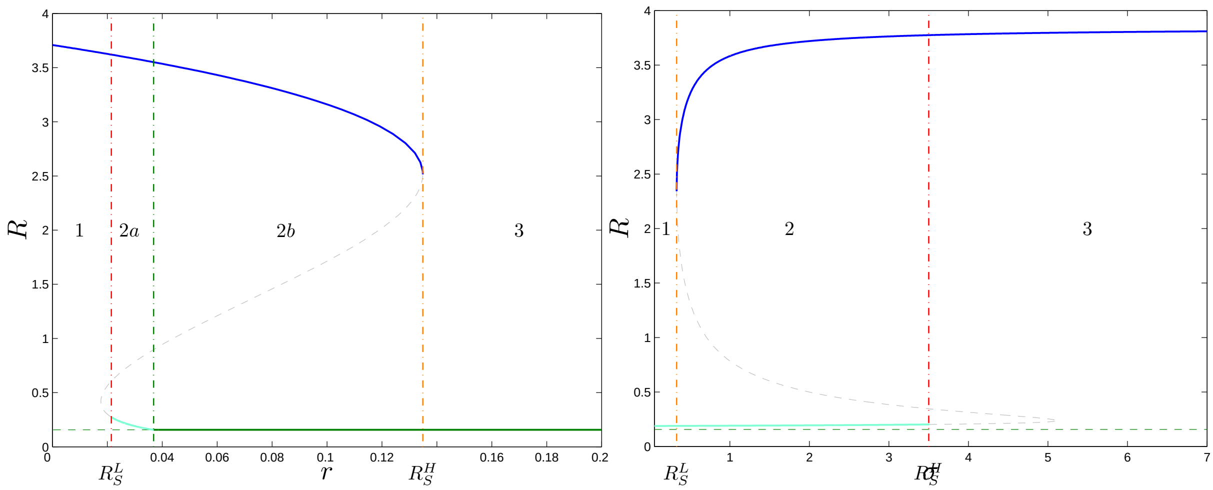
Region 2: There are two locally stable steady states, an empire and a small country, separated by a (weak) Skiba point:

- **a:** The small country is feasible.
- **b:** The small country becomes infeasible and has to be replaced by the boundary steady state (vanishing country).

$\mathbf{R_S^H}$: A saddle-node bifurcation separating region 2 and 3.

Region 3: There is only the vanishing country.

In the base case, small countries and large empires coexist only in a relatively small range, $0.02 < r < 0.035$. The small country turns unstable (i.e., converges to the boundary steady state) for larger discounting while empires are more robust in this respect.



(a) Discount rate r.

(b) Benefit σ.

Figure 6: Bifurcation diagram with respect to the interest rate r (a) and the benefit σ for the base case parametrization from Table 1. Blue: high equilibrium R^H; cyan: low equilibrium R_L; dashed green: boundary R_1; solid green: optimal boundary equilibrium R_1; dashed gray: non-optimal steady states; red: low boundary of the Skiba region R_S^L; orange: high boundary of the Skiba region R_S^H.

5.5.2 Bifurcation diagram in the preferences σ

Varying the preference parameter σ for holding territory (versus consumption per capita) along the base case leads to the bifurcation diagram in Figure 6b with three different domains:

Region 1: There is a unique globally stable small country.

$\mathbf{R_S^L}$**:** A saddle-node bifurcation separating Regions 1 and 2.

Region 2: There are two locally stable steady states, an empire and a small country are separated by a (weak) Skiba point.

$\mathbf{R_S^H}$**:** A heteroclinic bifurcation separating Regions 2 and 3.

Region 3: There is a unique globally stable empire.

The economic interpretation is quite intuitive. For $\sigma \to 0$ an emperor (asymptotically) does not care about the size of his empire, but only about consumption per capita, which is optimal only for a small country where costs for transport are low. For $\sigma \to +\infty$ a king (asymptotically) neglects consumption of his people and cares only about the size of his empire, which renders only a large empire optimal[9].

Further bifurcation analyses with respect to the other model parameters confirm that this outcome (small countries and large empires separated by a threshold) is a generic outcome of our model and not confined to just a few constructed examples. Furthermore, they confirm economic intuition: low discount rates, little external pressure and a high preference for size foster large empires, which explains why Germany with its powerful neighbors never became an empire, while Russia and the United States had the possibility to expand at the expense of weak neighbors. Low productivity A, low values of α (i.e. a low speed for conquering), and high transport costs β can render a large empire unattractive.

6 Connecting the model results to history

The pattern observed in the base case example, i.e., a coexistence of large empires and small countries, proved to be robust including the different bifurcation analyses. Not only that, it complies with most maps at most points in time in known human history with the dynamic process of growing empires and disappearing countries (including empires). Of course, the following discussion must be confined (except for some remarks) to the times before the Industrial Revolution due to our assumption of a non-growing economy. We are all familiar with most of the large empires in history[10]. Although the first civilizations emerged around 8000 BC, empires did not start before 3000 BC (in Egypt, Old

[9] The same outcome is observed in the model of Yegorov (2018), where there is no concern about consumption. That dynamic model assumes spending of all surplus on defense and transport, and gives convergence to R_2 as the only stable solution of the dynamic equation (it is not a dynamic optimization problem!).

[10] Taylor (2008) is a compilation of the major empires in history, of which the British one was the largest with $S = 33.7 \Longrightarrow R = 5.80$, but it does not fit our assumption of pre-industrial economies. See also the related *YouTube* videos, https://www.youtube.com/watch?v=ymI5Uv5cGU4 .

Kingdom). Restricting ourselves to empires larger than one million square kilometers, i.e., $R > 1$, the New Kingdom in Egypt was the first such empire, established around 1500 BC, and both its economic efficiency and the low external pressure (the Nile area is surrounded by deserts) explain its long sustainability. Later (1500–1000 BC), empires developed in China and India, but they were not larger in size. In 700–600 BC, the Neo-Assyrian empire reached a size of around $R = 2$, while the Achaemenid empire reached a size of around $R = 3$ about 500 BC. Around 333 BC (battle of Issus), Alexander the Great conquered territories and formed an empire of a size that ultimately proved to be unsustainable. Later empires became larger: the Umayyad Caliphate (661-750) had about 15 million km^2, corresponding to $R = 3.87$, while the Mongol empire (1206–1368) has reached 33 million km^2, or $R = 5.74$). By contrast, the Roman empire proved sustainable for some time but its territory did not exceed 10 million km^2, $R \approx 3$.

What is interesting is that small countries coexisted with large empires, e.g. ancient Greece (together with its small city states) existed along with the Persian empire. Similarly, during the Middle Ages and in particular in Europe, many small city states (e.g., Venice, Florence etc.) can be found on historical maps together with regional empires. At the same time (intermediately sized) national states were formed in Britain, France, and Spain, while Russia and Sweden were ruling over Scandinavia. The small cities and regional empires presumably exist due to the demands of the local population for consumption and the external pressure from the neighboring region. The establishment of the European empires may be explained (aside from language and other cultural ties, but then why neither in Italy nor in Germany?) by emperors' preferences, low transport costs and little external pressure (Britain as an island and Spain after defeating the Moors).

Continuing with Europe and moving through the Renaissance until the end of the 17$^{\text{th}}$ century, some empires had a short lifetime (e.g., Sweden during the Thirty Years' War, Napoleon), but some continued to expand like the Ottomans, and Russia. One explanation of this expansion is that the 'transport' costs in terms of government were reduced by the economies of scale of a relatively loyal and well-organized bureaucracy (Austria-Hungary). This may also explain the colonial empires of Britain, Spain, Portugal, and the Netherlands who as seafaring nations had lower transport costs and faced little local pressure. Bifurcations can and did take place from small to large, and vice versa. In the language of the model, the switch from a small country to a large empire (and vice versa) can come about because of changing pressures (e.g., by invasions), economic development, and changing preferences of the emperor. As the roving bandit settled as an emperor, his preference for size and maybe also for local consumption changed according to Olson (1965). Many empires turned out to be unstable and collapsed, presumably due to 'transport costs' if unavoidable tensions between different ethnicities and religions within any large empire are also accounted for, the recent dissolution of the Soviet Union being an example. However, neither these two events nor the unification of Italy and Germany in the 19$^{\text{th}}$ century, nor the collapse of the Ottoman empire after World War I fit our story but were driven by the advent of nationalism. The nationalistic stories fit better, as the pressure at borders with hostile neighbors limits the size of the new national entities in a way that the creation of the 'empires' of Yugoslavia and to a lesser extent of Czechoslovakia turned out to be unsustainable.

Considering the dynamics of an empire, it takes about $t = 150$ (years) to reach a 10% neighborhood of the steady state R^H in the base case. After $t = 70$, which is probably the longest reign of a ruler, only about half of achieving the empire is accomplished. Therefore, only dynasties with similar preferences across generations are capable of building and maintaining an empire. A collapse can occur faster and for various reasons such as different preferences shown by a successor, or the rising power of a neighbor (shock in P_0), and of course reasons like growth in ethnic tensions, which are not addressed in our framework (only indirectly, e.g., a shock in the transport cost parameter β might reflect growing nationalism in multi-national empires).

7 Concluding remarks

The objective of this paper is to offer an economic explanation of the formation (and the collapse) of empires. Based on Yegorov's (2018) model of a stagnant economy and accounting for space, the (optimal) intertemporal decisions of rulers to fund the military for defending or expanding a country were derived. The model uses a homogeneous space, a single production factor, and accounts for increasing and decreasing returns to geographical scale, economic constraints on military expenditures, and the preferences of an emperor for a large country and the consumption of his people.

The crucial implication is that there are two viable long-run outcomes: a small country and a large empire. Both outcomes are separated by a threshold. They have different ecological niches in the parameter space, both may collapse due to changing environments, and depend on initial circumstances, fortunate ones for empires. For example (and presumably most likely), small countries collapse due to increased pressure from their neighbor(s). Many (but by far not all) empires need a long time to grow, e.g., promoted by similar preferences of a dynasty but they may collapse after a change in these preferences (expressed by a new ruler). And even large empires are endangered, e.g., because of increased military force at their borders and high transport costs.

This result on the multiplicity of equilibria (they are due to scale economies (in defense); see also Arthur (1994)) can explain why historical maps almost always include many small countries (such as Switzerland for centuries) coexisting with large empires. The expansion of empires can be attributed to weak neighbors and very different preferences of a new King like Alexander the Great (or Napoleon compared with the French Republic following the French Revolution).

Favorable factors for building up an empire are: low discounting, e.g., caring about one's dynasty and striving more for grandness than for the wellbeing of the population. On the other hand, low productivity, high transport costs, military weakness, and strong pressure from neighboring countries limit the size of empires. The growth of an empire is constrained although output grows in terms of its size (i.e., of the order R^2) and defense requirements depend only on border length (and are thus linear in R), because transport costs increase in the order R^3, which limits the size of an empire and will necessarily lead to collapse for any empire that is too large.

Since we offer just a first try at what we think is an interesting and interdisciplinary topic, there are many areas for extension and elaboration and we will mention just a few here: including elements of economic growth instead of assuming a stagnant economy, which would allow to address how economic growth can contribute to the establishment of an empire (such as the British one); accounting for higher heterogeneity of the population as the empire expands (indeed many empires fell prey to nationalism, e.g., the Habsburg and the Ottoman empires, and recently the Soviet Union, Yugoslavia, and even Czechoslovakia); adding uncertainty because it is a crucial characteristic of any war; strategic considerations, i.e., to endogenize the pressure exerted by a neighboring ruler.

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