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Splitting a Random Pie: Nash-Type Bargaining with Coherent Acceptability Measures

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We propose an axiomatic solution concept for cooperative stochastic games where risk averse players bargain for the distribution of profits from a joint project that depends on management decisions by the agents. We model risk preferences by coherent acceptability functionals and show that in this setting the axioms of Pareto optimality, symmetry, and strategy proofness fully characterize a bargaining solution, which can be efficiently computed by solving a stochastic optimization problem. Furthermore, we demonstrate that there is no conflict of interest between players about management decisions and characterize special cases where random payoffs of players are simple functions of overall project profit. In particular, we demonstrate that for players with distortion risk functionals, the optimal bargaining solution can be represented by an exchange of standard options contracts with the project profit as the underlying. We illustrate the concepts in the paper by an extended example of risk averse households that jointly invest into a solar plant.

Key words: Stochastic bargaining games; coherent risk measures; stochastic programming; photovoltaics

History:

1. Introduction

Most business decisions entail uncertain future profits for the involved agents who in many cases are risk averse. If there is only one party involved in the project interacting with other economic entities only via markets, the theory of risk averse decision making can be used to evaluate and optimize decisions (Föllmer and Schied 2004, Pflug and Römisch 2007).

If a project is undertaken jointly by several agents and there is no efficient market for individual contributions, the question of how to distribute the project's profits arises. Examples for such situations are investment decisions or joint ventures.

The case when revenues are deterministic is well studied in cooperative game theory. Solution approaches include bargaining games and coalitional games. There is a rich literature on theoretical

properties of these games as well as on practical applications (e.g., Osborne and Rubinstein 1994, Maschler et al. 2013).

In this paper, we consider the situation of multiple risk averse agents that undertake a project together and bargain for the distribution of the resulting random profits. Sharing amongst risk averse players is a problem which arises in several fields of application including banking and insurance (e.g., Fragnelli and Marina 2003), engineering (e.g., Melese et al. 2017), as well as the rapidly growing *sharing economy*, which is currently transforming, among others, the hotel industry and transportation services (e.g., Zervas et al. 2017).

Our approach is designed with applications in the energy sector in mind. In the course of the global transition to clean, renewable energy there are many situations where a number of *small players* cooperate in projects that would be too costly or otherwise too demanding for a single entity. Examples are households that jointly invest in solar power plants or community storage (Chakraborty et al. 2019a,b) or the case of small companies that own renewable generation assets and pool their production in a virtual power plant for cost efficient market access as well as for diversification of market and production risks (Baeyens et al. 2013, Han et al. 2018, Nguyen and Le 2018a,b, Shin and Baldick 2018, Kovacevic et al. 2018, Gersema and Wozabal 2018, Han et al. 2019).

Most authors model project outcomes as deterministic and therefore do neither capture the inherent uncertainty in investment and operation of renewable electricity generation and energy storage nor the influence of risk preferences on the result of joint decisions (e.g. Chakraborty et al. 2019a,b, Han et al. 2019)

The few papers that treat the stochastic case either do not compute the optimal split of random profits (see for example Baeyens et al. 2013, Nguyen and Le 2018a,b, Shin and Baldick 2018, Gersema and Wozabal 2018) or use an ex-post distribution of profits based on the Shapley value (Ma et al. 2008, Han et al. 2018, E et al. 2014).

A principled approach that considers both randomness and risk aversion and yields an explicit ex-ante distribution of random payoffs is lacking. In this paper, we aim to fill this gap and investigate bargaining games under uncertainty where agents have risk preferences that can be expressed by coherent acceptability functionals. Based on a set of axioms, we characterize a unique distribution of acceptabilities which can be efficiently computed.

In the following, we briefly review the literature on stochastic cooperative game theory and point out the differences to our approach to better frame our contribution.

Many papers in stochastic cooperative game theory assume risk neutral decision making, neglecting the impact of risk aversion on the outcome of the game (e.g. Charnes and Granot 1976, 1977,

Xu and Veinott 2013, Parilina and Tampieri 2018). Among the authors that consider risk preferences most do not consider coherent risk measures but other forms of risk quantification – most notably expected utility (Suijs et al. 1999, Fragnelli and Marina 2003, Timmer et al. 2005, Habis and Herings 2011, Melese et al. 2017, Németh and Pintér 2017). To the best of our knowledge there are only very few papers that use coherent risk measures in cooperative games (e.g., Boonen et al. 2016, Toriello and Uhan 2017, Asimit and Boonen 2018).

Furthermore, most papers in stochastic cooperative game theory focus on theoretical properties such as existence and stability of solutions, but do not derive an explicit split up of random profits (e.g., Habis and Herings 2011, Xu and Veinott 2013, Németh and Pintér 2017, Toriello and Uhan 2017, Parilina and Tampieri 2018).

Lastly, most authors impose different properties on solutions of the game such as envy free allocations (Fragnelli and Marina 2003), maximal overall utility (Melese et al. 2017), or, most prominently, membership in the core of a coalitional game (Habis and Herings 2011, Xu and Veinott 2013, Toriello and Uhan 2017, Németh and Pintér 2017, Asimit and Boonen 2018). The latter concept is intimately linked to coalitional games, whereas the focus in this paper is on bargaining games.

Summarizing, our paper contributes to the literature on stochastic cooperative game theory in the following ways: Firstly, different from most of the literature, we model risk preferences by coherent acceptability functionals. Secondly, we frame the problem as a bargaining problem with axiomatic foundations resembling Nash bargaining. Finally, we are able to characterize a unique solution that can be efficiently computed.

In Section 2, we frame the problem as a bargaining game with a fixed set of players. In particular, we assume that if the project is realized, all players participate, i.e., players gain no bargaining power from the threat of forming subcoalitions.

We impose the axioms of Pareto optimality, symmetry, and strategy proofness for a solution. While the symmetry axiom encapsulates a notion of fairness that makes our approach similar to Nash bargaining, strategy proofness avoids the problem of strategic splitting of agents. We argue that these axioms are sensible in our context and show that they uniquely characterize a bargaining solution.

As opposed to most of the extant literature, we require that agents agree on a distribution of payoffs before the randomness realizes. Our solution thus entails both a distribution of acceptabilities and the pointwise distribution of profits in every possible future scenario. We show that there is a unique distribution of acceptabilities which can be computed solving convex stochastic optimization problems, which are computationally tractable in the case of discrete randomness. In particular, our approach fully integrates with two-stage and multi-stage stochastic programming

in that it does not only derive the distribution of profits and acceptabilities but allows for the optimization of auxiliary managerial decision variables needed in order to optimally execute the project.

In Section 3, we derive structural properties of the bargaining solution by analyzing the optimality conditions of the associated stochastic optimization problems. In particular, we obtain necessary and sufficient conditions for the existence of a solution in terms of the subgradient sets of the coherent risk functionals of the players. We demonstrate that the optimization of the auxiliary managerial decisions is not a contested choice in the bargaining problem and that optimal project management can therefore be decoupled from the problem of distributing the random project profits.

Furthermore, we demonstrate that in realistic special cases optimal bargaining solutions can be specified as simple functions of random project profits. This is conducive to the implementation of the calculated distribution of profits in contractual agreements and thereby facilitates the real world implementation of the bargaining outcome.

In particular, we show that affine and linear allocation rules as used in Suijs et al. (1999) and Timmer et al. (2005) can only be optimal, if there is a *least risk averse* player. Furthermore, we prove that in the much more realistic special case where the risk preferences of all agents can be described by distortion functionals, the optimal distribution of profits can be achieved by the exchange of finitely many standard option contracts.

In Section 4, we use the problem of jointly constructing a solar roof for a shared building as an illustrative example for the proposed methods, while Section 5 concludes the paper and discusses avenues for further research.

2. The Bargaining Problem

In this section, we introduce an axiomatic framework for a class of bargaining problems under uncertainty. Section 2.1 outlines the general setting, introduces notation, and discusses our choice of risk preferences. In Section 2.2, we introduce and motivate three axioms for risk averse bargaining under uncertainty and show that our axiomatic framework uniquely characterizes a bargaining solution. Lastly, we show that the bargaining solution can be represented as the solution to an optimization problem, which is reminiscent of classical Nash bargaining with bargaining power.

2.1. Motivation & Setting

We consider a project that is jointly undertaken by n risk averse players. We assume that all random quantities are defined on a common probability space $\mathcal{Y} = (\Omega, \mathcal{F}, \mathbb{P})$.

Each participant $i \in N := \{1, \dots, n\}$ obtains an outcome $R_i : \mathcal{Y} \rightarrow \mathbb{R}$ when playing alone and not as a member of the group, while the whole group of n players that cooperates receives a joint random profit of $M : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$. M may depend on joint managerial decisions $x \in \mathcal{X}$, which we indicate by writing $M(x, \omega)$ when it is important to emphasize the dependence on $\omega \in \Omega$. Positive M and R_i indicate profits, whereas negative values model losses. Throughout the paper, we assume that $x \mapsto M(x, \omega)$ is concave almost surely (a.s.).

The group decides on $x \in \mathcal{X}$ and aims at distributing the joint outcome M among the participants in a “fair way”, which leads to individual (random) payoffs $L(x, \omega) = (L_1(x, \omega), \dots, L_n(x, \omega))$ such that

$$M(x, \omega) = \sum_{i=1}^n L_i(x, \omega), \quad a.s. \quad (1)$$

It is worthwhile to point out that in the above setting, the distribution of profits cannot be decided *ex-post*, i.e., after the uncertainty realizes. This is made clear by the following example.

EXAMPLE 1. Let there be two risk averse players and two equally likely future states of the world, i.e., $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2\}$, and $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = 0.5$. Assume further that $R_1(\omega_j) = -R_2(\omega_j)$ for $j \in \{1, 2\}$. If the players are risk averse, they would prefer a certain payment of 0 to their endowments R_i and thus have an incentive to agree to *pool their risks* by defining a payout rule

$$L_i(\omega) = \frac{1}{2}(R_1(\omega) + R_2(\omega)), \quad \forall i \in N, \quad (2)$$

which yields a certain revenue of 0 for both players in both states of the world.

Note that, in agreeing to (2), the players shift wealth between future state of the world to achieve a better risk profile *ex-ante*, much like when entering an insurance contract. They are, however, not necessarily better off *ex-post*, since in each of the two states of the world, there is one player who regrets having pooled her risks. For this reason, a solution which is based on the distribution of wealth after the realization of randomness cannot achieve an *ex-ante* optimal allocation, as players would not agree to shift wealth between the scenarios after the randomness realized. Hence, in the above example an *ex-post* allocation would likely result in the same pay-offs as a solution without any cooperation and therefore in *ex-ante* welfare losses.

Consequently, we have to consider the risk preferences of agents in deciding about random *ex-ante* payoffs L . We assume that the risk preferences of player i can be expressed by a coherent acceptability functional \mathcal{A}_i defined on the Lebesgue space $\mathcal{L}^p(\mathcal{Y})$ of p -integrable random variables. Throughout this paper we will assume that $p \geq 1$.

DEFINITION 1. $\mathcal{A} : \mathcal{L}^p(\mathcal{Y}) \rightarrow \mathbb{R}$ is a coherent acceptability functional if for all $X, Y \in \mathcal{L}^p(\mathcal{Y})$, $\lambda \in \mathbb{R}^+$, and $c \in \mathbb{R}$ the following holds:

1. Monotonicity: If $X \leq Y$ almost surely, then $\mathcal{A}(X) \leq \mathcal{A}(Y)$.
2. Positive homogeneity: $\mathcal{A}(\lambda X) = \lambda \mathcal{A}(X)$.
3. Super-additivity: $\mathcal{A}(X + Y) \geq \mathcal{A}(X) + \mathcal{A}(Y)$.
4. Translation invariance: $\mathcal{A}(X + c) = \mathcal{A}(X) + c$.

While economists traditionally prefer to think about risk in terms of expected utility, coherent acceptability functionals have recently gained popularity because of their ease of interpretation and analytical tractability. In particular, coherent acceptability measures can be interpreted in terms of monetary units, which is why they sometimes also called *monetary utility functions* (Föllmer and Schied 2004, Pflug and Römisch 2007).

In this paper, we assume that projects can only be undertaken jointly by all players. In particular, we are considering a bargaining setup, where players cannot obtain a higher share of M based on the threat of forming subcoalitions.

An alternative approach would be to frame the game as coalitional game and allow for subcoalitions as is done, for example, when computing the Shapley value. However, this requires an explicit assessment of the values of all possible subcoalitions and makes the computational complexity grow exponentially in the number of players. Typically, this renders the Shapley value approach infeasible already for problems with more than 25 players, see van Campen et al. (2018).

We therefore focus on bargaining situations where the formation of subcoalitions is either practically or legally impossible or where there are too many players for an approach such as the Shapley value to be feasible.

2.2. An Axiomatic Approach

This subsection develops an axiomatic theory of bargaining solutions, which is close to the Nash bargaining approach, but, as opposed to classical theory, allows profits L_i and opportunity losses R_i to be random.

DEFINITION 2. An *instance* of the bargaining problem is a triple (R, M, \mathcal{A}) with $R = (R_1, \dots, R_n)$, $R_i \in \mathcal{L}^p(\mathcal{Y})$ ($i \in N$), $M : \mathcal{X} \rightarrow \mathcal{L}^p(\mathcal{Y})$, and $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ where \mathcal{A}_i are coherent acceptability measures for all $\forall i \in N$.

For a given instance (R, M, \mathcal{A}) , we define the set of possible *utility allocations* as

$$\mathcal{U}(R, M, \mathcal{A}) = \{u \in \mathbb{R}^n \mid \exists x \in \mathcal{X} \exists L : \sum_{i=1}^n L_i \stackrel{a.s.}{=} M(x), L_i \in \mathcal{L}^p(\mathcal{Y}), \mathcal{A}_i(R_i) \leq u_i \leq \mathcal{A}_i(L_i), i \in N\}. \quad (3)$$

The set $\mathcal{U}(R, M, \mathcal{A})$ is the set of possible utility vectors (u_1, \dots, u_n) , where the utility u_i of player i is bounded from above by the acceptability of the payoff assigned to player i in case of cooperation,

and bounded from below by the acceptability of the payoff that player i can achieve without cooperation.

We assume that the values $v_i := \mathcal{A}_i(R_i)$, representing the personal evaluations of the opportunity losses, are always strictly positive. The instance (R, M, \mathcal{A}) is called *feasible* if the set $\mathcal{U}(R, M, \mathcal{A})$ is non-empty. A sufficient condition for feasibility of an instance (R, M, \mathcal{A}) is $\sum_{i=1}^n R_i \leq M(x)$ almost surely for some x . We restrict ourselves to feasible instances, i.e., to cases where cooperation has a non-negative value for the players.

DEFINITION 3. A *bargaining solution* is a mapping F that assigns a vector $u = (u_1, \dots, u_n) \in \mathcal{U}(R, M, \mathcal{A})$ of utilities to each feasible instance (R, M, \mathcal{A}) .

In the following we discuss three axioms and argue why they should be fulfilled in the context of the bargaining games outlined in this paper. The first axiom is Pareto optimality, which is a basic efficiency requirement also used in the classical Nash bargaining approach.

AXIOM 1 (PAR). The utility allocation $u^* = F(R, M, \mathcal{A})$ prescribed by the bargaining solution is *Pareto optimal*. In particular, if $u \in \mathcal{U}(R, M, \mathcal{A})$ is another utility allocation, then

$$\exists i \in N : u_i > u_i^* \Rightarrow \exists j \in N : u_j < u_j^*.$$

Clearly, any allocation that does not fulfill PAR could be improved, at least for some players, without making anybody else worse off. Choosing such an allocation is obviously wasteful and therefore undesirable.

The next axiom is symmetry, which captures the essence of the notion of *fairness* employed by the Nash bargaining approach. The axiom requires that if all players are indistinguishable in every relevant aspect of the game, they should obtain the same acceptability value. More formally, we can state the symmetry axiom as follows.

AXIOM 2 (SYM). If $\mathcal{A}_i(R_i) = \mathcal{A}_j(R_j)$ for all $i, j \in N$, and if for every permutation σ of N ,

$$(u_1, \dots, u_n) \in \mathcal{U}(R, M, \mathcal{A}) \Rightarrow (u_{\sigma(1)}, \dots, u_{\sigma(n)}) \in \mathcal{U}(R, M, \mathcal{A}),$$

then $u_i^* = u_j^*$ for all $i, j \in N$.

Lastly, we add an axiom that ensures what is sometimes called *strategy proofness* of the allocation rule. The idea is that players should not benefit from strategically splitting up or merging. This is made sure by requiring that larger investments (larger opportunity losses) should entail larger shares of the profit. Similar ideas have been applied in different forms in the literature, e.g., in the concept of the proportional Shapley value (Feldman et al. 1999, Béal et al. 2018).

More formally, we require the following to hold.

AXIOM 3 (STR). If (R', M, \mathcal{A}') is a game associated with (R, M, \mathcal{A}) by $N' = N \setminus \{i\} \cup \{(i, k) \mid k = 1, \dots, p\}$ and

$$\begin{aligned} R'_j &= R_j \quad \forall j \neq i, & R'_{ik} &= R_i/p \quad (k = 1, \dots, p), \\ \mathcal{A}'_j &= \mathcal{A}_j \quad \forall j \neq i, & \mathcal{A}'_{ik} &= \mathcal{A}_i \quad (k = 1, \dots, p), \end{aligned}$$

then the utility allocations $u = F(R, M, \mathcal{A})$ and $u' = F(R', M, \mathcal{A}')$ are interrelated by

$$u'_j = u_j \quad \forall j \neq i, \quad u'_{ik} = u_i/p \quad (k = 1, \dots, p).$$

Thus, in the new game, where player i “splits herself” into p subplayers, splitting also her opportunity losses in equal parts, the benefits should be split as well. Note that, because of positive homogeneity of the acceptability measures \mathcal{A}_i , it makes no difference whether we split up L_i or the assigned utility $u_i = \mathcal{A}_i(L_i)$.

In the following, we give a characterization of those bargaining solutions that satisfy Axioms 1 – 3. To this end, we will require the following assumption, which we assume to hold for the rest of the paper.

ASSUMPTION 1. *The set \mathcal{X} is a compact topological space and $M(x, \cdot) : \mathcal{X} \rightarrow (\mathcal{L}^p, \sigma)$ is continuous for all $x \in \mathcal{X}$, where σ is the weak topology in \mathcal{L}^p .*

Using this assumption, we are able to show the following technical lemma, which ensures that there is always a solution to every feasible bargaining game. The proof can be found in Appendix A.

LEMMA 1. *Under Assumption 1, the set $\mathcal{U}(R, M, \mathcal{A})$ is compact.*

Building on Lemma 1, we prove that \mathcal{U} is a convex polyhedron in the next result.

LEMMA 2. *The set $\mathcal{U}(R, M, \mathcal{A})$ is the polytope*

$$\left\{ u \in \mathbb{R}^n \mid \forall i \in N : u_i \geq \mathcal{A}_i(R_i), \sum_{i=1}^n u_i \leq z \right\} \quad (4)$$

with

$$z = z(R, M, \mathcal{A}) = \max_{u, L, x} \left\{ \sum_{i=1}^n u_i \mid \sum_{i=1}^n L_i = M(x), \mathcal{A}_i(R_i) \leq u_i \leq \mathcal{A}_i(L_i), \forall i \in N \right\}. \quad (5)$$

Proof. The set $\mathcal{U}(R, M, \mathcal{A})$ defined by (3) is obviously just the u -projection of the feasible set $\mathcal{U}^+(R, M, \mathcal{A})$ of the maximization problem (5). Since the objective function of (5) only depends on u , it is immediately seen that

$$\sup_{u, L, x} \left\{ \sum_{i=1}^n u_i \mid (u, L, x) \in \mathcal{U}^+(R, M, \mathcal{A}) \right\} = \sup_u \left\{ \sum_{i=1}^n u_i \mid u \in \mathcal{U}(R, M, \mathcal{A}) \right\}.$$

$\mathcal{U}(R, M, \mathcal{A})$ is compact due to Lemma 1, therefore (5) attains its maximum.

We show that if (u, L, x) is feasible for (5), $u'_i \geq v_i \forall i$, and $\sum_{i=1}^n u'_i \leq \sum_{i=1}^n u_i$, then there is an L' and an x' such that also (u', L', x') is feasible. To see this, set $u''_i = u'_i + \frac{1}{n} \left(\sum_{j=1}^n u_j - \sum_{j=1}^n u'_j \right) \geq u'_i$ for all $i \in N$. Then, clearly, $\sum_{i=1}^n u''_i = \sum_{i=1}^n u_i$ and setting $L'_i = L_i - u_i + u''_i$ and $x' = x$, we have $\sum_{i=1}^n L'_i = M(x')$. Using translation invariance of \mathcal{A}_i , we get

$$\mathcal{A}_i(L'_i) = \mathcal{A}_i(L_i + (u''_i - u_i)) = \mathcal{A}_i(L_i) + (u''_i - u_i) \geq u_i + u''_i - u_i = u''_i \geq u'_i,$$

which establishes feasibility of (u', L', x') .

Let now (u^*, L^*, x^*) be an optimal solution of (5) with optimal value z . Since (u^*, L^*, x^*) is a feasible solution, it follows from the above that every $u \in \mathbb{R}^n$ with $u_i \geq v_i \forall i$ and $\sum_{i=1}^n u_i \leq \sum_{i=1}^n u_i^* = z$ can be extended by a suitable L and a suitable x to a feasible (u, L, x) . Therefore, the set (4) is a subset of the u -projection of $\mathcal{U}^+(R, M, \mathcal{A})$. Conversely, suppose that for some feasible (u, L, x) , the component u does *not* lie in the set (4). This could only be the case if $\sum_{i=1}^n u_i > z$, but then there would be a better solution to the optimization problem (5) than (u^*, L^*, x^*) . Thus, there cannot be a feasible u lying outside of (4). \square

In the next theorem, we show that Axioms 1 – 3 characterize a unique utility allocation.

THEOREM 1. *A bargaining solution F satisfies Axioms 1 – 3 if and only if for every instance (R, M, \mathcal{A}) the utility allocations $u = F(R, M, \mathcal{A})$ are given by*

$$u_i = \frac{v_i}{\sum_{j=1}^n v_j} \cdot z(R, M, \mathcal{A}), \quad \forall i \in N, \quad (6)$$

where z is defined by (5) and $v_i = \mathcal{A}_i(R_i)$ for all $i \in N$.

Proof. Let F be a bargaining solution that satisfies Axioms 1 – 3. For $u^* = F(R, M, \mathcal{A}) \in \mathcal{U}$, we must have $\sum_{i=1}^n u_i^* = z$ with $z = z(R, M, \mathcal{A})$ from (5), since $\sum_{i=1}^n u_i^* < z$ would be in contradiction to PAR. This shows that $u^* = F(R, M, \mathcal{A})$ must be an element of

$$\mathcal{U}_0 = \left\{ u \in \mathbb{R}^n \mid u_i \geq v_i, \forall i \in N, \sum_{i=1}^n u_i = z \right\}.$$

To show that (6) holds first assume $v_i \in \mathbb{Q}$ for all i , i.e., $v_i = p_i/q$ with $q \in \mathbb{N}$ and $p_i \in \mathbb{N}$ for all $i \in N$. Define an instance (R', M, \mathcal{A}') by splitting each player i into p_i players (i, k) with $k = 1, \dots, p_i$, $i \in N$, as in Axiom STR. Then all players have identical values $v'_{ik} = \mathcal{A}'_{ik}(R'_{ik})$, since, by the positive homogeneity of the \mathcal{A}_i ,

$$\mathcal{A}'_{ik}(R'_{ik}) = \mathcal{A}_i(R_i/p_i) = \frac{1}{p_i} \mathcal{A}_i(R_i) = \frac{v_i}{p_i} = \frac{1}{q}.$$

Successive application of Axiom STR for $i = 1, \dots, n$ yields

$$u'_{ik} = \frac{u_i^*}{p_i} \quad \forall k = 1, \dots, p_i, \forall i \in N.$$

Because of the identical values v'_{ik} , the feasible set \mathcal{U}' of (R', M, \mathcal{A}') is invariant under permutations of the coordinates. Thus both conditions of Axiom SYM are satisfied, and we can conclude that all players get the same utility $u'_{ik} = z \cdot \left(\sum_{j=1}^n p_j\right)^{-1} = \tilde{z}$ and hence

$$u_i^* = p_i \cdot \tilde{z} = \frac{p_i}{\sum_{j=1}^n p_j} \cdot z = \frac{v_i}{\sum_{j=1}^n v_j} \cdot z$$

as required.

For $v_i \notin \mathbb{Q}$, the assertion follows because of the continuity of the functions on the right hand side of (6) by a straightforward convergence argument, representing $v_i \mathbb{R}$ as limits of rational numbers. The continuity of the quotient $v_i / \sum_j v_j$ is clear, while the continuity of $z(R, M, \mathcal{A})$ is proved in Lemma 8 in Appendix A.

To show the converse, we assume that the bargaining solution F satisfies (6) and show that Axioms (1) – (3) are satisfied.

Axiom PAR: Suppose that, contrary to PAR, the utility allocation $u^* = F(R, M, \mathcal{A}) \in \mathcal{U}(R, M, \mathcal{A})$ is dominated by some other $u \in \mathcal{U}(R, M, \mathcal{A})$, i.e., $u_i > u_i^*$ for some i and $u_j \geq u_j^* \forall j \in N$. Then

$$\sum_{j=1}^n u_j > \sum_{j=1}^n u_j^* = z(R, M, \mathcal{A}),$$

which contradicts $u \in \mathcal{U}(R, M, \mathcal{A})$ because of (4).

Axiom SYM: In view of (4), the set $\mathcal{U}(R, M, \mathcal{A})$ can only be invariant under permutations of coordinates, if $v_i = v_j \forall i, j \in N$. In this case, (6) implies identical utilities $u_i = z/n \forall i \in N$.

Axiom STR: Let the conditions of STR be satisfied for the two instances (R, M, \mathcal{A}) and (R', M, \mathcal{A}') . Then, in particular,

$$v'_j = v_j \quad (j \neq i), \quad v'_{ik} = \frac{v_i}{p} \quad (k = 1, \dots, p),$$

where the last equality follows by means of the positive homogeneity of the \mathcal{A}_i . Therefore (6) implies

$$u'_j = \frac{v_j}{\sum_{\ell \neq i} v_\ell + \sum_{k=1}^p \frac{v_i}{p}} \cdot z = u_j \quad \forall j : j \neq i$$

and

$$u'_{ik} = \frac{\frac{v_i}{p}}{\sum_{\ell \neq i} v_\ell + \sum_{k=1}^p \frac{v_i}{p}} \cdot z = \frac{u_i}{p} \quad \forall k : k = 1, \dots, p,$$

as claimed by Axiom STR. □

We call a bargaining solution F satisfying Axioms 1 – 3 a *Nash-type bargaining solution* or Nash bargaining solution, for short. We shall outline a closer connection to classical Nash bargaining below. To that end, we discuss two mathematical programs that directly yield implementable solutions (u, L, x) without going through the stepwise procedure of first solving (5) and then applying Theorem 1 to find optimal u_i .

THEOREM 2. *Let the bargaining solution F fulfill Axioms 1 – 3. Then $u = F(R, M, \mathcal{A})$ can be obtained from the optimal solution (u, L, x) to the following convex optimization problem*

$$\begin{aligned} & \max_{u_i, L_i, x} \sum_{i=1}^n (u_i - v_i) \\ \text{s.t.} \quad & u_i = \frac{v_i}{\sum_{j=1}^n v_j} \sum_{j=1}^n u_j \\ & u_i \leq \mathcal{A}_i(L_i), \quad i \in N \\ & u_i \geq v_i, \quad i \in N \\ & \sum_{i=1}^n L_i \leq M(x) \\ & x \in \mathcal{X}. \end{aligned} \tag{7}$$

Proof. We write $c_i = v_i / (\sum_j v_j)$ and note that by Theorem 1 and Lemma 2, the bargaining solution satisfying the axioms is given as

$$u_i = c_i \cdot z \quad \forall i \in N, \quad \text{with } z = \max_{u \in \mathcal{U}} \sum_{i=1}^n u_i, \tag{8}$$

where $z = z(R, M, \mathcal{A})$ and $\mathcal{U} = \mathcal{U}(R, M, \mathcal{A})$. It is easy to see that the maximization problem $\max_{u \in \mathcal{U}} \sum_{i=1}^n u_i$ has the same optimal objective as the problem

$$\begin{aligned} & \max \sum_{i=1}^n u_i \\ \text{s.t.} \quad & u_i = c_i \sum_{j=1}^n u_j \\ & u \in \mathcal{U} \end{aligned} \tag{9}$$

Indeed, it is clear that the addition of the constraint $u_i = c_i \sum_j u_j$ cannot increase the objective. On the other hand, the special allocation u defined by $u_i = c_i \cdot z$ satisfies $u \in \mathcal{U}$, since

$$c_i \cdot z \geq \frac{v_i}{\sum_j v_j} \cdot \sum_j v_j = v_i \quad \forall i \in N$$

and since $\sum_i c_i \cdot z = z$. Moreover, u as defined above evidently maximizes the objective function $\sum_i u_i$ by attaining the optimal value z . Thus, (8) can be equivalently reformulated as

$$\begin{aligned} & \max \sum_{i=1}^n u_i \\ \text{s.t.} \quad & u_i = \frac{v_i}{\sum_{j=1}^n v_j} \sum_{j=1}^n u_j \\ & u_i \leq \mathcal{A}_i(L_i), \quad i \in N \\ & u_i \geq v_i, \quad i \in N \\ & \sum_{i=1}^n L_i = M(x) \\ & x \in \mathcal{X}. \end{aligned} \tag{10}$$

As the v_i are constants, the variables u_i in the objective function can be replaced by $(u_i - v_i)$. Due to the monotonicity of \mathcal{A}_i , we may convert the second last constraint to an equality constraint resulting in (7). \square

REMARK 1. In Theorem 2, the optimization of x is integrated with that of u_i and L_i . However, as Theorem 1 shows, the case where x can be negotiated by the players does not add any complexity to the bargaining problem itself: Since for $M = M(x)$, the dependence on x only impacts the factor $z(R, M, \mathcal{A})$ and the optimization with respect to x can thus be fully decoupled from the bargaining process. In particular, there is no conflict of interest between the players over managerial decisions, since x , loosely speaking, only influences the absolute size of the pie but not the relative shares.

REMARK 2. Note that the solution to (7) needs not be unique, i.e., it can happen that different choices of L and/or x produce the same optimal solution u . Hence, uniqueness of the bargaining solution can only be guaranteed on the level of the utilities u .

In order to highlight the connections of our approach to the classical theory of Nash bargaining, we present an alternative optimization problem that is more in the vein of the original theory of Nash bargaining in its deterministic context, where the proposed bargaining solution is expressed as the solution of a nonlinear optimization problem with a product-form objective function.

THEOREM 3. *If the bargaining solution F fulfills Axioms 1 – 3, $u = F(R, M, \mathcal{A})$ is the optimal solution to the optimization problem*

$$\begin{aligned} & \max \prod_{i=1}^n (u_i - v_i)^{v_i} \\ & \text{s.t. } u \in \mathcal{U}(R, M, \mathcal{A}). \end{aligned} \tag{11}$$

Proof. According to Lemma 2, $\mathcal{U} = \mathcal{U}(R, M, \mathcal{A}) = \{u \in \mathbb{R}^n \mid u_i \geq v_i \ (i \in N), \sum_{i=1}^n u_i \leq z\}$. By Axiom 1, the feasible set can be restricted to those u that satisfy $\sum_{i=1}^n u_i = z$. Except in the trivial boundary case where $z = \sum_j v_j$, a feasible solution with $u_i > v_i$ for all i exists. Taking the logarithm of the objective function, we get

$$\begin{aligned} & \max \sum_{i=1}^n v_i \log(u_i - v_i) \\ & \text{s.t. } u_i \geq v_i, \quad \forall i \in N \\ & \quad \sum_{i=1}^n u_i = z. \end{aligned} \tag{12}$$

Obviously, this is a convex optimization problem, as the objective function to be maximized is concave and the constraints are linear. The Lagrange function of (12) without the inequality constraint reads

$$\mathcal{L} = \sum_{i=1}^n v_i \log(u_i - v_i) - \lambda \left(\sum_{i=1}^n u_i - z \right).$$

The first order condition with respect to u_i therefore yields

$$u_i = \frac{1 + \lambda}{\lambda} \cdot v_i, \quad \forall i \in N. \tag{13}$$

Because of $\sum_i u_i = z$, it follows that

$$\frac{1 + \lambda}{\lambda} = \frac{z}{\sum_{i=1}^n v_i} \geq 1 \tag{14}$$

and therefore $u_i \geq v_i$ is fulfilled. Plugging (14) into (13) yields

$$u_i = \frac{v_i}{\sum_j v_j} \cdot z,$$

which is the solution identified by Theorem 1. □

Problem (11) is similar to the classical Nash bargaining solution adapted to our context. Apart from the stochasticity of the game, the only structural difference are the exponents v_i in the objective. This modification ensures that Axiom 3 is respected, which implies that the opportunity loss of agent i can be interpreted as her *bargaining power*. If agent i splits into several agents, the bargaining power decreases and the agents get the same as i gets in the original game.

Beginning with Roth (1979) and Binmore (1980) there is a large literature on Nash bargaining with bargaining power. Bargaining power in this literature is represented by nonnegative numbers γ_i which are used as exponents in the objective function $\prod_{i=1}^n (u_i - v_i)^{\gamma_i}$. In many papers on bargaining power γ_i are chosen more or less arbitrary. A notable exception is Binmore et al. (1986), where bargaining power is related to the bargainer's time preferences and to the risk of a breakdown of negotiations in dynamic bargaining models. In our setup, bargaining power (as well as the opportunity loss) is related to the acceptability $\mathcal{A}_i(R_i)$ or lost opportunities, which is a natural point of view for investment problems.

Note that setting $\gamma_i \neq 1$ in an arbitrary way, usually leads to the violation of the axiom SYM. However, in our setup symmetry is still ensured, because $\gamma_i = v_i$ implies equal bargaining power γ_i for players with equal opportunity losses v_i .

We conclude with a discussion of the two axioms *invariance with respect to affine transformations* (INV) and *invariance with respect to irrelevant alternatives* (IIA), which are imposed in classical Nash bargaining but are absent from our approach.

We do not impose INV, since coherent risk functionals are not closed with respect to affine transformations, i.e., it does not make sense to talk about a bargaining problem with modified acceptability functionals, say, $a_i \mathcal{A}_i(\cdot) + b_i$ (with $a \geq 0$ and $b \in \mathbb{R} \setminus \{0\}$), since the transformed functionals are not homogeneous, even if \mathcal{A} is homogeneous. Moreover, strategy proofness cannot be maintained simultaneously with INV, since a strategic split of a player might become profitable if the utilities of all players would be scale-invariant. This can be seen by inspecting (6): scaling v_i by a_i does not in general lead to a scaling of u_i by the same factor, which implies that INV is in general violated.

We also note that IIA is automatically fulfilled in our setup: The role of IIA in the classical theory is to deal with nonlinear boundaries of the bargaining set; in our context, however, only linear boundaries occur (see Lemma 2).

3. Characterizing the Optimal Allocation

This section is devoted to a structural understanding of solutions to problem (7). In Section 3.1, we derive optimality conditions in a simplified setting where M does not depend on x and derive some properties of the solution. In Section 3.2, we develop an alternative view of the overall problem,

which allows us to break up (7) into the problem of determining M and x and the problem of finding an optimal distribution L in Section 3.3. Finally, in Section 3.4, we discuss special cases and show that if all \mathcal{A}_i are distortion functionals, the form of the solution is particularly suitable for practical applications.

3.1. Optimality Conditions for the Bargaining Game

We start by introducing the following additional running assumption.

ASSUMPTION 2. *The coherent acceptability functionals \mathcal{A}_i are proper and upper semicontinuous.*

From Assumption 2, it follows that all $\mathcal{A}_i : \mathcal{L}^p(\mathcal{Y}) \rightarrow \mathbb{R}$ possess conjugate representations (e.g. Pflug and Römisch 2007, Theorem 2.30)

$$\mathcal{A}_i(X) = \inf\{\mathbb{E}[\zeta_i X] : \zeta_i \in \mathcal{Z}_i\}, \quad (15)$$

where \mathcal{Z}_i are convex sets in $\mathcal{L}^q(\mathcal{Y})$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $p \geq 1$. In consequence, \mathcal{A}_i is the support function $\sigma_{\mathcal{Z}_i}$ of \mathcal{Z}_i defined by the right hand side of (15) and is therefore completely characterized by the set \mathcal{Z}_i .

Assumption 2 is fulfilled for practically all relevant coherent acceptability functionals and guarantees that the infimum in (15) is attained. Note that in order to ensure that \mathcal{A}_i defined by (15) is a coherent acceptability functional, the set \mathcal{Z}_i has to additionally fulfill (e.g. Pflug and Römisch 2007)

$$\mathcal{Z}_i \subseteq \{\zeta_i \in \mathcal{L}^q(\mathcal{Y}) : \zeta_i \geq 0 \wedge \mathbb{E}[\zeta_i] = 1\}. \quad (16)$$

We start our investigation of (7) by introducing the new variable $z = \sum_{i=1}^n u_i$, which allows us to eliminate u_i by writing

$$u_i = \frac{v_i}{\sum_{i=1}^n v_i} z. \quad (17)$$

Moreover, we drop the constant part $\sum_{i=1}^n v_i$ from the objective function, resulting in an optimization problem with a constraint $z \geq \sum_{i=1}^n v_i$, which can be removed: if the problem is feasible this constraint does not change the solution. Feasibility can be verified offline by comparing the optimal z to $\sum_{i=1}^n v_i$. Finally, note that because of the monotonicity of \mathcal{A}_i , there must be optimal allocations such that $\sum_{i=1}^n L_i = M(x)$.

Together these simplifications lead to the following equivalent problem

$$\begin{aligned} & \max_{z, L_i, x} z \\ & \text{s.t.} \quad z \frac{v_i}{\sum_{i=1}^n v_i} \leq \mathcal{A}_i(L_i(\omega)) \\ & \quad \sum_{i=1}^n L_i(\omega) = M(x, \omega), \quad \text{a.s.} \\ & \quad x \in \mathcal{X}. \end{aligned} \quad (18)$$

We start by considering the case where M does not depend on x . This assumption will be relaxed in Section 3.3.

THEOREM 4. *If M does not depend on decision x , (L, z) is optimal for (18) if and only if*

A) $\sum_{i=1}^n L_i = M$ almost surely;

B) $z \frac{v_i}{\sum_{i=1}^n v_i} = \mathcal{A}_i(L_i) = \min\{\mathbb{E}[\zeta_i X] : \zeta_i \in \mathcal{Z}_i\}$ for all i ;

C) there exists a $\Lambda \in \mathcal{L}^q(\mathcal{Y})$ such that $\Lambda = \arg \min\{\mathbb{E}[\zeta_i L_i] : \zeta_i \in \mathcal{Z}_i\}$ for all i .

Moreover, if $z \geq \sum_{i=1}^n v_i$ then u obtained by (17) is optimal for (7).

Proof. In the following, we derive the KKT conditions for (18). Because the problem is convex, the KKT conditions are sufficient for optimality. To show that the KKT conditions are also necessary, note that it is possible to choose a feasible combination $L_i = M/n$ and (any) $z \in \mathbb{R}$ such that $z \frac{v_i}{\sum_{i=1}^n v_i} < \frac{\mathcal{A}_i(M)}{n}$ for all $i \in N$. Therefore a general Slater condition can be applied as a constraint qualification (see e.g., Bot et al. 2009, Theorem 3.2.9).

Given the conjugate representations (18), the Lagrange function of (15) can be written as

$$H(L, u, \Lambda, \eta, \Gamma, \lambda) = z + \mathbb{E} \left[\Lambda \left(M - \sum_{i=1}^n L_i \right) \right] + \sum_{i=1}^n \mathbb{E}[\Gamma_i L_i] - \frac{z}{\sum_{i=1}^n v_i} \sum_{i=1}^n \lambda_i v_i, \quad (19)$$

where $\Lambda \in \mathcal{L}^q(\mathcal{Y})$, $\Gamma_i \in \mathcal{L}^q(\mathcal{Y})$ and $\Gamma_i \geq 0$ for all $i \in N$, and $\lambda_i \in \mathbb{R}_0^+$. Note that the random variables Γ_i are chosen such that $\Gamma_i = \lambda_i \zeta_i$, where $\zeta_i \in \mathcal{Z}_i$ is optimal in (15) for \mathcal{A}_i . For an optimal solution, we therefore require

$$\frac{\Gamma_i}{\lambda_i} \in \arg \min \{ \mathbb{E}[\zeta_i L_i] : \zeta_i \in \mathcal{Z}_i \}, \quad (20)$$

which, because of (16), implies

$$\lambda_i = \mathbb{E}[\Gamma_i]. \quad (21)$$

Differentiating H with respect to z yields the following optimality condition

$$\frac{\partial H}{\partial z} = 1 - \frac{\sum_{i=1}^n \lambda_i v_i}{\sum_{i=1}^n v_i} = 0 \quad (22)$$

Taking Gateaux derivatives $DH(L_i, P)$ of H with respect to L_i in the direction of $P \in \mathcal{L}^q(\mathcal{Y})$ yields $DH(L_i, P) = -\mathbb{E}[\Lambda P] + \mathbb{E}[\Gamma_i P] = 0$ for all $P \in \mathcal{L}^q(\mathcal{Y})$, which is equivalent to $\Gamma_i = \Lambda$ almost surely for all i . This together with (21) immediately implies

$$\lambda_i = \mathbb{E}[\Lambda]. \quad (23)$$

Plugging (23) into (22) leads to $\mathbb{E}[\Lambda] = 1$ and hence

$$\lambda_i = 1. \quad (24)$$

Using $\Gamma_i = \Lambda$ and $\lambda_i = 1$ in (20) implies C).

The complementarity constraint for the inequality constraint in (18) is given by

$$0 \leq \lambda_i \perp \left(\mathcal{A}_i(L_i) - z \frac{v_i}{\sum_{i=1}^n v_i} \right) \geq 0 \text{ for all } i. \quad (25)$$

Together with (24) this implies B). Finally, A) is just the corresponding constraint in (18). \square

Condition C) above directly implies the following corollary.

COROLLARY 1. *An optimal solution of the bargaining problem fulfilling Axioms 1 – 3 can only exist if*

$$\bigcap_{i=1}^n \mathcal{Z}_i \neq \emptyset. \quad (26)$$

Moreover, the optimality conditions imply that the optimal dual variables ζ_i , which are supergradients of \mathcal{A}_i at L_i , are the same for all players i . This resembles standard optimality conditions often found in economics that require the marginal utilities of all players to coincide for an optimal allocation of goods. It is also similar to the conditions found for competitive Nash equilibria in markets where agents have coherent risk preferences (e.g., Heath and Ku 2004, Ralph and Smeers 2015, Philpott et al. 2016).

Condition C) in Theorem 4 also implies that L_i and L_j are comonotone, which will be an instrumental result in Section 3.4.

THEOREM 5. *If an allocation L is optimal, then L_i are comonotone, i.e., for any pair (i, j)*

$$(L_i(\omega) - L_i(\omega'))(L_j(\omega) - L_j(\omega')) \geq 0 \quad a.s. \quad (27)$$

Proof. We show that for any pair of optimal assignments L_i, L_j the set

$$A(i, j) = \{\omega \in \Omega : \exists \omega' \in \Omega : L_j(\omega) \geq L_j(\omega') \wedge L_i(\omega) < L_i(\omega')\} \quad (28)$$

has probability zero.

To this end, assume that $\mathbb{P}(A(i, j)) > 0$ and consider optimal L_i, L_j , an arbitrary $\omega \in A(i, j)$, and a related ω' fulfilling the defining condition of $A(i, j)$, i.e.,

$$L_j(\omega) \geq L_j(\omega') \wedge L_i(\omega) < L_i(\omega'). \quad (29)$$

Consider the conjugate representations (15) and the related optimal dual variables ζ_i and ζ_j for L_i and L_j . From (16) we know that $\mathbb{E}(\zeta_i) = \mathbb{E}(\zeta_j) = 1$ and $\zeta_i \geq 0$ and $\zeta_j \geq 0$ must hold almost surely. It follows from the optimality of ζ_i, ζ_j and (29) that $\zeta_j(\omega) \geq \zeta_j(\omega')$ and $\zeta_i(\omega) < \zeta_i(\omega')$. This holds for almost all $\omega \in A(i, j)$, which has positive probability as assumed above. But this is in violation of condition C) in Theorem 4 which states that $\zeta_i = \zeta_j$ almost surely implying that L_i, L_j cannot be optimal. It follows that $A(i, j)$ must have probability 0 and because i, j were arbitrary, the same is true for the set $A(j, i)$. Together this yields that L_i and L_j are comonotone. \square

3.2. An Alternative Characterization of Optimality

In this section, we show that the optimal solution to (7) is fully characterized by an acceptability functional \mathcal{A}^{\max} , which arises from condition C) in Theorem 4 and the joint returns M for all players.

Using the notation $\mathcal{Z}_i^*(\Lambda)$ for the dual cone of \mathcal{Z}_i at Λ , condition C) reads

$$L_i \in \mathcal{Z}_i^*(\Lambda), \quad (30)$$

which is equivalent to all L_i and Λ fulfilling the variational inequalities

$$\mathbb{E}[L_i(Z - \Lambda)] \geq 0 \text{ for all } Z \in \mathcal{Z}_i. \quad (31)$$

We introduce an acceptability functional \mathcal{A}^{\max} by

$$\mathcal{A}^{\max}(X) = \inf_{\zeta \in \mathcal{L}^q} \{\mathbb{E}[\zeta X] : \zeta \in \mathcal{Z}^{\max}\} = \sigma_{\mathcal{Z}^{\max}}, \quad \forall X \in \mathcal{L}^p(\mathcal{Y}) \quad (32)$$

with $\mathcal{Z}^{\max} = \bigcap_{i=1}^n \mathcal{Z}_i$.

The next theorem shows that the functional \mathcal{A}^{\max} is a proper, upper semicontinuous, coherent acceptability functional, which implies that the infimum in (32) is attained.

THEOREM 6. *If $\mathcal{Z}^{\max} \neq \emptyset$, then \mathcal{A}^{\max} is a proper upper semicontinuous coherent acceptability functional.*

Proof. \mathcal{A}^{\max} is concave and positive homogeneous by definition as support function $\sigma_{\mathcal{Z}^{\max}}$. Because all \mathcal{Z}_i fulfill (16), this must also be true for their intersection. Together this shows that \mathcal{A}^{\max} is a coherent risk measure.

Applying Theorem 2 and Theorem 4 of Section 6.2 in Chapter IV of Bourbaki (1989) to the family of upper semicontinuous (in fact even continuous) functionals $X \mapsto \mathbb{E}[\zeta X]$ for $\zeta \in \mathcal{Z}^{\max}$ shows that the functional $X \mapsto \inf\{\mathbb{E}[\zeta X] : \zeta \in \mathcal{Z}^{\max}\}$ is upper semicontinuous. Hence $\mathcal{A}^{\max}(X)$ is upper semicontinuous by definition.

Clearly, $\mathcal{A}^{\max}(X) \geq \mathcal{A}_i(X)$ for any i and therefore $\mathcal{A}^{\max}(X) > -\infty$, whenever this holds for $\mathcal{A}_i(X)$ and therefore the effective domain of \mathcal{A}^{\max} is not empty. Because $\mathcal{Z}^{\max} \neq \emptyset$, we have $\mathcal{A}^{\max}(X) < \infty$ for all $X \in \mathcal{L}^p(\mathcal{Y})$. Together this shows that \mathcal{A}^{\max} is a proper functional. \square

As pointed out in Corollary 1, \mathcal{Z}^{\max} must be nonempty if there is optimal solution for problem (18). The following lemma gives a characterization of \mathcal{Z}^{\max} in terms of the individual dual cones \mathcal{Z}_i^* , which we will use in Theorem 7 to define the optimality conditions for a modified version of problem (18) in terms of \mathcal{A}^{\max} .

LEMMA 3. Let $\mathcal{A}_i : \mathcal{L}^p(\mathcal{Y}) \rightarrow \mathbb{R}$ with $1 < p < \infty$, then

$$(\mathcal{Z}^{\max})^*(\zeta) = \left(\bigcap_{i=1}^n \mathcal{Z}_i \right)^*(\zeta) = \sum_{i=1}^n \mathcal{Z}_i^*(\zeta), \quad (33)$$

for any $\zeta \in \mathcal{Z}^{\max} \neq \emptyset$.

Proof. By Assumption 2, the functionals \mathcal{A}_i are upper semicontinuous and concave. Hence, their hypographs are (norm) closed and convex. By the Hahn-Banach separation theorem, it follows that they are weakly-closed as well. Since $p > 1$, $\mathcal{L}^p(\mathcal{Y})$ is reflexive and the weak topology is equivalent to the weak* topology. Therefore the hypographs are weak*-closed.

The (positive homogeneous) functionals \mathcal{A}_i are by definition identical with the support functions $\sigma_{\mathcal{Z}_i}$ of the defining sets \mathcal{Z}_i . Because the hypographs of the individual support functions are weak*-closed, the sum

$$\sum_{i=1}^n \text{hypo } \sigma_{\mathcal{Z}_i}, \text{ is weak*-closed.} \quad (34)$$

Finally, by Theorem 3.1 in Burachik and Jeyakumar (2005), (33) follows from (34). \square

We are now in a position to prove the following theorem yielding a deeper characterization of the optimal dual variable Λ as the supergradient of \mathcal{A}^{\max} at M . In fact, we show that if $p > 1$, any minimizing supergradient for (32) at M is an optimal dual variable Λ . Moreover, we show that $\mathcal{A}^{\max}(M) = z = \sum_{i=1}^m u_i$ for the optimal u in (7).

THEOREM 7. *If there exists an optimal solution (L, z, Λ) for (18), then*

$$\Lambda \in \arg \min \{ \mathbb{E}[M\zeta] : \zeta \in \mathcal{Z}^{\max} \}. \quad (35)$$

Conversely, if $M \in \mathcal{L}^p(\mathcal{Y})$ with $p > 1$ and $\mathcal{Z}^{\max} \neq \emptyset$ then any $\Lambda \in \mathcal{L}^q(\mathcal{Y})$ such that (35) holds is optimal for (18), i.e., there exist $L_i \in \mathcal{L}^p(\mathcal{Y})$ and z fulfilling (30) and the arg min-set in (35) is not empty.

In both cases, the optimal $z = \sum_{i=1}^n \mathcal{A}_i(L_i)$ is given by

$$z = \min \{ \mathbb{E}[M\zeta] : \zeta \in \mathcal{Z}^{\max} \} = \mathcal{A}^{\max}(M). \quad (36)$$

Proof. Let (L, z, Λ) be an optimal solution according to Theorem 4. Then L, Λ fulfill the variational inequality (31). Because Λ is in the intersection of all the argmin-sets, (31) holds in particular for all $Z \in \mathcal{Z}^{\max}$. Therefore, for any $Z \in \mathcal{Z}^{\max}$, we can sum and infer

$$0 \leq \mathbb{E} \left[\sum_{i=1}^n L_i(Z - \Lambda) \right] = \mathbb{E}[M(Z - \Lambda)] \quad (37)$$

and hence

$$M \in (\mathcal{Z}^{\max})^*(\Lambda), \quad (38)$$

which is equivalent to (35). Therefore

$$\mathcal{A}^{\max}(M) = \mathbb{E}[\Lambda M] = \sum_{i=1}^n \mathbb{E}[\Lambda L_i] = \sum_{i=1}^n \mathcal{A}_i(L_i) = \sum_{i=1}^n \frac{v_i}{\sum_{i=1}^n v_i} z = z. \quad (39)$$

For the other direction, assume $1 < p < \infty$, $\mathcal{Z}^{\max} \neq \emptyset$ and that Λ fulfills (35), which is equivalent to (38). Applying Lemma 3 to (38) yields

$$M \in \sum_{i=1}^n \mathcal{Z}_i^*(\Lambda), \quad (40)$$

which shows that for any Λ defined by (35) there exist $L_i \in \mathcal{Z}_i^*(\Lambda)$ which sum up to M .

The additional optimality condition B), which is equivalent to the condition

$$\mathbb{E}[\Lambda L_i] = \frac{v_i}{\sum_{i=1}^n v_i} z \quad (41)$$

for $L_i \in \mathcal{Z}_i^*(\Lambda)$, does not lead to a contradiction if z is chosen as $z = \mathbb{E}[M] = \mathcal{A}^{\max}(M)$. Hence, given a Λ fulfilling (35) there exist L, z such that L, Λ, z are optimal. Finally, if $\bigcap_{i=1}^n \text{dom } \mathcal{A}_i \neq \emptyset$ then the argmin-set in (35) is not empty by Theorem 6.

If $p = \infty$, we apply the above argument for an arbitrary $1 < p' < \infty$. Since $M \in \mathcal{L}^{p'}(\mathcal{Y})$ and (40) ensures that the resulting L_i are in $\mathcal{L}^\infty(\mathcal{Y})$, the argument still goes through. \square

Equations (35) and (36) show that the optimal value z can be rewritten as

$$z = \mathcal{A}^{\max}(M) = \mathbb{E}[\Lambda M], \quad (42)$$

where Λ is optimal for (35). This implies that the Lagrange multiplier Λ can in fact be interpreted as the (stochastic) shadow price of a change in M .

Because the shadow price Λ can be calculated without knowledge of the reference values v , (42) implies that the optimal z does not depend on v and hence that the shadow price of v_i is zero. It follows that z is continuous in v , which yields an alternative proof of Lemma 8 in Appendix A.

Altogether, if $p > 1$ the problem of finding the optimal value z and an optimal acceptability allocation u can be completely decoupled from the problem of finding the optimal splitting L of M . The values of z and u can be obtained together with the shadow prices Λ by (17) and by (42) and (35), which only needs M . Then any optimal allocation L must fulfill

$$\sum_{i=1}^n L_i = M \quad (43)$$

$$\mathbb{E}[\Lambda L_i] = u_i \quad \text{for all } i \in N \quad (44)$$

$$\mathbb{E}[L_i(Z - \Lambda)] \geq 0 \quad \text{for all } Z \in \mathcal{Z}_i, i \in N. \quad (45)$$

3.3. Maximizing Project Profits

So far we have considered fixed profits M . If M depends on decisions $x \in \mathcal{X}$, all results achieved so far stay valid if one replaces M by $M(x)$ whenever needed, as shown in the next result.

COROLLARY 2. *If $M(x, \omega)$ depends on decisions $x \in \mathcal{X}$, then optimal solutions (x, L, Λ, z) of (18) fulfill*

$$x = \arg \max_{x \in \mathcal{X}} \{\mathcal{A}^{\max}(M(x))\}. \tag{46}$$

together with A), B), C), where M is set to $M = M(x)$.

These optimality conditions are also sufficient if $x \mapsto \mathcal{A}^{\max}(M(x))$ is concave.

Proof. Replacing M with $M(x)$ in the Lagrangian (19), and using the fact that the optimal x must maximize the Lagrangian together with the KKT conditions, one gets

$$x = \arg \max_{x \in \mathcal{X}} \{\mathbb{E}[\Lambda M(x)]\} \tag{47}$$

and A), B) C), where $M=M(x)$. Using (35) and the definition of \mathcal{A}^{\max} , equation (47) can be rewritten as (46). □

REMARK 3. Note that the argument above is not predicated on the fact that $x \in \mathbb{R}^k$ for some $k \in \mathbb{N}$. Clearly, x can also be a random variable or even a stochastic process. This implies that the result remains valid in a stochastic two-stage or even multistage setup.

Corollary 2 is an alternative proof of Remark 1: The question of finding optimal decisions x can be decoupled from the questions of finding optimal acceptability values and allocations. In particular, it suffices to compute the acceptability of the aggregated profit $M(x)$, which implies that x is not a contested choice between the players, considerably simplifying the analysis. While in deterministic bargaining games it is straightforward that maximizing the profit first is optimal, this is not obvious in the stochastic case and is heavily based on the fact that translation invariant acceptability functionals are used.

The possibility to separate profit maximization, i.e., the optimization of x , and the determination of L_i also has practical consequences for the computation of optimal solutions. In particular, we note that if it is feasible to optimize $\mathcal{A}^{\max}(M(x)) = z = \sum_{i=1}^n u_i$, then it is also feasible to solve (18) without the first constraint (which defines the allocation) yielding an optimal combination of x , $M(x)$ and z . In particular, the fitting optimal Λ and L can be found using A)-C), or alternatively the system (43)-(45).

3.4. Special Cases: Affine Allocations and Options

Up to now, we analyzed general optimality conditions. In this section, we discuss two special cases for risk functionals \mathcal{A}_i , which allow for simple functional characterizations of the individual profits in terms of M . This in turn makes the design of contracts that implement the bargaining solution practically feasible and facilitates a real world implementation of the results in this paper.

We start by analyzing the case when there is a \mathcal{Z}_i such that $\mathcal{Z}_i \subseteq \mathcal{Z}_j$ for all j . This, for example, occurs if the functionals \mathcal{A}_i are of the same type with the \mathcal{Z}_i controlled by a single parameter, e.g., the *Average Value-at-Risk* (AVaR_α) with differing α for each participant. In this case the optimal L_i are affine functions of M as the next result shows.

COROLLARY 3. *If there is an $i \in \{1, \dots, n\}$ such that*

$$\mathcal{Z}_i \subseteq \mathcal{Z}_j \text{ for all } j \in \{1, \dots, n\} \quad (48)$$

and $H = \{j : \mathcal{Z}_j = \mathcal{Z}_i\}$, then any L, Λ, z with $z = \mathcal{A}_i(M)$ and $\Lambda \in \arg \min \{\mathbb{E}[M\zeta] : \zeta \in \mathcal{Z}_i\}$ with $L_i = a_i + b_i M$, where

$$a_i = \left(\frac{v_i}{\sum_{i=1}^n v_i} - b_i \right) \text{ and } b_i = \begin{cases} \beta_i & i \in H \\ 0 & \text{else} \end{cases} \quad (49)$$

is optimal if $\beta_j \geq 0$, $\sum_{j \in H} \beta_j = 1$.

Proof. Condition (48) implies $\mathcal{Z}^{\max} = \mathcal{Z}_i$, hence $\mathcal{A}^{\max} = \mathcal{A}_i$. Therefore (36) implies $z = \mathcal{A}_i(M)$. If $\Lambda \in \arg \min \{\mathbb{E}[\zeta M] : \zeta \in \mathcal{Z}_i\}$, it must fulfill (45), which is equivalent to

$$b_i \mathbb{E}[M(Z - \Lambda)] \geq 0 \text{ for all } Z \in \mathcal{Z}_i, i \in N \quad (50)$$

for $L_i = a_i + b_i M$, since $\mathbb{E}[\Lambda] = \mathbb{E}[Z] = 1$ due to (16).

Any choice $b_i = 0$ for $i \notin H$ and $b_i \geq 0$ for $b_i \in H$ therefore fulfills the variational inequalities. If b_i for $i \in H$ is chosen such that $\sum_{i \in H} b_i = 1$, then (49) ensures both (44), and $\sum_{i=1}^n a_i = 0$. It follows that the third optimality condition (43) is also fulfilled. \square

Therefore, it turns out that under the conditions of Corollary 3 the least risk averse take all the risk, while the other players receive deterministic side payments and bear no risk in the optimal allocation.

REMARK 4. Note that even under the assumptions of Corollary 3 no affine allocation can be optimal without side payments, as long as there is some $i \notin H$, i.e. $\mathcal{A}_i(M) < \mathcal{A}_{\max}(M)$. This is in particular true for the “naive” allocation, where all players receive a share of the revenue according to their share of invested capital.

This also implies that the proportional split up rules proposed, for example, in Timmer et al. (2005), Baeyens et al. (2013) are never optimal in our setting. Moreover, it is easy to show that

(48) is a necessary condition for an affine allocation. Hence, even approaches that allow for affine distributions with deterministic side payments, such as Suijs et al. (1999), are optimal only in the narrow set of circumstances described in Corollary 3.

Clearly, the case where one agent is most risk averse in the sense of (48) is a rare special case. Next, we consider the more realistic setting when the class of acceptability functionals is restricted to so called distortion functionals, and the project profit is bounded below.

ASSUMPTION 3. $\text{ess\,inf } M = C > -\infty$ and all \mathcal{A}_i are distortion functionals, i.e., of the form

$$\mathcal{A}_i(X) = \int_0^1 \text{AVaR}_\alpha(X) \, dm_i(\alpha),$$

where m_i are arbitrary probability measures on $[0, 1]$.

In the above definition, AVaR_α refers to the *Average Value-at-Risk*, defined as

$$\text{AVaR}_\alpha(X) = \int_0^\alpha F_X^{-1}(t) \, dt$$

with F_X the cumulative distribution function of X . See Föllmer and Schied (2004), Pflug and Römisch (2007) for more details on the AVaR and distortion functionals.

We show that, under Assumption 3, the optimal allocation L of the bargaining game can be represented as a basket of standard options plus deterministic side payments. This particularly simple form of L is conducive to a real world implementation of the bargaining solution. Such an implementation would otherwise be complicated by the requirement of a contract that specifies payoffs $L_i(\omega)$ for all agents i in all future states ω of the world, possibly without any further structure to it.

Using Assumption 3, we can dissect the L_i into slices represented by random variables of the form $\mathbb{1}_{\{x \geq s\}}(M)\xi_i(s)$ and then show that the acceptability of L_i can be build up from the acceptability of the slices. In particular, we establish the following lemma.

LEMMA 4. *Under Assumption 1 – 3, the following holds:*

1. *The optimal L_i in (18) can be written*

$$L_i(M) = \xi_i(C)C + \int_{-\infty}^{\infty} \mathbb{1}_{\{x \geq s\}}(M)\xi_i(s) \, ds \tag{51}$$

for some functions $\xi_i : \mathbb{R} \rightarrow [0, 1]$ with $\sum_{i=1}^n \xi_i(s) = 1$ for $s \geq C$.

2. *If \mathcal{A}_i is a distortion functional, then*

$$\mathcal{A}_i(L_i) = \int_{-\infty}^{\infty} \mathcal{A}_i(\mathbb{1}_{\{x \geq s\}}(M))\xi_i(s) \, ds.$$

Proof. By Theorem 5 above and Proposition 5.16 in McNeil et al. (2005), L_i can be written as a monotonous function of M , i.e., $L_i = g_i(M)$ with $g_i : \mathbb{R} \rightarrow \mathbb{R}$ monotonously increasing. Since $M = \sum_{i=1}^n L_i$, the mappings g_i have to be absolutely continuous.

Therefore, it follows from the fundamental theorem of calculus that

$$L_i(\omega) = g_i(M(\omega)) = \int_C^{M(\omega)} g_i'(s) ds + g_i(C) = \int_{-\infty}^{\infty} \mathbf{1}_{\{x \geq s\}}(M(\omega)) \xi_i(s) ds + g_i(C)$$

with

$$\xi_i(s) = \begin{cases} g_i'(s), & \text{if } s > C \\ 0, & \text{if } s < C. \end{cases}$$

Also note that by the condition $M = \sum_{i=1}^n L_i$ it follows that for $s \geq C$, $\xi_i(s) \geq 0$ and $\sum_{i=1}^n \xi_i(s) = 1$ almost surely, establishing the first part of the theorem.

To prove the second part, note the random variables $\mathbf{1}_{x \geq s}(M)$ are comonotone for all $s \in \mathbb{R}$. Now define a set of points $(s_i)_{i \in \mathbb{N}}$ such that

$$\sum_{i=1}^N \mathbf{1}_{\{x \geq s_i\}}(M) \xi_i(s_i) \xrightarrow[N \rightarrow \infty]{a.s.} \int_{-\infty}^{\infty} \mathbf{1}_{\{x \geq s\}}(M) \xi_i(s) ds.$$

We then get

$$\begin{aligned} \mathcal{A}_i(L_i) &= \mathcal{A}_i \left(\lim_{N \rightarrow \infty} \sum_{i=1}^N \mathbf{1}_{\{x \geq s_i\}}(M) \xi_i(s_i) + \xi_i(C)C \right) = \lim_{N \rightarrow \infty} \mathcal{A}_i \left(\sum_{i=1}^N \mathbf{1}_{\{x \geq s_i\}}(M) \xi_i(s_i) \right) + \xi_i(C)C \\ &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \mathcal{A}_i(\mathbf{1}_{\{x \geq s_i\}}(M) \xi_i(s_i)) + \xi_i(C)C = \int_{-\infty}^{\infty} \mathcal{A}_i(\mathbf{1}_{\{x \geq s\}}(M)) \xi_i(s) ds + \xi_i(C)C \end{aligned}$$

where the second equality follows from Theorem 3.2 in Wozabal and Wozabal (2009) and the third one by comonotone additivity of \mathcal{A}_i (e.g., Pflug and Römisch 2007, Proposition 2.49). \square

Using this representation, we can show the following theorem.

THEOREM 8. *In a game fulfilling Assumptions 1 – 3 and Axioms 1 – 3 there is always an optimal L_i for which the ξ_i in the representation (51) are of the form*

$$\xi_i(s) = \begin{cases} \delta_i(s), & \mathcal{A}_i(\mathbf{1}_{\{x \geq s\}}(M)) = \max_j \mathcal{A}_j(\mathbf{1}_{\{x \geq s\}}(M)) \\ 0, & \text{otherwise,} \end{cases} \quad (52)$$

for almost all $s > C$ with $\delta_i(s) \in \{0, 1\}$ such that $\sum_{i=1}^n \delta_i = 1$ and $\xi_i(C)$ such that $\sum_{i=1}^n \xi_i(C) = 1$ and

$$\xi_i(C)C + \mathcal{A}_i \left(\int_{-\infty}^{\infty} \mathbf{1}_{\{x \geq s\}}(M) \xi_i(s) ds \right) = \frac{v_i}{\sum_{i=1}^n v_i} \mathcal{A}^{\max}(M). \quad (53)$$

Proof. We first prove the statement for $\delta_i \in [0, 1]$. Suppose that the statement is false and there exist optimal L_i , which violate (52). Then for a set $S \subseteq \mathbb{R}$ with positive probability there are players i and j such that

$$\mathcal{A}_i(\mathbf{1}_{\{x \geq s\}}(M)) > \mathcal{A}_j(\mathbf{1}_{\{x \geq s\}}(M))$$

for $s \in S$ and $\xi_i(s) < \xi_j(s)$.

Defining $\xi'_i(s) = \xi_i(s) + \mathbb{1}_S(s)\xi_j(s)$ and $\xi'_j(s) = \xi_j(s) - \mathbb{1}_S(s)\xi_j(s)$, we obtain a feasible $L' = (L_1, \dots, L'_i, \dots, L'_j, \dots, L_n)$ with

$$\begin{aligned} \sum_{i=1}^n \mathcal{A}_i(L'_i) - \sum_{i=1}^n \mathcal{A}_i(L_i) &= \mathcal{A}_i(L'_i) - \mathcal{A}_i(L_i) + \mathcal{A}_j(L'_j) - \mathcal{A}_j(L_j) \\ &= \int_S \xi_j(s) (\mathcal{A}_i(\mathbb{1}_{\{x \geq s\}}(M)) - \mathcal{A}_j(\mathbb{1}_{\{x \geq s\}}(M))) ds > 0, \end{aligned}$$

by Lemma 4. This contradicts the assumption of optimality of L by Theorem 2.

It follows that if for some $s > C$ the set $\mathcal{I}(s) = \arg \max_j \mathcal{A}_j(\mathbb{1}_{\{x \geq s\}}(M))$ has only one element, i.e., $\mathcal{I}(s) = \{i\}$, then $\delta_i = 1$ and $\delta_j = 0$ for $j \neq i$. If $|\mathcal{I}(s)| > 1$, any ξ'_i with $\sum_{i \in \mathcal{I}} \delta'_i(s) = 1$ and $\delta'_i(s) \in [0, 1]$ for $s > C$ is also optimal if $\xi_i(C)$ is modified for $i \in \mathcal{I}$ such that condition B) and (35) is still fulfilled. We thus can randomly pick an $i \in \mathcal{I}$ and set $\delta'_i = 1$ to obtain an optimal solution as described in the statement of the theorem. \square

The functions $M \mapsto L_i$ are thus monotonically increasing and piecewise affine with interchanging constant pieces and pieces with slope 1. The payoff can therefore be written as partition of the worst case payoff C plus a portfolio of standard call and put options on M , which are either received by or written to the community of the other players.

The next theorem establishes that in many practically relevant cases, finitely many options are enough to represent the split M into $L = (L_1, \dots, L_n)$.

THEOREM 9. *Under Assumption 1 – 3, the players can reach an optimal distribution of payoffs L by exchanging a finite number of options on M as an underlying if*

1. *the measures m_i are discrete with finitely many atoms or*
2. *the distribution of M is discrete and has only finitely many atoms.*

Proof. To prove the first point note that for any α

$$\text{AVaR}_\alpha(\mathbb{1}_{\{x \geq s\}}(M)) = \frac{1}{\alpha} \int_0^\alpha F_{\mathbb{1}_{\{x \geq s\}}(M)}^{-1}(t) dt = \max\left(\frac{F^{-1}(\alpha) - s}{\alpha}, 0\right). \quad (54)$$

It follows that if m_i has only finitely many atoms $\alpha_1, \dots, \alpha_m$ with probabilities p_1, \dots, p_m , then

$$\mathcal{A}_i(\mathbb{1}_{\{x \geq s\}}(M)) = \sum_{i=1}^m p_i \text{AVaR}_{\alpha_i}(\mathbb{1}_{\{x \geq s\}}(M)),$$

which together with (54) implies that $s \mapsto \mathcal{A}_i(\mathbb{1}_{\{x \geq s\}}(M))$ are piecewise affine functions with finitely many pieces. Since these functions can cross only finitely often, \mathbb{R} can be divided into finitely many intervals where $i^* = \arg \max_j \mathcal{A}_j(\mathbb{1}_{\{x \geq s\}}(M))$ remains constant. In these intervals player i^* receives all the additional payments from the project, i.e., $M \mapsto L_{i^*}(M)$ has slope 1 while $L_i(M)$ has slope

0 for all $i \neq i^*$. This structure of payments can be achieved by combining finitely many standard call and put options.

Similarly, to prove the second point, note that if M has finite support there are only finitely many distinct $\mathbb{1}_{\{s \leq s\}}$, i.e., the sets $\arg \max_j \mathcal{A}_j(\mathbb{1}_{\{x \geq s\}}(M))$ can only change finitely often, leading to L which can be expressed by finitely many options. \square

4. An Application Example: Construction of a Solar Roof

In this section, we use the theory developed in the previous sections to analyze the investment in 7.5 kW peak (kWp) of photovoltaic (PV) panels on a duplex house near Munich that is jointly owned by a young couple and a family consisting of two kids and two parents, one of whom is working. We stress that the example is meant as an illustration of the principles discussed in this paper and not as a contribution to the literature on optimal planning of residential PV installations. Consequently, we give preference to simple modeling wherever possible.

We describe the setup, the assumptions, and the models for stochastic variables in Section 4.1 and discuss the results of the bargaining game in Section 4.2.

4.1. Parameters and Modeling

In the following, we distinguish variables between the two households by the subscripts f (family) and c (couple). For our calculation we assume that the lifetime of the panels is 20 years and that investment costs are €1300 per kWp (Kost et al. 2018). For simplicity, we assume zero maintenance cost.

The two owners of the duplex house jointly invest in the project and can both use electricity generated from the panels. Electricity that is not directly consumed is sold to the grid for a feed-in tariff F that is fixed at 10 cents/kWh for the lifetime of the plant, which roughly corresponds to the current subsidy regime in Germany (see Fraunhofer ISE 2020).

The joint profit from installing the PV panels is thus the sum of profits generated from selling to the grid for the feed-in tariff and the avoided cost of self consumption. Since we assume identical electricity prices for both households, it is inconsequential for the joint profit who consumes the electricity. Therefore, we assume that electric current obeys physical laws and flows to the households dependent solely on consumption patterns. Hence, the savings from self-consumption are distributed more or less randomly and we assume that the two parties agree to make the necessary financial transactions in order to reach the agreed upon distribution of profits.

The reward of the project is random, since electricity production P of the PV panels, average household power prices G , hourly demand C_c and C_f , as well as the returns of the alternative investments R_c and R_f are random. We assume these factors to be independent and, for our

calculations, represent them by $S = 1000$ equally probable scenarios, which are sampled from the models outlined below.

For simplicity, we assume that the investments are split up proportional to average electricity consumption of the two households simulated from the models described below, which can be seen as a proxy for apartment size. This results in investments of €4505 by the family and €5245 by the couple. The opportunity losses are defined by the 20 year random profits R_c and R_f from investing in a diversified portfolio of German stocks. We model the price of the portfolio as a geometric Brownian motion with a yearly drift of 6.93% and a volatility of 21.83%, which we estimate based on daily (01/1988-12/2019, ignoring missing values) closing prices for the DAX performance index \hat{GDAXI} , freely available from *Yahoo! Finance*.

To generate samples for average household electricity prices, we fit a GBM process to average yearly household prices from 2000 – 2018 obtained from BDEW (2019) and simulate 20 years of yearly price changes starting from a price of 30.43 cents per kWh, which was the average price in 2018.

We model a yearly profile of solar irradiation for one square meter of ground in Munich in hourly resolution following Twidell and Weir (2015), Chapter 4. In particular, given values for the total radiation G_t and the diffuse radiation G_d , the produced energy per square meter is given by

$$\eta [(G_t - G_d) \cos \theta + G_d]. \quad (55)$$

Here, η denotes the efficiency of the panel and θ is the angle of incidence, i.e., the angle between the sun beam and the tilted surface of the solar panel. The angle of incidence varies with time and depends on several components:

1. The angle β between the panel and the horizontal (tilt) where $0^\circ \leq \beta \leq 90^\circ$ if the surface is directed towards the equator and $90^\circ < \beta \leq 180^\circ$ otherwise.
2. The angle γ between the normal to the solar panel surface, projected to the horizontal, and the local longitude meridian (azimuth). If $\gamma = 0^\circ$ the panel faces south, if $\gamma = 90^\circ$ the panel faces west, and if $\gamma = -90^\circ$ the panel faces east.
3. The latitude ϕ and the longitude ψ of the location of the panel.
4. The declination (northern hemisphere)

$$\delta = 23.45^\circ \sin \frac{360^\circ (284 + d)}{365}$$

where d denotes the day of the year with $d = 1, \dots, 365$.

5. The rotation angle w (hour angle) since the last solar noon,

$$w = (15^\circ h^{-1})(t_{zone} - 12h) + (\psi - \psi_{zone}).$$

Here t_{zone} is the civil time of the time zone containing longitude ψ and ψ_{zone} is the longitude where civil time t_{zone} and the solar time coincide. For simplicity, we neglect the so called equation of time, which is a small correction term.

Based on these components, the angle of incidence, respectively its cosine used in (55), fulfills

$$\cos \theta = (A - B) \sin \delta + [D \sin w + (E + F) \cos w] \cos \delta,$$

where

$$A = \sin \phi \cos \beta, \quad B = \cos \phi \sin \beta \cos \gamma, \quad D = \sin \beta \sin \gamma, \quad E = \cos \phi \cos \beta, \quad F = \sin \phi \sin \beta \cos \gamma.$$

In order to calculate usable electricity from (55) for a location in Munich, we use the relevant geographic information $\phi = 48.137154^\circ$, $\psi = 11.576124^\circ$, $\psi_{zone} = 15^\circ$. Moreover, we derive the local radiation values G_t and G_d from the global radiation maps published by DWD (2020) (German Weather Service). Finally we assume that 8m^2 of area are required per kWp capacity and that the efficiency of the panels is $\eta = 0.1$.

For a combination of angles $x = (\beta, \gamma)$ we calculate the nominal production $P(x)$ by (55) and the production in scenario $s = 1, \dots, S$ as

$$P_s(x) = P(x)\epsilon_s$$

where ϵ_s are independent normally distributed errors with mean 1 and standard deviation 0.2, which model multiplicative deviations from the long-term mean.

To generate hourly electricity demands, we use the free tool *LoadProfileGenerator*¹ using default profiles CHR33 and CHR44 for 2018 for the couple and the family, respectively. We let devices be generated randomly and assume that the couple owns an electric car which is charged at home and used for a 30km commute every day. To obtain random consumption profiles, we randomize the generated profiles by resampling days. More specifically, for a given day, we sample from the days in the generated profile that are in the same month and on the same weekday.

We assume that the risk preferences of the two households are given by

$$\mathcal{A}_c(X) = 0.6 \mathbb{E}(X) + 0.4 \text{AVaR}_{0.1}(X), \quad \mathcal{A}_f(X) = 0.7 \mathbb{E}(X) + 0.3 \text{AVaR}_{0.05}(X).$$

Note that the above risk preferences are distortion functionals with $m_f(1) = 0.7$, $m_f(0.05) = 0.3$, $m_c(1) = 0.6$, and $m_c(0.1) = 0.4$.

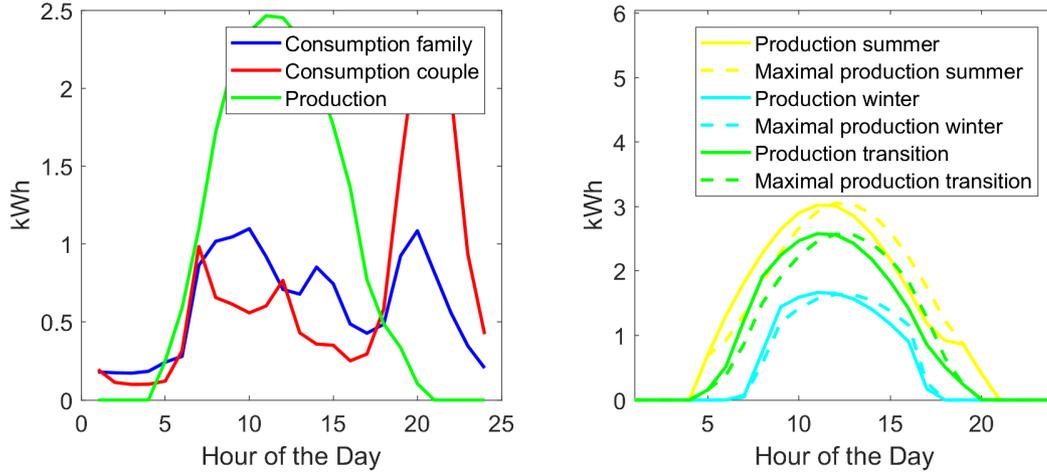


Figure 1 Consumption of the two investors against production of PV panels (left), optimal consumption versus configuration maximizing overall production.

4.2. Results

For simplicity, we solve the hourly planning problem, which is the basis for the distribution of profits and the alignment of the PV panels, only for one year of operation and multiply the result by 20 to obtain an estimate for the profits over the whole lifetime of the installation. Hence, we implicitly assume that the considered year is typical.

To obtain scenarios G_s for household electricity prices, we simulate trajectories G_s^t for $t = 1, \dots, 20$, $s = 1, \dots, S$ for the 20 year lifetime of the plant from the yearly electricity price process discussed above. We then obtain scenarios for average average household prices $G_s = 20^{-1} \sum_{t=1}^{20} G_s^t$, which we use in our calculation.

Apart from the decision on the split up of project profits, the investors have to decide on the azimuth γ and the panel tilt β for the solar panels, which together constitute the decision $x = (\beta, \gamma)$ in our framework. We therefore use $\mathcal{X} = [-180, 180] \times [0, 90]$ as the feasible set of our problem.

Since the production depends in a non-convex way on x , we discretize \mathcal{X} using a grid \mathcal{X}^g with mesh size 1° in both dimensions and solve the problems

$$\Pi(x) = \begin{cases} \max_{L, u, M} \mathcal{A}_c(L_c) + \mathcal{A}_f(L_f) \\ \text{s.t.} & M_s = \min(P_s(x), C_{cs} + C_{fs})G + \max(P_s(x) - C_{cs} - C_{fs}, 0)F, \forall s = 1, \dots, S \\ & M_s = L_{cs} + L_{fs}, \forall s = 1, \dots, S \\ & u_c = \frac{v_c}{v_f + v_c} (\mathcal{A}_c(L_c) + \mathcal{A}_f(L_f)) \\ & u_f = \frac{v_f}{v_f + v_c} (\mathcal{A}_c(L_c) + \mathcal{A}_f(L_f)) \\ & \mathcal{A}_c(L_c) \geq v_c \\ & \mathcal{A}_f(L_f) \geq v_f \end{cases} \quad (56)$$

¹ See <https://www.loadprofilegenerator.de/>

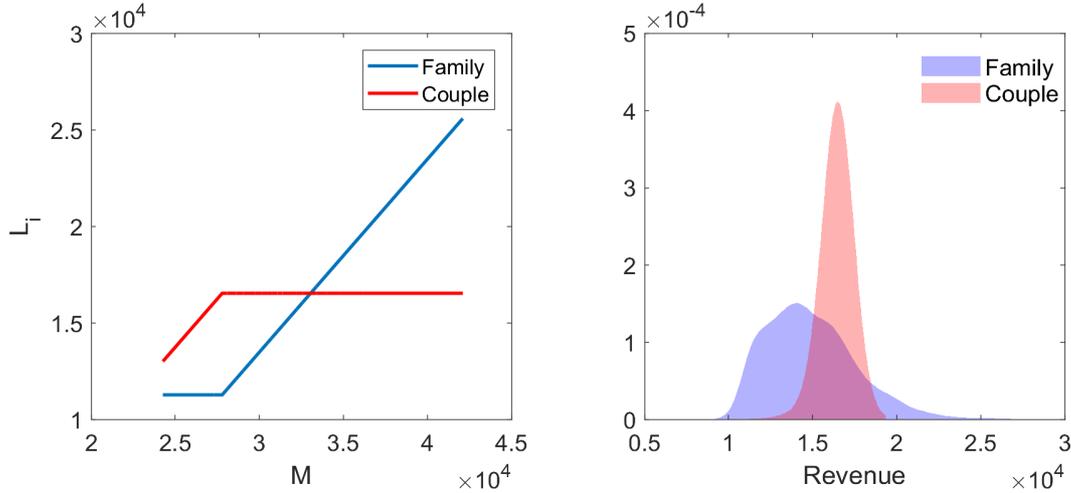


Figure 2 Distribution of revenues as a function of M (left) and as density plots (right).

for all $x \in \mathcal{X}^g$ and then find

$$x^* = \arg \min_{x \in \mathcal{X}^g} \Pi(x).$$

The optimal M , L , and u can then be found from the optimal solution of the problem $\Pi(x^*)$. Note that, for a fixed x , the above problem can be efficiently solved as a linear optimization problem.

The optimal angles are $x^* = (-18^\circ, 45^\circ)$ which deviates significantly from the angles $x^+ = (4^\circ, 41^\circ)$ maximizing overall electricity production, which is an interesting result in its own right. The production curves in summer, winter, and the transition periods (spring and autumn) are depicted in Figure 1. Relative to x^+ , the panels are facing eastward to receive more sunlight in the morning to match the consumption pattern of the households better. Furthermore, the higher tilt of 45° ensures increased production in winter as compared to x^+ . Although this choice leads to less overall production, it maximizes acceptability, since self consumption is strictly preferred to feeding into the grid.

The split up of the profits is plotted in the left panel of Figure 2. In accordance with the theory in Section 3.4, the payoffs are split up into simple options on the underlying M : the family receives a call option while the couple is short a put option on top of the distribution of the minimal value of M . Clearly, a contract specifying these payoffs as a function of M is legally feasible and easily understood by the parties.

Inspecting the density plot of the profits in the right panel of Figure 2, we see that the distribution ensures that the risk averse couple gets a moderate reward for sure which is capped once M reaches the strike price of the put option. Consequently, the distribution of the rewards of the couple is concentrated around its mean with less downside risk but also limited upside potential. Contrary to that, the family bears most of the downside risk of the project but also has a more pronounced upside potential due to the payout structure of the call option.

The differences in the investments and consequently in the conflict points lead to a split up of acceptabilities of $u_c = 16,145$ and $u_f = 13,788$ for the couple and the family, respectively. Due to the low prices of the panels and the rather high prices of grid electricity in Germany, these acceptabilities clearly exceed the acceptabilities of in the alternative investment which are $v_c = 6030$ and $v_f = 5150$.

5. Conclusions

We formulate a bargaining game for risk averse players that face uncertain profits from cooperation and whose risk preferences can be described by coherent acceptability functionals. Besides the allocation of acceptability values, we put special emphasis on finding ex-ante agreements on the distribution of uncertain profits. Additionally, the players also have to agree on an optimal usage of their resources by taking optimal managerial decisions.

We impose three axioms for a “fair solution” and show that these uniquely characterize a distribution of acceptabilities. Furthermore, we show that bargaining solutions can equivalently be characterized as solutions of stochastic optimization problems, which can be used to efficiently compute the bargaining allocations. In particular, we derive a stochastic programming formulation that establishes a close link of the proposed approach to Nash bargaining with bargaining power.

We study necessary and sufficient optimality conditions of these problems and show that profits of players have to be comonotone and that managerial decisions can be separated from the question of a fair distribution of profits. Furthermore, we show that in the case where agents have comonotonously additive risk preferences, the distribution of profits can be characterized as an exchange of standard options contracts between the agents, which makes the approach practically feasible for real world applications.

We present an illustrative case study of a joint investment in a solar roof on a duplex house. Two households with different consumption patters, risk preferences, and investment size optimize the alignment of the solar panels and search for a fair allocation of the profits from selling electricity to the grid and from the reduction of their energy bills by self consumption. In order to maximize self-consumption, households choose an alignment that deviates substantially from the alignment with maximal energy production. The optimal allocation of profits can be reached by fixed payments and a long position in a call option on the project profit for the player with smaller risk aversion and a short position in a put option for the more risk averse player.

Interesting topics for further research include profit sharing problems in energy applications such as jointly controlling a virtual power plant or managing a community storage as well as in other areas such as finance and insurance.

In the present paper, we studied bargaining games. It would be interesting to apply similar ideas to investigate coalitional games under uncertainty taking into account the implications of coalitional rationality for players whose risk aversion can be modeled using coherent acceptability measures.

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Appendix A: Auxiliary Results

LEMMA 5. *Let X and Y be two topological spaces. If Y is compact, then the projection*

$$\text{proj}_1 : X \times Y \rightarrow X, \text{ with } \text{proj}_1(x, y) = x$$

is a closed mapping.

LEMMA 6. *For every X , there is a $\alpha_0(X)$ such that*

$$\mathcal{A}_i(X) = \text{AVaR}_{\alpha_0(X)}(X).$$

If $\mathcal{A}_i \neq \mathbb{E}$ and X is not constant, then $\alpha_0(X) < 1$ is unique.

Proof. Note that $\alpha \mapsto \text{AVaR}_\alpha(X)$ is continuous and strictly monotonically increasing if X is not almost surely constant. If X is almost surely constant, then $\text{AVaR}_\alpha(X)$ is constant as well and α_0 can be chosen arbitrarily and in particular $\text{AVaR}_1(X) = \mathbb{E}(X) = \mathcal{A}_i(X)$.

For the non-constant case, clearly,

$$\text{AVaR}_0(X) = \text{ess inf } X \leq \mathcal{A}_i(X) \leq \mathbb{E}(X) = \text{AVaR}_1(X)$$

and strict monotonicity yields a unique $\alpha_0(X)$ by the intermediate value theorem. □

LEMMA 7. *Let $\mathcal{A}_i \neq \mathbb{E}$ be coherent acceptability measures, then the set*

$$K = \left\{ (L_1, \dots, L_n) : \exists x \in \mathcal{X} \text{ with } \sum_i L_i \leq M(x), \mathcal{A}_i(L_i) \geq v_i > -\infty \right\}$$

is relatively weakly compact in $\times_{i=1}^n \mathcal{L}^p$.

Proof. By the theorem of Banach-Alaoglu, K is relatively weakly compact if it is norm-bounded, where we use

$$\|(L_1, \dots, L_n)\| = \sum_i \|L_i\|_p$$

as the norm in $\times_{i=1}^n \mathcal{L}^p$.

Suppose there is a sequence $(L^k)_{k \in \mathbb{N}} \subseteq K$ with corresponding $(x^k)_{k \in \mathbb{N}} \subseteq \mathcal{X}$, such that the former is unbounded. Possibly by selecting a subsequence, we can find a j such that $\|L_j^k\|_1 \rightarrow \infty$. Similarly,

we can assume without loss of generality that there is a player i for whom there are sets $A_i^k \subseteq \Omega$ with

$$\int_{A_i^k} L_i^k d\mathbb{P} \xrightarrow{n \rightarrow \infty} -\infty. \quad (57)$$

Let $\alpha^k = \alpha_0(L_i^k, \mathcal{A}_i)$. If $\alpha_0^k \not\rightarrow 1$, then there exists an accumulation point $\bar{\alpha}_0 < 1$ and sets $B_i^k \subseteq \Omega$ such that $\mathbb{P}(A_i^k \cup B_i^k) = \alpha^k < 1$ and $L_i^k(\omega') \geq L_i^k(\omega)$ for all $\omega \in A_i^k \cup B_i^k$, $\omega' \in \Omega \setminus (A_i^k \cup B_i^k)$ and

$$\frac{1}{\alpha^k} \int_{A_i^k \cup B_i^k} L_i^k d\mathbb{P} \geq v_i,$$

implying that $\int_{B_i^k} L_i^k d\mathbb{P} \rightarrow \infty$ and therefore for any other $j \neq i$

$$\begin{aligned} \mathcal{A}_j(L_j^k) &\leq \int L_j^k d\mathbb{P} \leq \int_{A_i^k \cup B_i^k} (M(x^k) - L_i^k) d\mathbb{P} + \int_{\Omega \setminus (A_i^k \cup B_i^k)} (M(x^k) - L_i^k) d\mathbb{P} \\ &\leq \sup_{x \in \mathcal{X}} \mathbb{E}(|M(x)|) - v_i - \int_{\Omega \setminus (A_i^k \cup B_i^k)} L_i^k d\mathbb{P} \xrightarrow{n \rightarrow \infty} -\infty, \end{aligned}$$

since $\sup_{x \in \mathcal{X}} \mathbb{E}(|M(x)|) = \sup_{x \in \mathcal{X}} \|M(x)\|_1 < \infty$ because of Assumption 1. This violates the assumption that $\mathcal{A}_j(L_j^k) \geq v_j > -\infty$ and thus shows that $L^k \notin K$ eventually.

What is left, is the case $\alpha^k \rightarrow 1$. If $\mathbb{E}(L_i^k) \rightarrow \infty$, then clearly for $j \neq i$

$$\mathcal{A}_j(L_j^k) \leq \mathbb{E}(L_j^k) \leq \mathbb{E}(M(x^k) - L_i^k) = \mathbb{E}(M(x^k)) - \mathbb{E}(L_i^k) \xrightarrow{n \rightarrow \infty} -\infty < v_j.$$

Hence, we can assume that $\mathbb{E}(L_i^k)$ remains bounded above and we therefore can find a finite $C \in \mathbb{R}$, such that for every $0 < \alpha \leq 1$ and $0 \leq \gamma < \alpha$

$$\int_{\gamma}^{\alpha} F_{L_i^k}^{-1}(t) dt \leq \int_{L_i^k \geq 0} L_i^k d\mathbb{P} \leq C, \quad \forall n \in \mathbb{N}. \quad (58)$$

Now for $0 < \alpha \leq 1$

$$\text{AVaR}_{\alpha}(L_i^k) = \frac{1}{\alpha} \left(\int_0^{\mathbb{P}(A_i^k)} F_{L_i^k}^{-1}(t) dt + \int_{(\mathbb{P}(A_i^k), \alpha]} F_{L_i^k}^{-1}(t) dt \right) \xrightarrow{n \rightarrow \infty} -\infty,$$

since the first term in the brackets diverges to $-\infty$ because of (57), while the second term is bounded above by C due to (58). However, this in particular, implies that

$$\lim_{n \rightarrow \infty} \mathcal{A}_i(L_i^k) \leq \lim_{n \rightarrow \infty} \mathbb{E}(L_i^k) = \lim_{n \rightarrow \infty} \text{AVaR}_1(L_i^k) = -\infty$$

and therefore $\mathcal{A}_i(L_i^k) \geq v_i$ is eventually violated, which leads to a contradiction to $L^k \in K$, proving the claim. \square

Proof of Lemma 1 Since $\mathcal{A}_i(M) \leq \mathbb{E}(M)$, the set is clearly bounded.

To show that $\mathcal{U}(v, M, \mathcal{A})$ is closed, we write $\mathcal{U}(v, M, \mathcal{A}) = \text{proj}_1(\mathcal{V})$ with

$$\begin{aligned} \mathcal{V} &= \text{hypo}(\mathcal{A}) \cap \left(\{u : u_i \geq v_i\} \times \prod_{i=1}^n \mathcal{L}^p \right) \cap \left(\mathbb{R}^n \times \left\{ L \in \prod_{i=1}^n \mathcal{L}^p : \exists x \in \mathcal{X} \text{ with } \sum_i L_i = M \right\} \right) \\ &\subseteq \mathbb{R}^n \times \bar{K}^\sigma. \end{aligned}$$

The first set is the hypograph of \mathcal{A} which is closed in $\prod_{i=1}^n \mathcal{L}^p$ since \mathcal{A} is upper semi-continuous. Since the set is also convex, it is also closed in $\mathbb{R}^n \times (\prod_{i=1}^n \mathcal{L}^p, \sigma)$, where σ is the weak topology in $\prod_{i=1}^n \mathcal{L}^p$. Clearly, the second set is also closed in the same topology. To analyze the third set note that by Assumption 1 the set $M(\mathcal{X})$ is weakly compact in \mathcal{L}^p . Define the linear function $f(L) = \sum_{i=1}^n L_i$ from $\prod_{i=1}^n \mathcal{L}^p$ to \mathcal{L}^p . Since f is bounded, it is continuous and consequently the set $f^{-1}(M(\mathcal{X}))$ is closed in the norm topology in $\prod_{i=1}^n \mathcal{L}^p$ and since it is convex also in the weak topology.

It follows that \mathcal{V} is closed in $\mathbb{R}^n \times \bar{K}^\sigma$ and because \bar{K}^σ is weakly compact due to Lemma 7, Lemma 5 shows that $\mathcal{U}(v, M, \mathcal{A})$ is closed. \square

LEMMA 8. *The function $v \mapsto z(R, M, \mathcal{A})$ is continuous.*

Proof. Starting from the definition

$$z = \max_{L, x} \left\{ \sum_j \mathcal{A}_j(L_j) \mid \sum_j L_j = M(x), \mathcal{A}_j(L_j) \geq v_j (j \in N) \right\} \quad (59)$$

for some chosen $i \in N$ and $\epsilon > 0$, we define a *perturbed problem*

$$z' = \max_{L, x} \left\{ \sum_j \mathcal{A}_j(L_j) \mid \sum_j L_j = M(x), \mathcal{A}_j(L_j) \geq v_j (j \neq i), \mathcal{A}_i(L_i) \geq v_i - \epsilon \right\} \quad (60)$$

Clearly, problem (60) is a relaxation of problem (59), so in particular $z' \geq z$.

First, we show the following auxiliary result: Let L' be the optimal solution of (60). Then there is a feasible solution L of (59) with $|\mathcal{A}_i(L'_i) - \mathcal{A}_i(L_i)| \leq \epsilon, \forall i \in N$.

The assertion is obviously valid if L' is a feasible solution of (59). So let us assume now that this is not the case. Then, for suitable x ,

$$\sum_j L'_j = M(x), \quad \mathcal{A}_j(L'_j) \geq v_j (j \neq i), \quad \mathcal{A}_i(L'_i) \geq v_i - \epsilon, \text{ and } \mathcal{A}_i(L'_i) < v_i.$$

Setting $u'_j = \mathcal{A}_j(L'_j)$ ($j \in N$) and $0 \leq \delta = v_i - u'_i \leq \epsilon$, we have

$$\sum_{j \neq i} u'_j \geq \sum_{j \neq i} v_j + \delta, \quad (61)$$

since otherwise

$$z' = \sum_j u'_j = v_i - \delta + \sum_{j \neq i} u'_j < v_i - \delta + \sum_{j \neq i} v_j + \delta = \sum_j v_j,$$

i.e., z' would be smaller than z , in contradiction to the fact that (60) is a relaxation of (59).

Setting $\delta_j = u'_j - v_j \geq 0$ ($j \neq i$), it follows from (60) that

$$\sum_{j \neq i} \delta_j \geq \delta. \quad (62)$$

We define $L_i = L'_i + \delta$, $L_j = L'_j - \delta'_j$ for $j \neq i$ with $0 \leq \delta'_j \leq \delta_j$ ($j \neq i$) and $\sum_{j \neq i} \delta'_j = \delta$, which is possible because of (62). Then

$$\sum_j L_j = (L'_i + \delta) + \sum_{j \neq i} L'_j - \sum_{j \neq i} \delta'_j = \sum_j L'_j + \delta - \sum_{j \neq i} \delta'_j = \sum_j L'_j = M(x).$$

Furthermore, using translation invariance of the measures \mathcal{A}_i , we get:

$$\mathcal{A}_i(L_i) = \mathcal{A}_i(L'_i + \delta) = \mathcal{A}_i(L'_i) + \delta = u'_i + \delta = v_i,$$

as well as

$$\mathcal{A}_j(L_j) = \mathcal{A}_j(L'_j - \delta'_j) = \mathcal{A}_j(L'_j) - \delta'_j = u'_j - \delta'_j \geq u'_j - \delta_j = v_j \quad (63)$$

for each $j \neq i$. Hence, L is feasible solution of (59), with the value for x a part of the optimal solution (60). Finally,

$$|\mathcal{A}_i(L'_i) - \mathcal{A}_i(L_i)| = |\mathcal{A}_i(L'_i) - \mathcal{A}_i(L'_i + \delta)| = |-\delta| \leq \epsilon,$$

and because of (63), for $j \neq i$,

$$|\mathcal{A}_j(L'_j) - \mathcal{A}_j(L_j)| = |-\delta'_j| = \delta'_j \leq \sum_{k \neq i} \delta'_k = \delta \leq \epsilon.$$

This proves the auxiliary statement.

Now suppose that some $\epsilon > 0$ is given, and consider the perturbed problem (60) for the given ϵ and some component i . By the auxiliary result, we can associate to the optimal solution of (60) a feasible solution of the basic problem (59) for which the objective value $\sum_i \mathcal{A}_i(L_i)$ is at most worse by $n \cdot \epsilon$ compared to the solution value of (60). As problem (60) relaxes problem (59), this means that the solution values z and z' can differ by not more than $n \cdot \epsilon$.

Let us write $\zeta(v)$ for $z(R, M, \mathcal{A})$ with $\mathcal{A}_i(R_i) = v_i$ ($i \in R$). Then what has been shown above is that for two vectors v and v' ,

$$v_j = v'_j \quad \forall j \neq i \text{ and } |v_i - v'_i| \leq \epsilon \Rightarrow |\zeta(v) - \zeta(v')| \leq n\epsilon.$$

Consider now, to given v , a vector \bar{v} with $|v_j - \bar{v}_j| \leq \epsilon$ for all j . Then

$$\begin{aligned} |\zeta(v) - \zeta(\bar{v})| &\leq |\zeta(v) - \zeta(\bar{v}_1, v_2, \dots, v_n)| + |\zeta(\bar{v}_1, v_2, \dots, v_n) - \zeta(\bar{v}_1, \bar{v}_2, v_3, \dots, v_n)| \\ &\quad + |\zeta(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n-1}, v_n) - \zeta(\bar{v})| \leq n^2\epsilon, \end{aligned}$$

showing the continuity of $\zeta(v)$ as claimed. \square