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The Mathematics of Ageing

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Abstract Age is a crucial variable in social sciences and particularly in population dynamics. In the first part of this paper, a two-state optimal control model is proposed to explain the substantial variations of scientific production over the life cycle of researchers. We identify conditions under which typical hump-shaped age-specific patterns of scientific production turn out to be optimal for individual researchers. The second part of the paper deals with the ageing of learned societies. In a nutshell, the dilemma of a learned society is that keeping young, i.e. electing young entrants, has the drawback of reducing the replacement rate of members. It turns out that electing a mix of young and old members delivers the optimal solution of the problem, i.e. guaranteeing a young age structure, while ensuring a high recruitment rate.

Keywords: Age-structured models, optimal control, scientific production over the life cycle, optimal recruitment of learned societies.

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1 Introduction

Age is one of the most important variables in social sciences, psychology, biology and other disciplines. By definition, age is the synchronous lapse of calendar time and individual time. Not only that most physical and cognitive abilities are crucially dependent on age (e. g. Skirbekk 2004), age also underlies the organisation of family, education, work and leisure. Hence, whether through physiological capabilities, explicit age-related rules or informal expectations, age structures individual life courses (Settersten 2003). Accordingly, the rate of occurrence of any demographic event varies strongly with age. Therefore, age—besides gender—is the core variable in demography. With some exaggeration one could say that demography is the science dealing with 'age'.

Age is not only important for individual life courses, but the composition by age is also crucial for the future development of aggregated entities such as nation states or the world population. Changes in a population's age structure will have implications on almost all sectors of a society. The United Nations write in their World Population Ageing report that "*[p]opulation ageing—the increasing share of older persons in the population—is poised to become one of the most significant social transformations of the twenty-first century, with implications for nearly all sectors of society, including labour and financial markets, the demand for goods and services, such as housing, transportation and social protection, as well as family structures and inter-generational ties*" (United Nations 2017a, p. 1).

However, while a couple of decades ago, population ageing has been concentrated in high-income countries, increasing shares of older population have been observed also in other world regions such as Asia and Latin America (see Figure 1).

The middle-income countries within these regions are projected to become as aged as, or even more aged than, many of today's high-income countries but at an even higher pace than the latter. Looking at an example, China in the 1950s displayed the typically pyramid shape, consistent with an expanding population, where children outnumbered consecutive higher age groups (see Figure 2, left graph). Due to substantial gains in longevity, the number of persons in older age groups has substantially increased in the meantime. On the other hand, China's one-child policy resulted in sharp reductions in fertility. As a consequence, the number of older persons has been growing faster than the numbers of people in any younger age group, resulting in shifting relative sizes of the various age groups over time, the latter implying an increase in the proportions of older-aged populations while the age pyramid has been narrowing at its base (cf. Figure 2, middle graph). The growth of the share of populations at older ages is projected to even accelerate during the coming decades. For example in China, the share of population aged 60 years or more is projected to rise from 16 to 35 per cent from 2017 to 2050, i.e. it will more than double just within

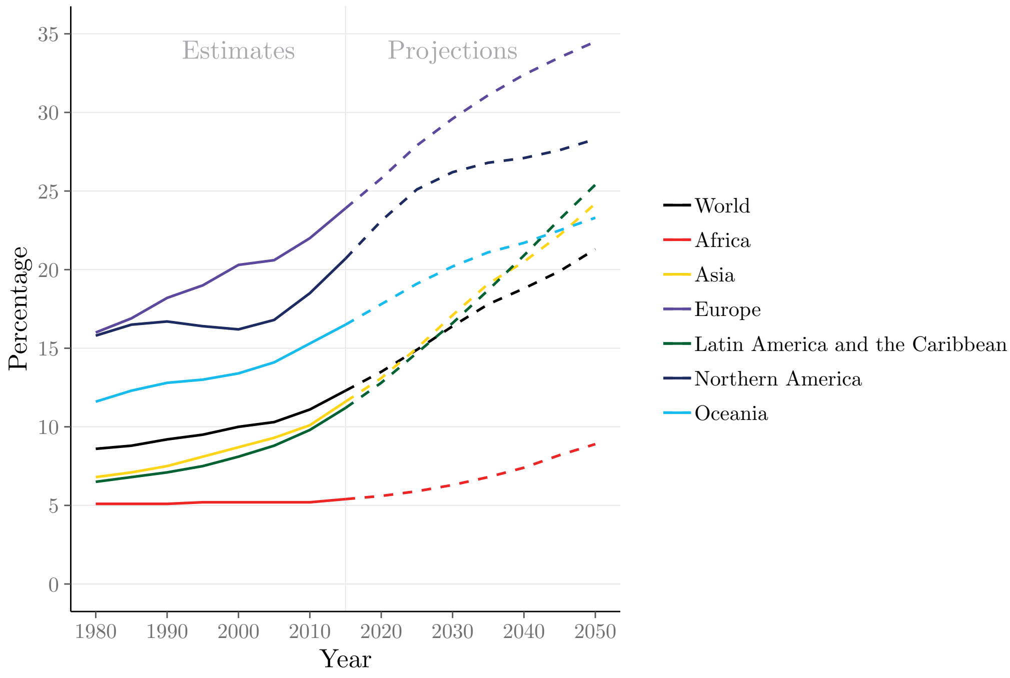


Fig. 1: Percentage of the population aged 60 years or over for the world and regions, 1980–2050 (cf. United Nations 2017a, p.31). Data source: United Nations (2017b).

33 years (ibid, p. 29). Accelerated ageing is also visible in the projected urn-shaped population pyramid for China in 2050 (see Figure 2, right graph).

However, not only nation states are faced with the challenges of population ageing. Various kinds of age-structured organisations such as universities or armies are confronted with an increasing average age of their members/employees. As it has been argued that 'old' organisations might be seen as not innovative enough and no longer well-informed, ageing organisations—regardless of whether they are firms, faculties, societies, political bodies, teams, or national academies—seek ways to rejuvenate. Instead of fertility and mortality rates, it is more generally rates of entering or leaving the population which affect the future age structure of the population.

The interrelationships between fertility and mortality rates with population age structure have been formalised within the stable population theory. Alfred J. Lotka, born in 1880 in Lemberg (at the time in Austria, now Lviv in Ukraine), can be considered as founder of formal demography, in particular of the stable population theory (Lotka 1907, 1922, 1939; see also Keyfitz 1968), the core concept of population dynamics.[1] Although it was formalised by Lotka in the early 20th century, some features of the stable population theory had been known already much earlier. In fact, it was Nathan Keyfitz, the second most influential formal demographer of the past century, who discovered that already Leonhard Euler, the most productive mathematician of the 18th century, had been aware of the concept of asymptotic stable populations

[1] The central role of stable populations in population dynamics has been compared with those of the normal distribution in statistics.

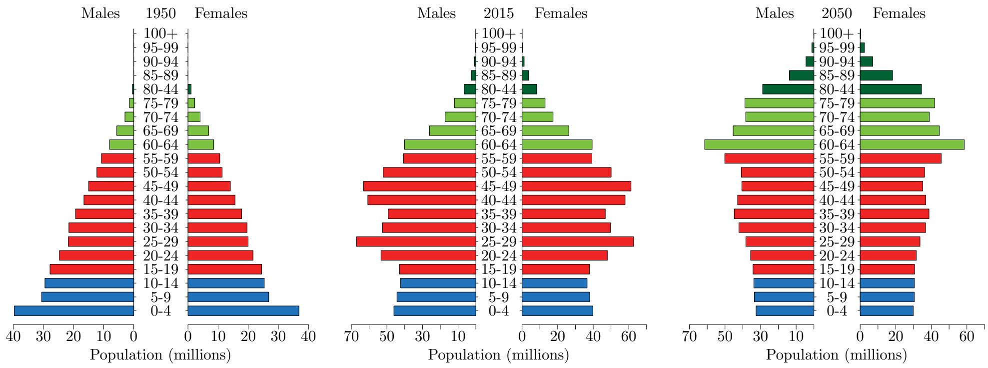


Fig. 2: Population pyramids for China in 1950, 2015 and (projected) 2050. The colors denote broad age groups: 0–14 (blue), 15–59 (red), 60–79 (light green), and 80+ (dark green). See also `https://population.un.org/ProfilesOfAgeing2017/index.html`. Data source: United Nations (2017b).

(Euler 1760). The most crucial result of the theory is strong ergodicity: it roughly says that in the long run, a population 'forgets' its past age structure if it is subject to constant age-dependent mortality and fertility rates over time (see, e. g. Preston et al. 2000; Feichtinger 1979).

It is well-known that mathematical methods play an important role in demography and population dynamics. The backbone of the latter is a renewable aggregate of individuals, the investigation of which is part of renewal theory that also plays a key role in Operations Research.

Indeed, formal demography and Operations Research—two seemingly unrelated fields—share several methodological links. The purpose of this paper is to show how optimal control methods—or more specifically, intertemporal optimisation techniques—can be applied in conjunction with formal demography to solve problems of population ageing.[2] In this paper, we present two applications, one at the micro level and the other at a more aggregate level, both in the field of academia.

The first research question is set at the individual level and deals with an age-related topic every scientist is confronted with: the ability to produce scientific publications is not evenly distributed over her/his age. Scientific creativity tends usually to rise more or less rapidly to a peak and then gradually decreases. Typical life cycle patterns are not only observed in academia, but also in artistic production or in criminal behaviour (e. g. Kanazawa 2003). Not strikingly, there are many studies of career paths of creative people since the famous statistician Quetelet (1835) started researching this question almost 200 years ago. The aim of this chapter is to present an optimal control model which is able to explain the hump-shaped pattern of scientific production over age.

The second research question deals with the increasing ageing in age-structured populations with fixed size. Several years ago, the presidency of the Austrian Academy

[2] In contrast to demography, age-specific optimization methods play an important role in economics, e. g. in capital vintage models (see e. g. Feichtinger et al. 2006a; Feichtinger and Veliov 2007).

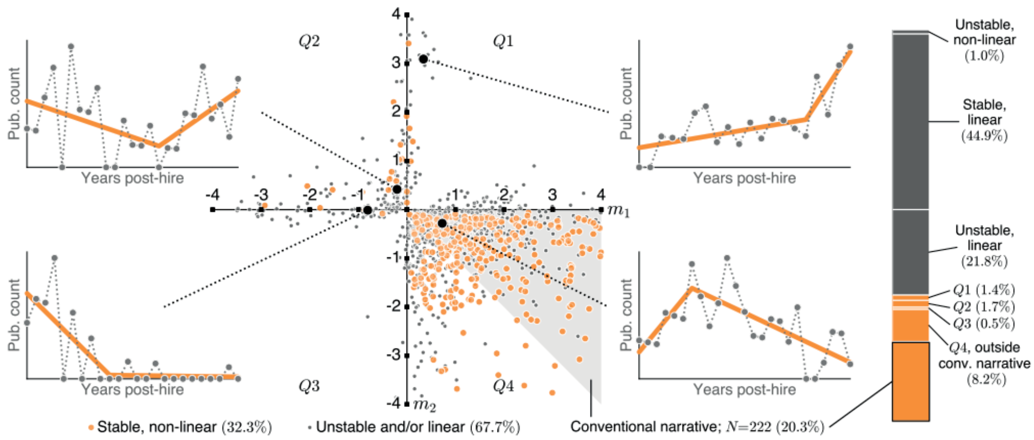


Fig. 3: Distribution of individuals' productivity trajectory parameters. Diverse trends in the individual productivity fall into four quadrants based on their slopes. Plots show example publication trajectories to illustrate general characteristics of each quadrant. The shaded triangular region (bottom center) corresponds to the conventional narrative of early increase followed by gradual decline Way et al. (Source: 2017). The permission to reproduce this figure by the first author of the cited paper is gratefully acknowledged.

of Sciences (in German "Österreichische Akademie der Wissenschaften, OEAW)", aware of the raising age of their members, asked the Vienna Institute of Demography to work out a proposal to counteract this 'over-ageing', or in other words, to rejuvenate the age structure under the restriction of keeping the fixed size of the organisation. The main purpose of this part of the paper is to illustrate how optimal control methods can be applied to solve problems arising in the dynamics of certain sub-populations. It should be stressed that the applicability of the approach is not restricted to learned societies, but could be extended to firms and other hierarchical organisations as well. Note that models of this kind belong to manpower (personal) planning, another important field of operations research.

2 Optimal scientific production over the life cycle

One way to measure the productivity of scientists is to count their publications. A well-known fact is the inequality of scientific output not only across individuals but also for each researcher over time. Typically, an initial rapid rise of scientific creativity is followed by a gradual decline.

There are many studies on the career paths of creative people illustrating this hump-shaped pattern. However, recently, Way et al. (2017) identified several other distributions of research productivity over the life cycle. Using a large dataset originating in computer science departments of the U.S. and Canada, they showed that an age-specific hump-shaped productions trajectory does not always occur. Figure 3 illustrates four productivity patterns found by Way et al. (2017).

Below we try to provide a theoretical underpinning of the four different research patterns detected by Way et al. (2017). In particular we show that the most frequent age-specific distribution of scientific work occurs if two conditions are fulfilled. A sufficiently high education level at the beginning of scientists' careers usually leads to an increase of sufficient productivity. If they do not bother much about their reputation at the end of their career, the number of publications then gradually decreases until retirement.

2.1 The model

The following deterministic continuous time-optimal control model has two state variables. The first is the stock of human capital, $K(t)$ an individual has accumulated control at age t, while the second is the reputation $R(t)$ of a scientist. The output of a scientist is publishing papers, P. A necessary condition to do so is having built up a stock of knowledge being strictly positive. Building up reputation can work as a leverage with respect to productivity. To model this we introduce the scientific production function

$$P = P(K,R) = K^{\alpha}(R+1)^{\beta}, \tag{1}$$

with α and β denoting positive constants smaller than one. The functional form reflects that one can be productive without working on reputation. Investing in knowledge at a rate $I(t)$ can be seen as a scientist's major activity. Omitting the time arguments, the dynamics of the human capital of an individual satisfies

$$\dot{K}(t) = g(K(t))I(t) - \delta_K K(t), \tag{2}$$

where $g(K)$ reflects that investment in knowledge is more fruitful if one has already built up some knowledge, being an increasing function, and δ_K is the obsolescence rate of the human capital.

Beside the major activity of knowledge production, the scientist is also embedded in a network of colleagues. Thus, there is a mutual influence in course of their common work. The second state variable we include in our model is the reputation $R(t)$, measuring a scientist's position within the scientific community. Denoting by $N(t)$ the second control variable, i.e. networking as collaboration with colleagues, conference presentations etc., the reputation develops according to

$$\dot{R}(t) = h(K(t))N(t) - \delta_R R(t), \tag{3}$$

where $h(K)$ measures the efficiency of networking depending on the personal human capital and δ_R is the obsolence rate of the reputation. Note the asymmetry of the right-hand sides of (2) and (3), as both $g(\cdot)$ and $h(\cdot)$ depend on K. Investing in knowledge is more effective if the researcher is already knowledgeable. Therefore $g(K)$ is increasing in K. In addition, investing in networking pays off if the scientist is knowledgeable. Then the scientist makes a good impression when presenting his research, talking to other researchers, writing emails and so on. Therefore, $h(K)$ is increasing in K.

It makes sense to assume both $g(K)$ and $h(K)$ as being S-shaped, i.e. as convex-concave (the latter referring to saturation effects). In particular, we use the following S-shaped functions

$$g(K) := \frac{a(l_1 + K^\theta)}{1 + K^\theta}, \tag{4}$$

$$h(K) := \frac{e(l_2 + K^\sigma)}{1 + K^\sigma}, \tag{5}$$

where θ, σ, a, e, l_1, and l_2 are positive parameters.

The goal of the scientist is to maximise the, with rate r discounted, stream of his or her scientific publication, net of the costs for investing both in knowledge and networking

$$\max_{I(\cdot),N(\cdot)} \int_0^T e^{-rt} \left(c_0 P\left(K(t), R(t)\right) - C_1(I(t)) - C_2(N(t))\right) \mathrm{d}t + e^{-rT} \left(\kappa R(T)\right) \tag{6}$$

s.t. to the system dynamics (2) and (3), the initial conditions

$$K(0) = K_0 \geq 0, R(0) = R_0 = 0 \tag{7}$$

and

$$I(t) \geq 0, N(t)) \geq 0. \tag{8}$$

An important feature of our model is the fact that doing research and networking usually create utility for a scientist as long as it is done 'to a reasonable extent'. Only if I and N exceed certain thresholds are these activities connected with disutilities, i.e. they must be seen as costly.

For simplicity we assume linear-quadratic functions $C_i(\cdot) \quad (i = 1, 2)$

$$C_1(I) := d_1 I^2 - c_1 I \quad \text{and} \quad C_2(N) := d_2 N^2 - c_2 N, \tag{9}$$

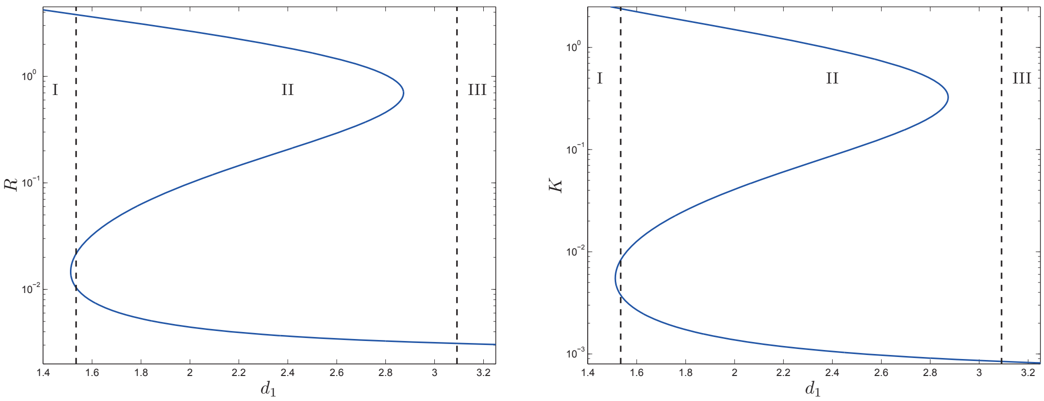
with c_1, c_2, d_1 and d_2 all positive.

The salvage value in (6) and the associated parameter κ reflects that the scientist assigns a positive value to reputation even after his career ($\kappa > 0$). However, the case of $\kappa = 0$ can be interpreted with the proverb *shrouds have no pockets*. Usually, the phrase refers to material goods, but in this case, there is of course the aspect of intellectual wealth.

2.2 Results

The application of Pontryagin's maximum principle (see e. g. Grass et al. 2008) delivers some interesting insights into the optimal investment patterns resulting in various patterns of scientific output. In particular, it can be established that the four patterns identified by Way et al. (2017) can be generated as optimal paths for appropriate parameter values.

We show that *typical and fading* patterns typically arise in scenarios where scientists themselves do not assign a too high positive value to being regarded as knowledgeable or having a high reputation at the end of their career. In such a case the



(a) bifurcation diagram (d_1, R) (b) bifurcation diagram (d_1, K)

Fig. 4: The solid line shows the bifurcation diagram in d_1 for the equilibria of the canonical system. For the parameter values between the dashed lines there exist Skiba solutions in finite time $T = 50$ with initial condition $R(0) = 0$. Note that in an interval on the right side of the Skiba region only one equilibrium exists. Thus, the existence of three equilibria is not necessary for the occurrence of a finite time Skiba solution.

scientist will opt for a typical pattern if the disutility for hard working is not too high. Here it could help that scientists during their studies obtain a lot of knowledge. This implies that when scientists start their career already being quite knowledgeable, any investments in knowledge and networking become more efficient.

If a scientist does assign a substantial positive value to being regarded as knowledgeable with a high reputation at the end of his or her career, the patterns *slump* and *busy* come into the picture. We show that a *slump* pattern, where the scientist is not very productive halfway through her career, can be avoided by high quality education. Again, starting the career with a lot of knowledge makes further investments in knowledge and networking more efficient. This raises productivity along the lifetime, resulting in the *busy* pattern. For details see Feichtinger et al. (2018).

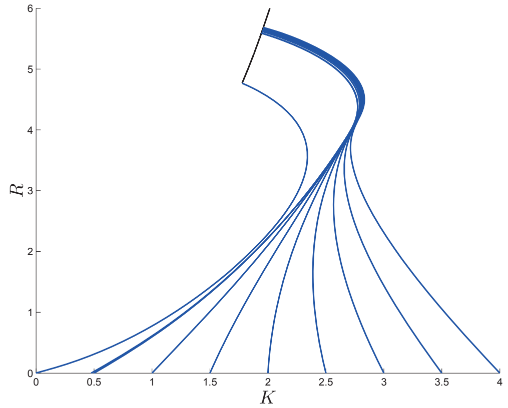
In the following we discuss the possibility of multiple equilibria whose basins of attraction are separated by Skiba thresholds. Starting at the threshold, the so-called Skiba point, the scientist is indifferent what career to choose.

In Figure 4 the bifurcation parameter is d_1, which reflects the cost of investment in knowledge. This means that d_1 specifies how fast the marginal utility of the scientist declines. For a large value of d_1 it is therefore costly to increase knowledge and only convergence to the small steady state takes place.

In the following we consider three typical examples for solutions lying in the different regions given by the bifurcation diagram Figure 4. Each of these figures shows the solution paths in the state space with initial values satisfying $R(0) = 0$ and $K(0) \in [0, 4]$. Additionally the manifold of the endpoints, i.e.

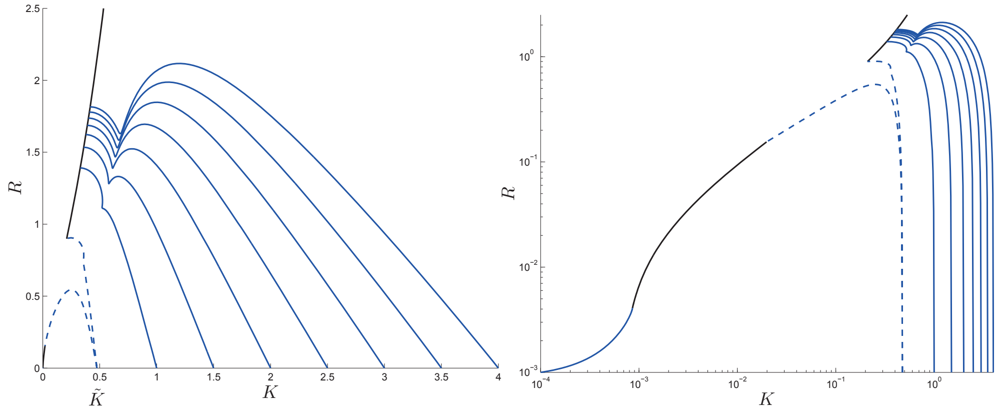
$$\{(K(T), R(T) \colon K(0) > 0, R(0) = 0, T = 50\},$$

is depicted as a grey curve. This manifold can be seen as the counterpart to a steady state of the infinite time horizon problem. Figure 5a shows optimal solution paths



(a) State-space $d_1 = 1.4$ (only finite solution)

Fig. 5: This figure shows the phase portrait for $d_1 = 1.4$, lying left to the region with Skiba solutions. The single manifold of the endpoints lies in the upper right part of the state space, starting at $(1.78, 4.77)$.



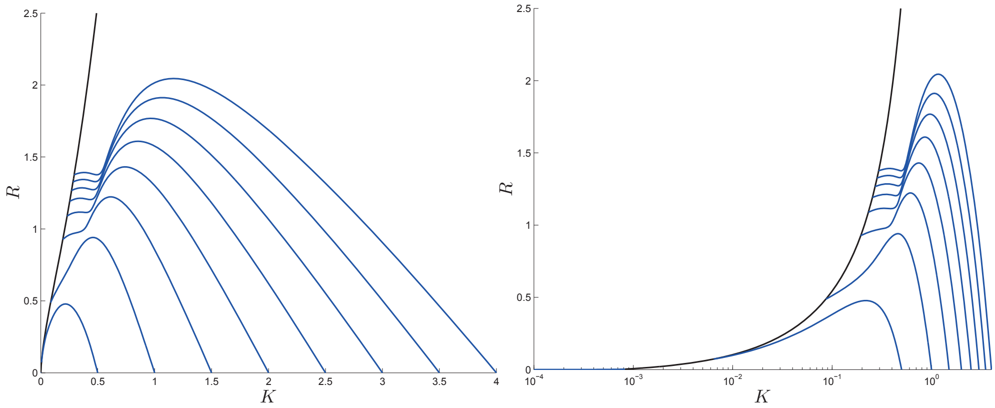
(a) State-space $d_1 = 3$ (only finite slution)

(b) State-space $d_1 = 3$ (only finite solution, log-scale)

Fig. 6: For the second scenario with $d_1 = 3$ there is a Skiba solution with $\tilde{K} = 0.475$. From this Skiba point a Skiba curve originates with initial values $R(0) > 0$ (dashed grey line). Left to the Skiba point $\tilde{K}$ the solution paths end at a manifold very near the origin and is therefore hardly visibly in panel (a). Therefore the K axis is logarithmically scaled in panel (b). This reveals that the manifold of the endpoints is separated into two distinct arcs, which is the counterpart to two different equilibria for a usual Skiba solution.

for a relatively small value of d_1. This means that it is easy for the scientist to create knowledge and the positive value of $\kappa_2 = 10$ indicates that the horizon date reputation is positively valued. Therefore the solutions end up with large values of reputation and values of knowledge above 1.5.

For the intermediate case (see Figure 6) with values of d_1 between 1.53 and 3.09 the solutions are history-dependent in the following sense: For small initial values of



(a) State-space $d_1 = 3.2$ (only finite slution) (b) State-space $d_1 = 3.2$ (only finite solution, log-scale)

Fig. 7: For the last scenario $d_1 = 3.1$ is chosen to lie in region III of Fig. 4, and therefore yields again unique solutions. In the logarithmically scaled figure panel (b) it becomes apparent that the two previously disconnected manifolds of endpoints are now combined to a single continuous manifold.

knowledge, i.e. $K(0) < \tilde{K} = 0.4754$, the solutions end up with very low reputation and knowledge (see Figure 6b). From an interpretative point of view the corresponding researchers invest more time in activities other than research, e. g. teaching or administration. On the other hand, for larger values than $\tilde{K}$ the researcher starts a scientific career and ends up with a high value of reputation and knowledge.

Finally for values of $d_1 > 3.09$ the spectrum of research careers is continuous, cf. Figures 7a and 7b. This means that with increasing initial knowledge the attractivity to start a scientific career increases continuously. Thus, other than in the previous Skiba case, researchers with some intermediate knowledge at the beginning also end up with average values of knowledge and reputation. There is no abrupt change in knowledge and reputation at the end is like it was at $\tilde{K}$. This is due to the relatively high costs of knowledge increase.

In Figure 8 the time paths of productivity for the three previously explained scenarios are plotted. To make the results comparable, the initial states are chosen equally, namely $R(0) = 0$ and $K(0) = \tilde{K}$, the Skiba value from $d_1 = 3$. In the first case, for the low value of $d_1 = 1.4$ productivity is steadily increasing until it reaches its maximum $P(R,K) = 4.84$ at $t = 44$. Finally it drops down to a rather high value of $P(R,K) = 4$, cf. Fig. 8a.

For comparable high costs $d_1 = 3.2$ the situation is quite different. The low maximum of productivity $P(R,K) = 0.24$ is already reached at $t = 5.27$ and drops down to the vanishing value of $P(R,K) = 0.004$, cf. Figure 8c.

For the Skiba case we find both patterns qualitatively repeated: a) the productive researcher, reaches his or her maximum at $t = 38.75$ and finally only loses around 30% of its productivity, whereas b) the less productive researcher reaches maximum productivity already at $t = 7.6$ and loses 94% of his or her highest productivity, cf. Figure 8b.

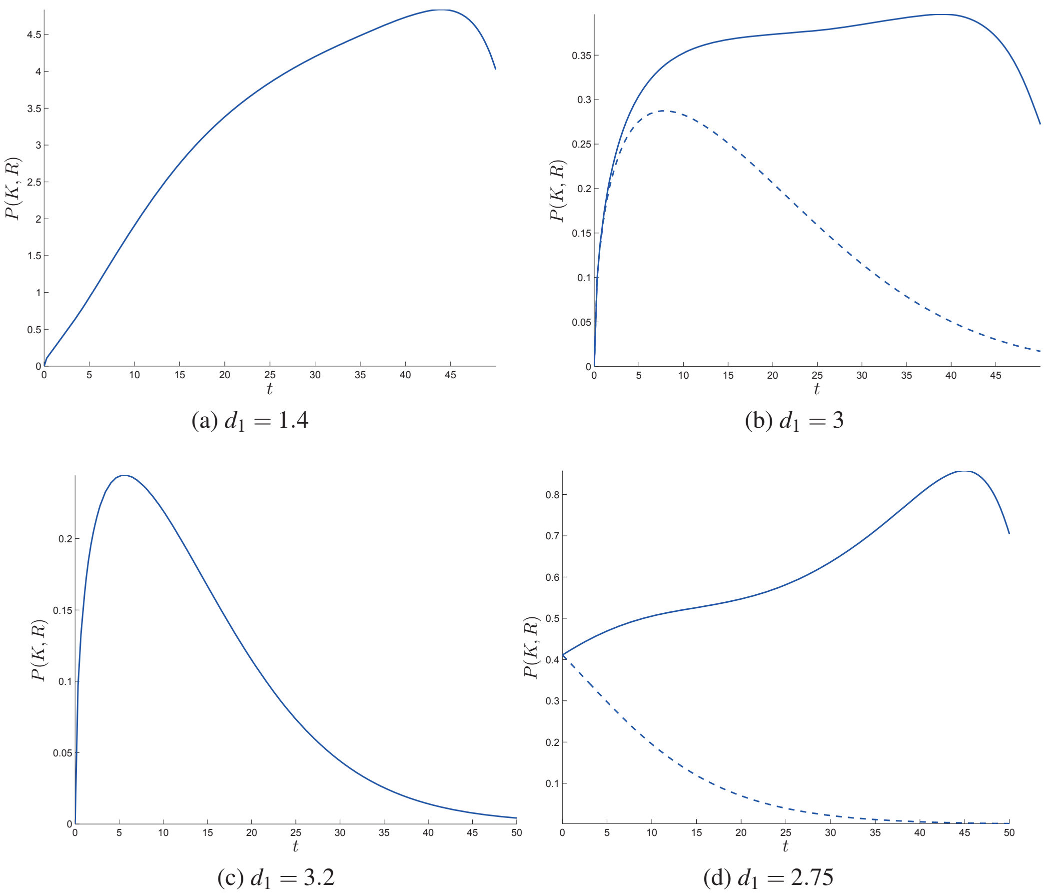


Fig. 8: Panels (a)-(c) show time paths of the production function $P(K,R)$ for the three cases, starting at the Skiba state $\tilde{K} = 0.475$. Panel (d) depicts a second example from the Skiba region II, with $d_1 = 2.75$.

The absolute value of the productive researcher in the Skiba cases is low compared to the researcher in Scenario I. This is due to the fact that we have chosen a high value $d_1 = 3$ for the Skiba case, a choice we made for pragmatic reasons, since in this way the manifold of low end points in Figure 6 was better visible.

Considering a lower value of $d_1 = 2.75$ in region II we find a qualitatively different solution pattern for less productive researchers. In that case we find the so-called *fading* career, where researchers gradually reduce their research activities over time, cf. Figure 8d. For an explanation of the different behaviour for both Skiba cases, we have to note that for lower d_1 the Skiba point $\bar{K} = 0.107$ is smaller than the Skiba point for $d_1 = 3$ with $\tilde{K} = 0.475$. Therefore, the starting situation for both cases differs substantially.

3 The dilemma of learned societies

The second application in this paper deals with the gradually increasing ageing of learned societies, which has been documented in a series of studies on European and American academies of sciences such as the Académie des Sciences (Institut de France) (Leridon 2004), the Royal Danish Academy of Sciences and Letters (Matthiessen 1998), the Royal Netherlands Academy of Arts and Sciences (van de Kaa et al. 2008), the National Academy of Sciences [of the U.S.] (Cohen 2003), the Austrian Academy of Sciences (Feichtinger et al. 2007) and in an international comparison, the British Royal Society, the Russian Academy of Sciences, the Berlin-Brandenburg Academy of Sciences and Humanities, and the Norwegian Academy of Sciences and Letters (Riosmena et al. 2012). The studies examined the historic, present and future demographic developments of these learned societies with a special focus on the statutory restrictions governing size and election procedures of the academies. In fact, the byelaws of many learned societies specify a maximum size of members (under a certain statutory age), which allows elections only when places fall vacant (i.e. when members surpass that statutory age threshold). These restrictions create a dilemma for the learned societies in the context of population ageing, as Leridon (2004, p. 109) described it: "*To counteract the spontaneous trends in ag[e]ing in the institution, [...] new members would have to be elected at increasingly younger ages year after year, which would have the drawback of reducing the rate of population replacement.*" However, the latter strategy is in conflict with the academies' desire of being representative for all research fields and thus to be able to elect a sufficient number of young scientists from emerging research disciplines.

As mentioned in the introduction, the presidency of the OEAW got concerned on the ageing membership. In 2005, they asked the first author of the present paper on possible measures to be taken to counter the increasing ageing of its member population given the statutory conditions. In a research project funded by the Austrian Science Fund (FWF)[3], we thoroughly analysed the age dynamics of the Austrian Academy of Sciences, projected the future age distribution of the Academy members based on several alternative scenarios, and developed an age-structured optimal control model to determine the optimal trade-off between the rate of elections and the mean age of the academicians. In the following paragraphs, we present, discuss and update selected results achieved in the various research articles within the project (Dawid et al. 2009; Feichtinger and Veliov 2007; Feichtinger et al. 2007, 2012; Riosmena et al. 2012; Winkler-Dworak 2008).

In Feichtinger et al. (2007) the population of the OEAW has been studied in detail. The data come from the biographic records of the members of the Austrian Academy of Sciences (Hittmair and Hunger 1997; Österreichische Akademie der Wissenschaften 1996–2005). In December 2015, the presidency of the Academy asked for a re-evaluation of the ageing of the Academy and kindly provided updated records. The Austrian Academy of Sciences was funded in 1847 as "Kaiserliche Akademie der Wissenschaften in Wien" (Imperial Academy of Sciences in Vienna) under the auspices of Emperor Ferdinand I. The Academy is structured around two

[3] "Age Structured Populations with Fixed Size", contract no. P20408-G14.

sections: the section for mathematics and natural sciences and the section for humanities and social sciences. Membership distinguishes between honorary members, full members and corresponding members.[4] The bye-laws of the OEAW specify that the Academy comprises (at most) 90 full members, evenly divided between the two sections. Although membership is lifelong, members aged 70 years and above are, while fully retaining their rights, not considered for the maximum number of members. Finally, it is the full members who elect new Academy members.

In principle, the population dynamics of the Academy can be studied using the same methods as those employed when studying any other population. However, the Academy's vital events differ from those of a conventional population. There, the current generation of individuals will spawn the following one, and current academy members will also elect the next generation of academicians, but different from fertility, where the intake occurs in the lowest age groups, an Academy's intake may take place in all age groups, similar to immigration (see e. g. Espenshade et al. 1982; Feichtinger and Steinmann 1992). In addition, as the total number of members is limited by the bye-laws, the number of elections is strictly determined by the number of exits from the Academy, i.e. mortality, out-migration or leaving the Academy for other reasons, and retirement (i.e. surpassing the statutory age threshold of 70 years).

The next subsections thoroughly discuss the processes of in- and outflow into the Academy and their implications on the ageing of the Academy population. While the first section analyses the mortality of OEAW members and compares life expectancy of that group to recent values of the Austrian population, the focus on second subsection is on entrants into the Academy, i.e. on the age distribution of elections and its impact on the mean age of members. The third subsection formalises the trade-off between a young age structure and the number of elections and the fourth subsection finally derives an age-structure optimal control model for an optimal election policy given the dilemma of the academies.

3.1 The low mortality of learned societies

Academies of Sciences age more rapidly than national populations as members of learned societies have shown to exhibit a substantially lower mortality. This finding has been documented for various European academies[5] including the Austrian Academy of Sciences (Winkler-Dworak 2008). The latter author investigated the mortality of the members of the Austrian Academy of Sciences since its foundation in 1847 to 2005 and contrasted it to life table death rates for the Austrian popula-

[4] Full membership requires residence in Austria. If a full member moves abroad, his or her status changes to that of a corresponding member abroad. In a similar vein, corresponding membership distinguishes between residence in Austria and abroad. For an illustrative example of multiple changes in membership status see that of the famous physicist Ludwig Boltzmann in Feichtinger et al. (2006b).

[5] See Andreev et al. (2011) for the Royal Society and the Russian Academy of Sciences; Matthiessen (1998) for the Royal Danish Academy of Sciences and Letters, van de Kaa et al. (2008) for the Royal Netherlands Academy of Arts and Sciences, Leridon (2005) for the French Academy of Sciences, and Winkler-Dworak and Kaden (2013) for the Saxon Academy of Sciences and Humanities in Leipzig.

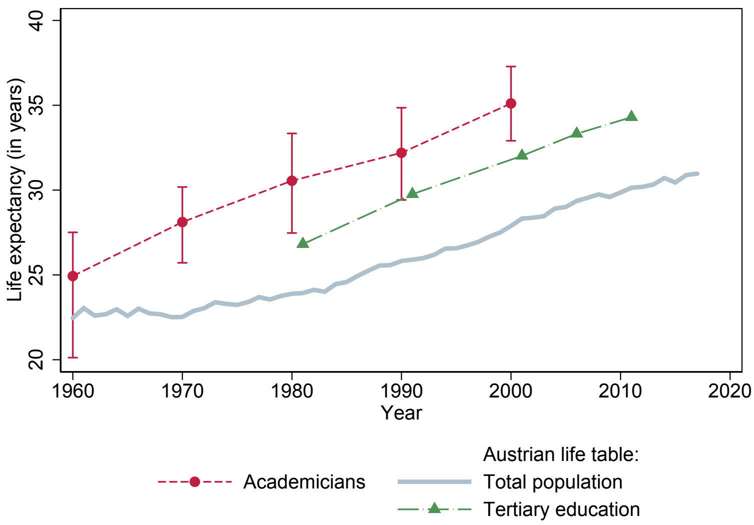


Fig. 9: Life expectancy at age 50 for members of the Austrian Academy of Sciences (with 95% confidence interval), Austrian general population and Austrian tertiary educated population (all only male populations). Source: Authors' own computations (academicians); Human Mortality Database (2019) (total population); Doblhammer-Reiter (1996), Doblhammer et al. (2005), Klotz (2007), Klotz and Asamer (2014) (tertiary-educated population).

tion.[6] During the first hundred years of the Academy's existence, the members did not exhibit a significantly lower mortality than the general Austrian population, but after World War II a clear survival advantage of the academicians emerged (Winkler-Dworak 2008).

Figure 9 replicates the analysis presented in (Winkler-Dworak 2008) for the life expectancy at age 50 in the second half of the 20th century and complements it with corresponding recent life expectancy values for the entire Austrian population and for those with tertiary education. Over the second half of the past century, academicians showed an increasingly higher life expectancy at age 50 than the Austrian male population, which is even more remarkable given the fact that also life expectancy for the latter has been steeply increasing since the 1970s. At the turn of the past century, Austrian academicians had a remaining life expectancy of about 35.1 years (95% confidence interval [32.9, 37.3]), about seven years more than the Austrian male population at the time. The survival advantage even persists, though at a smaller extent of about 3 years, compared to the population with tertiary education.

From a time perspective, the latest available value for the life expectancy at age 50 of the Austrian tertiary educated population in 2011 is still slightly lower than the corresponding value for the Academy members a decade earlier. What is even more striking is that only very recently average Austrian men have reached a life expectancy value at age 50 similar to that which academicians already enjoyed around

[6] The analysis concerned only members residing in Austria and, due to the historically low number of women among Academy members, was limited only to male members.

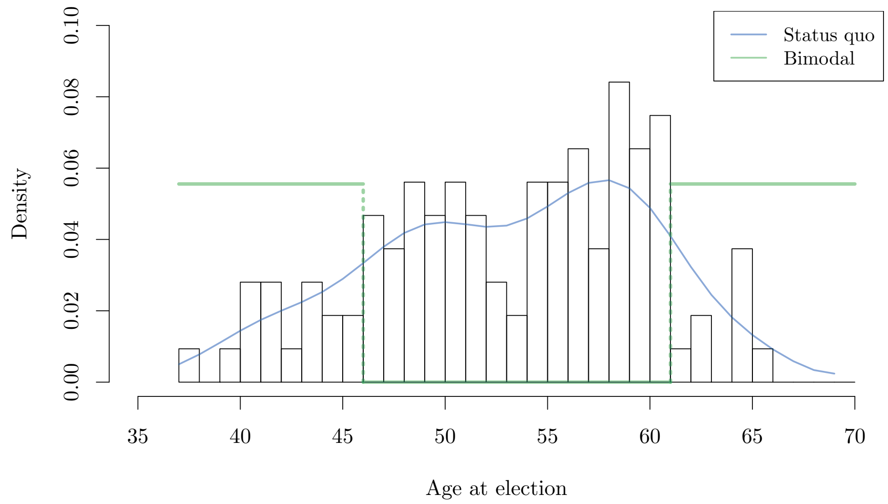


Fig. 10: Histogram of age distribution at election of members of the Austrian Academy of Sciences (both sections combined, 2000–2015) and density plots of alternative projection scenarios.

1980. Therefore, several authors concluded that academicians indeed represent a vanguard group in the achievement of longevity (Andreev et al. 2011; Winkler-Dworak and Kaden 2013), and thus, academies age more rapidly than other population groups or general populations. However, in addition to the exceptionally high and increasing longevity, many European academies have experienced an upward secular trend in the mean age at election contributing to the ageing of academies via the inflow (Riosmena et al. 2012). The next subsection will therefore explore the age distribution at election and its consequences on the age structure of the member population.

3.2 Projecting the impact of the age at election of Academy members

The population dynamics in hierarchical bodies in which the total membership size remains constant is determined by the rate of intake, the age distribution at entry into a given status, the number of exits (deaths or dismissals), the statutory retirement age and the population size. The intake itself is solely determined by the number of deaths and retirements. The only scope for modifying the age structure lies in the age distribution of entries, e. g. at election. Figure 10 shows the histogram of the age distribution of election into the OEAW for the years 2000–2015.

In the period 2000–2015, ages at election ranged over nearly 30 years of age, where the youngest member was elected at age 37 years, while it was age 65 years for the oldest member at election. The mean age of the age distribution at election was around 53.6 years with a standard deviation of 6.6 years. A Gaussian kernel estimate of the density function (see e. g. Hartung et al. 2002) yields a bimodal curve with the first mode around age 50 and the second mode around age 58 years (blue curve in Figure 10). In principle, a bimodal function at election may arise because of a conjunction of two motives: on the one hand, electing young members may signify rewarding excellence, while on the other, electing older ones means a recognition of lifetime achievement.[7]

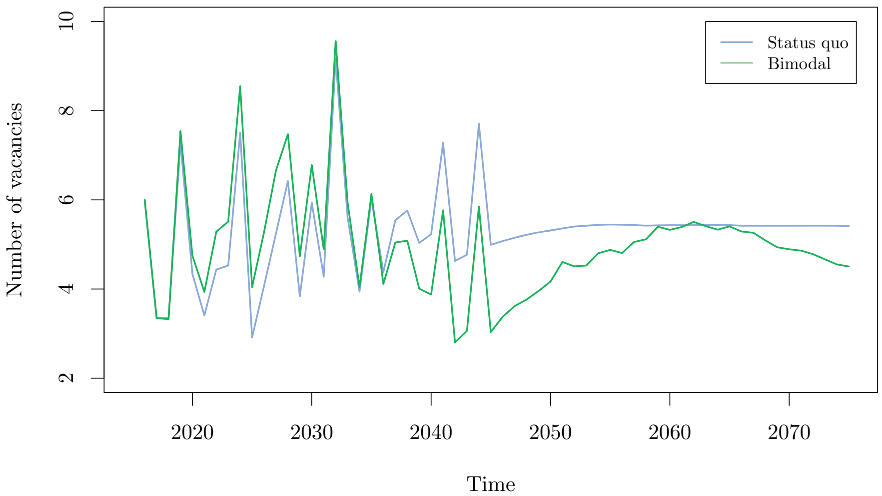
In order to study the impact of the age distribution at election on the structure of the Academy population, we use demographic projection methods. First, we project the number of members per section for each single-year age group alive in the next year. The survivorship rates are based on forecasted age-specific life table death rates from Statistik Austria (2015), which were adjusted for the lower mortality of the academicians (see subsection above; for more details on the adjustment see Feichtinger et al. 2007). The difference between the number of survivors and the maximum size of each section yields the number of vacant places in each section. We assume that vacant seats are immediately filled in the following year by electing new Academy members.

For the age distribution at election of new members, we consider two alternative scenarios. The first scenario represents the continuation of the *status quo* (blue curve in Figure 10) and is captured by the estimated density of the observed age distribution at election from the years 2000–2015 for both sections combined. Second, we model the two motives of rewarding excellence vs lifetime recognition in a strongly polarised pattern by assuming a *bimodal* age distribution (green curve in Figure 10), where members are uniformly elected only at very young ages (i.e. 37–47 years) and at older ages (i.e. 59–69 years). Note that such an election strategy is quite the opposite of current practice, as the vast majority of the members were elected at medium ages between 2000–2015 (cf. Figure 10). Nonetheless, the mean ages of both election scenarios are very close to each other, whereas the standard deviation of the *bimodal* scenario is almost the double of the *status quo* one.

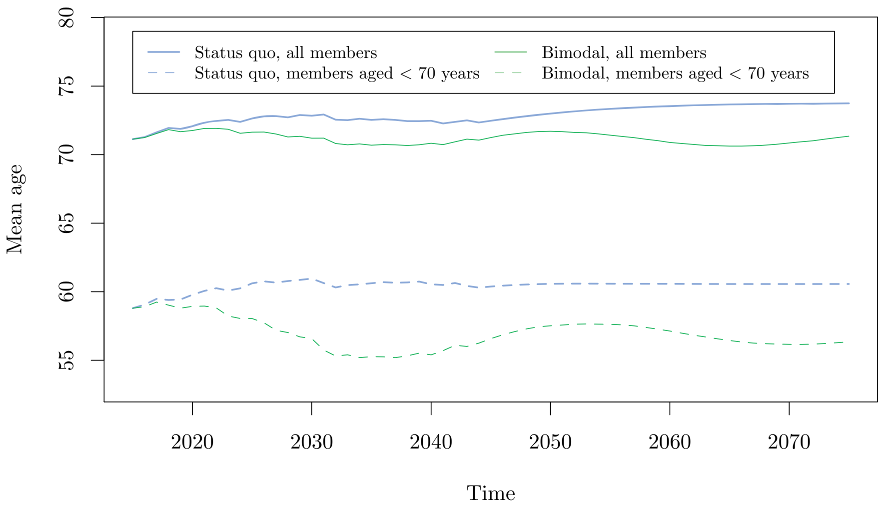
Figure 11 (top panel) depicts the projected number of vacancies from 2015–2075 for the two alternative scenarios and both sections combined.[8] Over the transitionary period ranging into the 2040s, the number of vacancies sharply fluctuates due to the initial age structure of members in 2015 and then stabilises for the *status quo* scenario around five elections per year. In contrast, the number of vacancies under the *bimodal* scenario seems to be characterised by longer-period waves in the later part of the projection horizon. Evidently, the number of vacancies in the first five years is almost solely determined by the age structure of members in 2015 and only afterwards small differences become visible between the alternative scenarios. In particular, the higher number of members elected close to the statutory age limit under the *bimodal* scenario results in a slightly higher projected number of vacancies than for the *status quo* scenario in the 2020s, while it is the opposite from mid-2030s onwards. The trough in the number of vacancies for the *bimodal* scenario in the early 2040s results from the fact that medium-aged scientists are not considered for election under the latter scenario. Only when the first very young elected members reach the

7 We are grateful to Warren Sanderson for this interpretation.

8 The Kolmogorov-Smirnov test did not yield any statistically significant differences in the age distribution at election between the sections ($p = 0.9$).



(a) Number of vacancies/elections



(b) Mean age of members

Fig. 11: Projected number of vacancies/elections (top panel) and mean age of members (bottom panel) for the Austrian Academy of Sciences, 2015–2075.

statutory age threshold does the number of vacancies start rising again, reaching the projected number of vacancies for the *status quo* scenario in the early 2060s. As we assumed that vacancies are immediately filled, the longer tenure associated with the very young new members implies a decrease in the projected number of vacancies towards the end of the projection horizon. The latter result clearly demonstrates the trade-off of the academies between a young age structure of the members and high number of vacancies.

The bottom panel of Figure 11 finally plots the projected mean age of members for the two scenarios, again for both sections combined (solid lines). In addition, the dashed lines represent the mean age of members only for those aged less than the statutory age threshold. Apart from around 2030 to 2040, the projected mean age of all members continuously increases over the projection period and eventually amounts to about 73.7 years for the *status quo* scenario. In contrast, the projected mean age under the *bimodal* scenario fluctuates around 71 years. Considering only members aged less than the statutory age threshold, the differences between the scenarios are more pronounced. While the projected mean age for the *status quo* stabilises around 60.6 years, the corresponding value for the *bimodal* scenario fluctuates between 55.2 and 57.6 years.

Summing up, the *bimodal* scenario would yield a substantially lower mean age for Academy members than a continuation of the current election practice, although both election policies exhibit a similar mean age. Intuitively, a lower/higher mean age at election should decrease/increase the mean age of the member population. However, the results of the projections suggest that other characteristics such as the spread of the age distribution at election substantially affect the mean age of the member population as well. In the next sections, we will formalise the trade-off between a young age structure and high number of recruitment in constant-sized populations and we derive a relationship between the mean age of members and the characteristics of the age distribution of members. Later we will develop an age-structured optimal control model to counteract the ageing of the Academy population while ensuring a sufficient number of vacancies.

3.3 Formalising the dilemma of the academies

Intuitively, to counteract the trend of ageing, new members have to be elected at increasingly young ages. As mentioned earlier, this would have the drawback of reducing the inflow of new members. Thus, there is a fundamental dilemma in a constant-sized, age-structured population, such as in an academy of sciences: the desire to maintain a young age structure, while ensuring a high recruitment rate.

The following thought experiment by OEAW member Gerhart Bruckmann (cited in Feichtinger et al. 2007) illustrates this trade-off: “If the Academy elects only 47.5 year old members, they stay—neglecting mortality and other possibilities for exit—22.5 years in the membership population decisive for the maximum size. The OEAW comprises 90 full members (45 in each section) below the statutory age, which yields $90 : 22.5 = 4$ entrants each year. If, on the other hand, only 55 year old members are elected, the same calculation results $90 : 15 = 6$ entrants per year.” Carrying the argument to extremes, if all members are elected at age 69, then there will be maximum recruitment every year.

These simple calculations of a constant-sized population are based on a fundamental identity in demography. Denoting by M the total size of the population, by R the number of annual new entrants and by T the mean duration in the population, the stationary state is characterised by the relation

$$M = RT. \tag{10}$$

For conventional populations, the stationary state arises for a constant flow of births and unchanging age-specific death rates over time. Then, R denotes the annual constant number of births and the average duration T equals life expectancy at birth of the stationary population. Hence, the identity connects the three most important indicators of a stationary population, namely population size (stock), the births (entrants) and life expectancy (average duration). Note that in queuing theory, which is based on birth-death processes, the identity (10) is known as Little's formula (see Hillier and Lieberman 1974, p. 384)

The stationary model is applicable to any kind of population. An example widely used in textbooks and lecture notes refers to student cohorts at university (see e. g. Preston et al. 2000): a graduate program enrols $R = 10$ students per year and has a student body of $M = 40$; it can be assumed to have a mean duration in graduate school of $T = 4$ years. If due to financial restrictions, the student body M must shrink, there are essentially two possibilities to reach this target: either the admissions R have to decrease or the mean length of the studies T must be reduced.

For the sake of simplicity, we consider in what follows only the Academy members below the statutory age threshold as it is the total size of that group which is limited by the bye-laws. They correspond to a fixed-sized organisation (i.e. equation (10) holds) with a prescribed retirement age omega.

In this case, Dawid et al. (2009) derive an interesting relation between the mean age of the academicians, the mean age of entrants and the variance of the recruitment distribution:

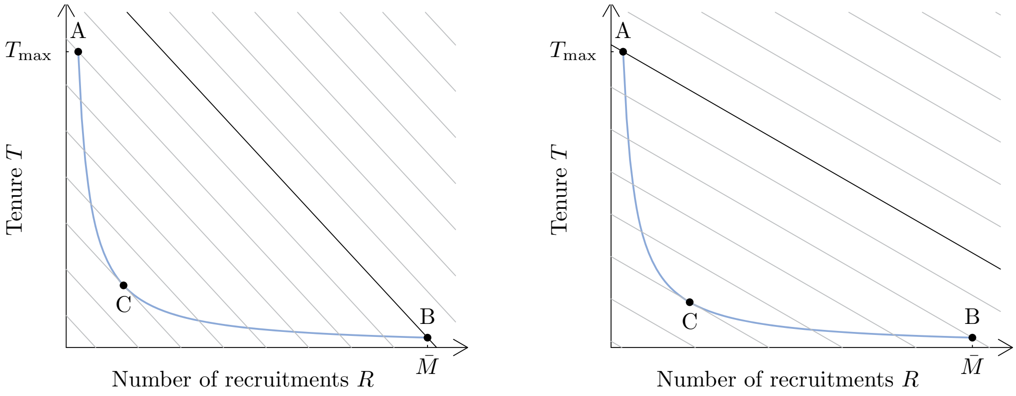
$$\bar{A} = \frac{1}{2}\left(\omega + m - \frac{\sigma}{\omega - m}\right), \tag{11}$$

where $\bar{A}$ denotes the mean age of the fixed-sized organisation, m the mean age of recruitment distribution, and σ^2 its variance.

On this formula, two issues are remarkable: (1) intuitively, the average age of the stock $\bar{A}$ increases with the mean age of entrants, m. However, it can be shown that the latter holds if and only if $\omega - m > \sigma$, which is numerically fulfilled for the age distributions at elections, which we considered. The latter case also implies that the mean age of the population $\bar{A}$ and the mean duration in the system (i.e. the average tenure) $T = \omega - m$ are inversely related. (2) The variance of the recruitment distribution, σ^2, influences $\bar{A}$ negatively as suggested by the difference in the mean age of members between the two projection scenarios above.

Remark: The remarkable property that the arithmetic mean $\bar{A}$ depends only on the first two moments of the recruitment distribution, m and σ^2 has an interesting analogue in queueing theory. For the single-channel queueing system M/G/1, i.e. exponentially distributed independent interarrival times and independent and identically distributed general service time distributions, the so-called Pollaczek-Khinchin formula is valid (Gross and Harris 1974). It says that the expected number of customers in the system depend exclusively on the first two moments of the service time distribution. More precisely, the length of an M/G/1 queue increases both with the mean duration of service as well as its variance. Note that the latter dependency is just opposite to the formula (11), where a concentrated entrance distribution yields the highest mean age.

As pointed out in the beginning of the subsection, academies are faced with two conflicting goals: to obtain a young age structure (or, mathematically equivalent, a high average duration), while ensuring a high recruitment rate. However, since the product of the right-hand side of identity (10) is constant, it is not possible to increase



(a) α dominates, thus B is optimal

(b) β dominates, thus A is optimal

Fig. 12: Illustration of maximising weighted sum of number of recruitments and mean length of tenure T subject to trade-off between recruitment R and average tenure T (blue curve) with alternative optimal corner solutions. The parallel grey lines represent indifference curves of equal objective value.

both R and T simultaneously. Hence, we define an objective function as a weighted mean of R and T, which we aim to maximise, i. e.

$$\max(\alpha R + \beta T), \tag{12}$$

where α and β are non-negative weights with $\alpha + \beta = 1$.

The maximisation of the objective function (12) subject to the condition (10) is depicted in Figure 12. While the side condition (10) is represented by a hyperbola in the state space, the parallel lines with slope α/β indicate the objective function with equal values. The higher the intercept of the lines, the higher the value of the objective function.

Figure 12 illustrates that corner solutions are optimal. If α dominates, then it is optimal to elect a maximum number of entrants, who stay for only one year (point B in Figure 12, left panel), while for large values of the weight β all entrants stay in the system for the maximal possible tenure (point A in Figure 12, right panel). Note that the tangent to the hyperbola (point C in Figure 12) refers to the smallest feasible value of the objective (12).

3.4 An optimal age-structured control model

Let $M(a,t)$ denote the number of members of a learned society at time t and age a. The dynamics of the age-structured population $M(a,t)$ can be expressed in form of the McKendrick equation used in formal demography (McKendrick 1926; Keyfitz and Keyfitz 1997).

$$M_t(t,a) + M_a(t,a) = -\mu(a)M(t,a) + R(t)u(t,a), \tag{13}$$

The population gains new members, not through birth or immigration, but by way of elections (recruitment of new members) indicated by the term $R(t)u(t,a)$.

$$R(t) = M(t,\omega) + \int_0^{\omega} \mu(a)M(t,a)\,\mathrm{d}a, \tag{14}$$

with the side conditions

$$M(0,a) = M_0(a), \quad M(t,0) = 0, \tag{15}$$

where we used the following notation:
$\mu(a)$ the time-invariant mortality rate of members at age a;
$R(t)$ the intensity of recruitment at time t;
$u(t,\cdot)$ is the age density[9] of recruitment at time t;
$M_0(\cdot)$ is the initial age-density of members;
ω is a fixed exit (retirement) age of members:
$M_t + M_a$ is the sum of the partial derivatives of M (strictly speaking, this is the derivative of M in the direction (1,1) in the (t,a)-plane, i.e. the change along a diagonal in the Lexis diagram).

The dynamics of the age structure of the learned society is given by the classical McKendrick equation (13), while (14) indicates that the size of the organisation is fixed and equals $\bar{M} = \int_0^{\omega} M_0(a)\,\mathrm{d}x$ (this can be easily seen by integarating (13) over a and utilizing the assumption for fixed size). Alternatively (14) can be understood as follows: At any time t the recruitment $R(t)$ is determined by the number of people reaching the threshold age ω (first term on the r.h.s.) and the number of deaths, where the latter is determined by the sum of age-specific deaths (second term on the r.h.s.).

The following constraints are posed for the recruitment density, $u(t,\cdot)$, which is considered as the control (decision) variable:

$$0 \le u(t,a) \le \bar{u}(a), \qquad \int_0^{\omega} u(t,a)\,\mathrm{d}a = 1, \tag{16}$$

where $\bar{u}(a)$ is an upper bound for the control.

As mentioned above, we focus our analysis on two objectives:

- the recruitment intensity, $R(t)$, which is to be maximised;
- the average age $\frac{1}{M}\int_0^{\omega} aM(t,a)\,\mathrm{d}a$, which is to be minimised.

Since two (conflicting) objectives are involved, we employ the Pareto optimisation framework, considering the aggregated objective function

$$\max \int_0^{\infty} e^{-rt} \left[\alpha R(t) - \beta \int_0^{\omega} aM(t,a)\,\mathrm{d}a \right] \mathrm{d}t, \tag{17}$$

where $r > 0$ is a time-preference rate, $\alpha > 0$ and $\beta \ge 0$ are weights attributed to the two objectives. The first objective is to maximise the recruitment intensity $R(t)$, while the second objective is to minimize the average age $\int_0^{\omega} aM(t,a)\,\mathrm{d}a$ of the members.

[9] To avoid misunderstanding we stress that $M(t,\cdot)$ need not be a probability density, while $u(t,\cdot)$ is assumed to be a probability (normalised) density, in the sense given by the equality in (16) below.

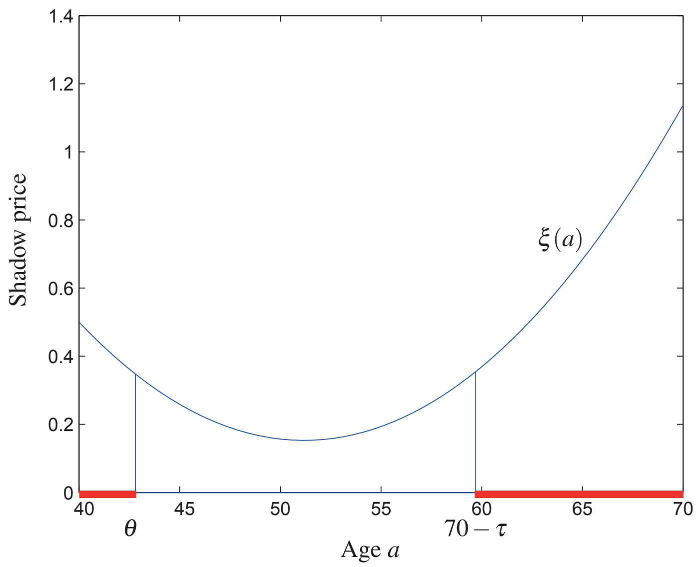


Fig. 13: Stationary shadow price $\xi(a)$ of a person elected at age a, where a varies between 40 and 70, for $\alpha = \beta = 0.5$; The red lines denote the lower and upper boundary age intervals in which persons are recruited.

For an exact solution to the system (13)–(15) we refer to Feichtinger et al. (2003) and Feichtinger and Veliov (2007).

The solution of the infinite-time age-distributed optimal control problem described above requires a special form of Pontryagin's maximum principle presented in Feichtinger and Veliov (2007). Crucial for such an analysis is the adjoint variable $\xi(a)$, i.e. the shadow price of a recruited individual at age a. More precisely, $\xi(a)$ measures the marginal value of a newly elected person at age a. It turns out that the shadow price shows a U-shaped pattern (see Figure 13). This remarkable fact implies that it is better to select younger and/or older individuals rather than middle-aged ones; compare Figure 11 illustrating the superiority of bi-polar recruitment policies.

To summarise our main result: the intertemporal optimisation procedure reveals that it is optimal to elect a mix of young and old entrants to guarantee a young Academy while avoiding a freeze of recruitment altogether. It should be noted that the election of medium-aged persons is the worst solution in terms of the proposed target (compare also Figure 12.)

4 Conclusions

The purpose of this paper was to illustrate the role of intertemporal optimisation in age-related problems. Among many issues two cases have been selected. The first deals with the production of a scientist over his or her life cycle, while in the second case one asks how a constant-sized age-structured organisation can be kept young.

A crucial characteristic of the economics of science is the extreme inequality of scientific productivity (compare e. g. Stephan 1996). Productivity patterns vary substantially across individuals and over the life cycle. Lotka (1926) stressed the highly skewed nature of scientific publications. In physics, for instance, he observed that six per cent of publishing scientists produced half of all papers. The inherent inequality of scientists has been formulated by Goodwin and Sauer (1995) as follows: “While some authors publish papers like a well-oiled machine, others produce at an erratic rate, and some others show early promise but become deadwood after a certain time.”

Several factors have been suggested to contribute to the inequality in productivity between scientists. While Symonds et al. (2006) refer to discrepancies between women and men appearing early in their scientific careers, the ’Matthew Effect’ in science (Merton 1968) states that past success in research usually acts as leverage for future productivity (’the winner takes it all’).

Moreover, the scientific productivity of individual researchers over their career is not evenly distributed in time. The predominant pattern exhibits an intuitively plausible course over the life cycle: scientific creativity tends to rise rapidly to a peak and then to gradually decline.

Hump-shaped life cycle patterns are not only observed in academia, but also in other fields as in artistic production, consumption of illegal drugs as well as other criminal behaviour. Demographers will note the similarity to the age-specific first marriage and fertility curves.

In ’The Wiley Handbook of Genius’, Simonton (2014) provides a rich collection of various forms of creativity. While almost all models dealing with the dynamics of scientific productivity are descriptive (see Rinaldi et al. 2000, for an interesting example), there are a few normative approaches using the human capital approach proposed by Becker (1962); see Levin and Stephan (1991) and Stephan (1996) for such examples.

The approach we have chosen in part 1 of the present paper may be seen in this line. Assuming that the scientific output depends not only on knowledge, i.e. the human capital accumulated by a researcher but also on the network of colleagues he or she is embedded in, we are able to explain various productivity patterns over the life cycle identified empirically by Way et al. (2017). The present paper can be seen as a theoretical foundation of these empirical facts.

In the intertemporal optimisation model we assumed that the scientists derive utility from publishing papers and from performing research and networking. However, working too hard causes disutility, i.e. too large investments in knowledge and networking are costly.

If scientists do not bother about their reputation at the end of their career, we show that a sufficient education level is needed for scientists to develop a typical research pattern where productivity increases in the beginning of their career while declining towards retirement. If the education level is not sufficient, a fading research pattern will result where productivity declines over time. On the other hand, when a scientist is eager to have a good reputation at the end of his or her career, sufficient education will result in increasing productivity over the career lifetime, preventing a midlife slump.

Let us briefly mention two possible extensions of the proposed model. While we did capture that the efficiency of research depends on the stock of knowledge already accumulated, the functions g and h in (2) and (3), respectively, might also depend on the quality of the colleagues in one's network, i. e. on the reputation of the scientists. Another extension would be that the efficiency of accumulating knowledge and reputation explicitly depends on age, i. e. on time t. Thus, in a more realistic scenario, the effects of ageing and/or learning should be included.

Scientists reaching a notably high level of reputation in their career are likely to be elected into an academy of sciences. The second part of the paper deals with the ageing of such learned societies. In fact, these institutions have been shown to age more rapidly than any national population. We investigated the accelerated ageing on the example of the Austrian Academy of Sciences from two sides. On the one hand, the Academy members exhibit a remarkable longevity. We have shown that the academicians' life expectancy at age 50 of around twenty years ago has still not been reached by the Austrian total population of today. It has been suggested that the exceptional longevity is not only the result of selectivity into election to membership of academicians (for an extensive discussion see Andreev et al. 2011; Winkler-Dworak and Kaden 2013). According to the literature on the social gradient in mortality (e. g. Mackenbach et al. 1999), factors beneficial to health, such as a high educational level, upper professional status associated with high income accumulate for academicians. Moreover, Academy members may even enjoy a further longevity advantage compared to average scientists by the social circumstances that the status confers (Link et al. 2013): election into an Academy not only indicates (and rewards) scientific excellence and an outstanding contribution to science, but it certainly entails an enlargement of one's personal academic network, providing further research opportunities and thus facilitating to stay scientifically active beyond retirement. Continued demanding mental activities to very high ages are clearly associated with less cognitive decline (Schooler and Mulatu 2001; Kliegel et al. 2004) and higher longevity (Ghisletta et al. 2006; Gondo and Poon 2007) and may therefore contribute to the outstanding longevity of the academicians.

On the other hand, the OEAW has experienced an upward secular trend in the age at election. By employing demographic projection, we investigated how the age structure of the Academy might evolve if the current election policy were to be continued. In a second step, we developed an age-structured optimal control model in order to derive an optimal trade-off between the two conflicting goals of minimising the average age of the learned society and maximising the number of recruitments per year. Our results indicate that it is best to elect new members who are either young or old, with as few middle-aged new entrants as possible. Although the current election policy displays a somewhat bimodal pattern, the modes are still more concentrated around the mean than in the derived optimal recruitment policy.

Feichtinger et al. (2012) consider further interesting extensions to the derivation of optimal recruitment policies for age-structured organisations. Learned societies such as the Russian Academy of Sciences, the Académie Française and OEAW recruit their members from the pool of corresponding members. The latter authors show that there is an approximately fifty per cent chance of a corresponding member to become a full member while the mean duration is 7.5 years (remarkably constant over time). Further research may analyse which kind of promotion policies with respect to quantum and timing should be implemented by an academy.

Clearly, there are many more mathematical issues in ageing. Let us briefly mention only one of them. In an article in the Economist (2014) it is presumed that half of all people who have ever been over 65 are alive today. Motivated by these discussions and by a paper by Cohen (2014), we asked the question *'How many old people have ever lived?'* By using formal demography together with historical data on population processes we could show that only 5–10 percent of all people aged 65 or more years are alive today (see Sánchez-Romero et al. 2017).

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