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Stability in affine optimal control problems constrained by semilinear elliptic partial differential equations*

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Abstract

This paper investigates stability properties of affine optimal control problems constrained by semilinear elliptic partial differential equations. This is done by studying the so called metric subregularity of the set-valued mapping associated with the system of first order necessary optimality conditions. Preliminary results concerning the differentiability of the functions involved are established, especially the so-called switching function. Using this ansatz, more general nonlinear perturbations are encompassed, and under weaker assumptions, than the ones previously considered in the literature on control constrained elliptic problems. Finally, the applicability of the results is illustrated with some error estimates for the Tikhonov regularization.

1 Introduction

We consider the following optimal control problem

$$\min_{u\in\mathcal{U}}\left\{\int_\Omega \Big[w(x,y)+s(x,y)u\Big]\,dx\right\}, \tag{1.1}$$

subject to

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big)+d(x,y) = \beta(x)u & \text{in} \quad \Omega \\ A(x)\nabla y\cdot\nu+b(x)y = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

The set $\Omega\subset\mathbb{R}^n$ is a bounded domain with Lipschitz boundary, where $n\in\{2,3\}$. The unit outward normal vector field on the boundary $\partial\Omega$, which is single valued a.e. in $\partial\Omega$, is denoted by ν. The control set is given by

$$\mathcal{U}:=\{u:\Omega\to\mathbb{R} \ \text{ measurable}: \ \ b_1(x)\le u(x)\le b_2(x) \ \text{ for a.e. } x\in\Omega\},$$

where b_1 and b_2 are bounded measurable functions satisfying $b_1(x)\le b_2(x)$ for a.e. $x\in\Omega$. The functions $w:\Omega\times\mathbb{R}\to\mathbb{R}$, $s:\Omega\times\mathbb{R}\to\mathbb{R}$, $d:\Omega\times\mathbb{R}\to\mathbb{R}$, $\beta:\Omega\to\mathbb{R}$ and $b:\partial\Omega\to\mathbb{R}$ are real-valued and measurable, and $A:\Omega\to\mathbb{R}^{n\times n}$ is a measurable matrix-valued function.

There are many motivations for studying stability of solutions, in particular for error analysis of numerical methods, see e.g., [30, 31]. Most of the stability results for elliptic control problems are obtained under a second order growth condition (analogous to the classical Legendre-Clebsch condition). For literature concerning this type of problems, the reader is referred to [18, 21, 22, 24, 25, 35] and the references therein. In optimal control problems like (1.1)–(1.2), where the control appears linearly (hence, called affine problems)

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this growth condition does not hold. The so-called bang-bang solutions are ubiquitous in this case, see [4, 9, 10]. To give an account of the state of art in stability of bang-bang problems, we mention the works [1, 28, 29, 33, 37] on optimal control of ordinary differential equations. Results for optimization problems constrained by partial differential equations have been gaining relevance in recent years, see [5, 8, 9, 10, 12, 34]. However, its stability has been only investigated in a handful of papers, see e.g., [12, 32, 34]. From these works, we mention here particularly [34], where the authors consider linear perturbations in the state and adjoint equations for a similar problem with Dirichlet boundary condition. They use the so-called structural assumption (a growth assumption satisfied near the jumps of the control) on the adjoint variable. This assumption has been widely used in the literature on bang-bang control of ordinary differential equations in a somewhat different form.

The investigations of stability properties of optimization problems, in general, are usually based on the study of similar properties of the corresponding system of necessary optimality conditions. The first order necessary optimality conditions for problem (1.1)–(1.2) can be recast as a system of two elliptic equations (primal and adjoint) and one variational inequality (representing the minimization condition of the associated Hamiltonian), forming together a *generalized equation*, that is, an inclusion involving a set-valued mapping called *optimality mapping*. The concept of *strong metric subregularity*, see [11, 16], of set-valued mappings has shown to be efficient in many applications especially ones related to error analysis, see [2]. This also applies to optimal control problems of ordinary differentials equations, see e.g., [15, 28].

In the present paper we investigate the strong metric subregularity property of the optimality mapping associated with problem (1.1)–(1.2). We present sufficient conditions for strong subregularity of this mapping on weaker assumptions than the ones used in literature, see Section 6 for precise details. The structural assumption in [34] is weakened and more general perturbations are considered. Namely, perturbations in the variational inequality, appearing as a part of the first order necessary optimality conditions, are considered; which are important in the numerical analysis of ODE and PDE constrained optimization problems. Moreover, nonlinear perturbations are investigated, which provides a framework for applications, as illustrated with an estimate related to the Tikhonov regularization. The concept of linearization is employed in a functional frame in order to deal with nonlinearities. The needed differentiability of the control-to-adjoint mapping and the switching function (see Section 3) is proved, and the derivatives are used to obtain adequate estimates needed in the stability results. Finally, we consider nonlinear perturbations in a general framework. We propose the use of the compact-open topology to have a notion of "closeness to zero" of the perturbations. In our particular case this topology can me metrized, providing a more "quantitative" notion. Estimates in this metric are obtained in Section 5.

2 Preliminaries

The euclidean space $\mathbb{R}^s$ is considered with its usual norm, denoted by $|\cdot|$. As usual, for $p \in [1, \infty)$, we denote by $L^p(\Omega)$ the space of all measurable p-integrable functions $\psi : \Omega \to \mathbb{R}^s$ with the norm

$$|\psi|_{L^p(\Omega)} := \Big(\sum_{i=1}^{s} \int_\Omega |\psi_i(x)|^p \, dx \Big)^{\frac{1}{p}}.$$

The space $L^\infty(\Omega)$ consists of all measurable essentially bounded functions $\psi : \Omega \to \mathbb{R}^s$ with the norm

$$|\psi|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |\psi(x)|.$$

We denote by $C(\bar\Omega)$ the space of continuous functions on Ω that can be extended continuously to $\bar\Omega$ equipped with the L^∞-norm. We denote by $H^1(\Omega)$ the space of functions $\psi \in L^2(\Omega)$ with weak derivatives in $L^2(\Omega)$ endowed with its usual norm. The space $H^1(\Omega) \cap C(\bar\Omega)$ is endowed with the norm

$$|\psi|_{H^1(\Omega)\cap C(\bar\Omega)} := |\psi|_{H^1(\Omega)} + |\psi|_{C(\bar\Omega)}.$$

A function $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if $\psi(\cdot, y)$ is measurable for every $y \in \mathbb{R}$, and $\psi(x, \cdot)$ is continuous for a.e. $x \in \Omega$. A function $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be locally Lipschitz uniformly in the first variable if for each $M > 0$ there exists $L > 0$ such that

$$|\psi(x, y_2) - \psi(x, y_1)| \le L|y_2 - y_1|$$

for a.e. $x \in \Omega$ and all $y_1, y_2 \in [-M, M]$. In order to abbreviate notation, we define $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, y, u) := \beta(x)u - d(x, y) \quad \text{and} \quad g(x, y, u) := w(x, y) + s(x, y)u.$$

The following assumption is supposed to hold throughout the remainder of the paper. It ensures that the mathematical objects related to problem (1.1)–(1.2) that we consider are well defined. Assumption 1 is quite standard in the literature, see the book [39].

Assumption 1. *The following statements are assumed to hold.*

(i) The set $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. The matrix $A(x)$ is symmetric for a.e. x in Ω, and there exists $\alpha > 0$ such that $\xi \cdot A(x)\xi \geq \alpha|\xi|^2$ for a.e. x in Ω and all $\xi \in \mathbb{R}^n$.

(ii) The functions w, s and d are Carathéodory, twice differentiable with respect to the second variable, and their second derivatives are locally Lipschitz uniformly in the first variable.

(iii) The functions $A, \beta, b, d(\cdot, 0), d_y(\cdot, 0), w_y(\cdot, 0)$ and $s_y(\cdot, 0)$ are measurable and bounded.

(iv) The function $d_y(\cdot, y)$ is nonnegative a.e. in Ω for all $y \in \mathbb{R}$. The function b is nonnegative a.e. in $\partial\Omega$ and $|b|_{L^\infty(\partial\Omega)} > 0$.

Items (*i*) and (*iv*) of Assumption 1 ensure that the partial differential equations appearing in this paper have unique solutions in the space $H^1(\Omega) \cap L^\infty(\Omega)$.

2.1 The elliptic operator

We consider the set $D(\mathcal{L})$ of all functions $y \in H^1(\Omega) \cap L^\infty(\Omega)$ for which there exists $h \in L^2(\Omega)$ such that

$$\int_\Omega A(x)\nabla y \cdot \nabla\varphi \, dx + \int_{\partial\Omega} b(x)y\varphi \, ds(x) = \int_\Omega h\varphi \, dx \quad \forall \varphi \in H^1(\Omega). \tag{2.1}$$

As usual, ds denotes the Lebesgue surface measure. It is easy to see that for each $y \in D(\mathcal{L})$ there exists a unique element $h \in L^2(\Omega)$ such that (2.1) holds. We define the operator $\mathcal{L} : D(\mathcal{L}) \to L^2(\Omega)$ by assigning each $y \in D(\mathcal{L})$ to the function $h \in L^2(\Omega)$ satisfying (2.1). By definition, a function $y \in H^1(\Omega) \cap L^\infty(\Omega)$ belongs to $D(\mathcal{L})$ if, and only if, it is the weak solution of the linear elliptic partial differential equation

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) = h & \text{in} \quad \Omega, \\ A(x)\nabla y \cdot \nu + b(x)y = 0 & \text{on} \quad \partial\Omega \end{cases}$$

for some $h \in L^2(\Omega)$. The following lemma is of trivial nature.

Lemma 2.1. *The set $D(\mathcal{L})$ is a linear subspace of $H^1(\Omega) \cap L^\infty(\Omega)$. Moreover, the operator $\mathcal{L} : D(\mathcal{L}) \to L^2(\Omega)$ is a well defined linear mapping.*

If $D(\mathcal{L})$ is endowed with the norm of $L^2(\Omega)$, then $\mathcal{L}$ is an unbounded operator from $D(\mathcal{L})$ to $L^2(\Omega)$. Since $A(x)$ is symmetric for a.e. $x \in \Omega$, by (2.1) we have

$$\int_\Omega \mathcal{L}y\varphi \, dx = \int_\Omega y\mathcal{L}\varphi \, dx \tag{2.2}$$

for all $y, \varphi \in D(\mathcal{L})$, the so-called integration by parts formula.

Remark 2.2. If $\partial\Omega$ is of class $C^{1,1}$, A is Lipschitz in $\bar{\Omega}$, and b is Lipschitz and positive in $\partial\Omega$, then

$$D(\mathcal{L}) = \{y \in H^2(\Omega) : A(\cdot)\nabla y \cdot \nu + b(\cdot)y = 0\},$$

and $\mathcal{L}y = -\operatorname{div}\big(A(\cdot)\nabla y\big)$ for all $y \in D(\mathcal{L})$, see [19, Theorem 2.4.2.6].

The following lemma shows the inclusion $D(\mathcal{L}) \subset C(\bar{\Omega})$. Its proof can be found in [39, Theorem 4.7] and follows the arguments in [4, 38].

Lemma 2.3. *Let $\alpha \in L^\infty(\Omega)$ be nonnegative and $h \in L^2(\Omega)$. There exists a unique function $y \in D(\mathcal{L})$ such that*

$$\mathcal{L}y + \alpha(\cdot)y = h \tag{2.3}$$

and this function belongs to $C(\bar{\Omega})$. Moreover, for each $r > n/2$ there exists a positive number c such that

$$|y|_{H^1(\Omega)\cap C(\bar{\Omega})} \leq c|h|_{L^r(\Omega)}$$

for all $\alpha \in L^\infty(\Omega)$ nonnegative, $y \in D(\mathcal{L})$, and $h \in L^2(\Omega) \cap L^r(\Omega)$ satisfying (2.3).

The following technical lemma can be deduced from Lemma 2.3, see the proof of [10, Lemma 3.4]. Its use in optimal control of elliptic partial differential equations dates from the paper [9, Lemma 2.6]. It has shown to be useful for diverse estimates, see [9, 34].

Lemma 2.4. *There exists a positive number c such that*

$$|y|_{L^2(\Omega)} \leq c|h|_{L^1(\Omega)}$$

for all $\alpha \in L^\infty(\Omega)$ nonnegative, $y \in D(\mathcal{L})$ and $h \in L^2(\Omega)$ satisfying (2.3).

The proof of the next result can be found in [7, Theorem 2.11] in the case of a Dirichlet problem, see also [20, Lemma 6.8]. Here we adapt the argument below Theorem 2.1 in [6, p. 618].

Lemma 2.5. *Let $\alpha \in L^\infty(\Omega)$ be nonnegative, $\{h_m\}_{m=1}^\infty$ be a sequence in $L^2(\Omega)$ and $h \in L^2(\Omega)$. For each $m \in \mathbb{N}$, let $y_m \in C(\bar{\Omega})$ be the unique function satisfying $\mathcal{L}y_m + \alpha(\cdot)y_m = h_m$, and let $y \in C(\bar{\Omega})$ be the unique function satisfying of $\mathcal{L}y + \alpha(\cdot)y = h$. If $h_m \rightharpoonup h$ weakly in $L^2(\Omega)$, then $y_m \to y$ in $C(\bar{\Omega})$.*

Proof. Let $p \in (2n/(n+2), n/(n-1))$. Then $W^{1,p}(\Omega)$ is compactly embedded in $L^2(\Omega)$ and consequently, by Schauder's Theorem, $L^2(\Omega)$ is compactly embedded in $W^{1,p}(\Omega)^*$. By the latter compact embedding, every weakly convergent sequence in $L^2(\Omega)$ converges also in $W^{1,p}(\Omega)^*$ to the same limit. Define $\mathcal{K} : L^2(\Omega) \to C(\bar{\Omega})$ by $\mathcal{K}h := y$, where $y \in C(\bar{\Omega})$ is the unique function satisfying $\mathcal{L}y + \alpha(\cdot) = h$. The result follows from [27, Theorem 3.14], since that theorem asserts that the linear operator $\mathcal{K}$ is continuous from $L^2(\Omega)$ endowed with the norm of $W^{1,p}(\Omega)^*$ to $C(\bar{\Omega})$. □

Remark 2.6. Using the definitions of the set $D(\mathcal{L})$ and the operator $\mathcal{L}$, we can write in a shorter way the partial differential equations involved in this paper. For example, given $u \in \mathcal{U}$, to say that y belongs to $D(\mathcal{L})$ and satisfies $\mathcal{L}y + d(\cdot, y) = \beta(\cdot)u$ is equivalent to say that y belongs to $H^1(\Omega) \cap L^\infty(\Omega)$ and satisfies the weak formulation of (1.2), that is

$$\int_\Omega A(x)\nabla y \cdot \nabla\varphi \, dx + \int_\Omega d(x,y)\varphi \, dx + \int_{\partial\Omega} b(x)y\varphi \, ds(x) = \int_\Omega \beta(x)u\varphi \, dx$$

for all $\varphi \in H^1(\Omega)$. This weak formulation makes sense since, by (*ii*) and (*iii*) of Assumption 1, for any $y \in L^\infty(\Omega)$, the function $d(\cdot, y)$ belongs to $L^\infty(\Omega)$.

2.2 The control model

Having in mind Remark 2.6, given a function $u \in \mathcal{U}$ we say that $y_u \in D(\mathcal{L})$ is the associated state to $u \in \mathcal{U}$ if

$$\mathcal{L}y_u = f(\cdot, y_u, u). \tag{2.4}$$

The following proposition shows that the mapping $u \to y_u$ from $\mathcal{U}$ to $D(\mathcal{L})$ is well defined. Its proof can be found in the standard literature; it follows from [39, Theorem 4.8], see also [39, p. 212].

Proposition 2.7. *For each $u \in \mathcal{U}$ there exists a unique state $y_u \in D(\mathcal{L})$ associated with $u \in \mathcal{U}$. Moreover, $\{y_u : u \in \mathcal{U}\}$ is a bounded subset of $H^1(\Omega) \cap C(\bar{\Omega})$ and for each $r > n/2$ there exists $c > 0$ such that*

$$|y_{u_2} - y_{u_1}|_{H^1(\Omega)\cap C(\bar{\Omega})} \leq c|u_2 - u_1|_{L^r(\Omega)}$$

for all $u_1, u_2 \in \mathcal{U}$.

We call the function $\mathcal{G} : \mathcal{U} \to H^1(\Omega) \cap C(\bar{\Omega})$ given by $\mathcal{G}(u) := y_u$ the control-to-state mapping. The functional $\mathcal{J} : \mathcal{U} \to \mathbb{R}$ given by

$$\mathcal{J}(u) := \int_\Omega g(x, y_u, u)\, dx$$

is called the objective functional of problem (1.1)–(1.2).

Definition 2.8. Let $\bar{u}$ belong to $\mathcal{U}$.

(i) We say that $\bar{u}$ is a global solution of problem (1.1)–(1.2) if $\mathcal{J}(\bar{u}) \leq \mathcal{J}(u)$ for all $u \in \mathcal{U}$.

(ii) We say that $\bar{u}$ is a local solution of problem (1.1)–(1.2) if there exists $\varepsilon_0 > 0$ such that $\mathcal{J}(\bar{u}) \leq \mathcal{J}(u)$ for all $u \in \mathcal{U}$ with $|u - \bar{u}|_{L^1(\Omega)} \leq \varepsilon_0$.

(iii) We say that $\bar{u}$ is a strict local solution of problem (1.1)–(1.2) if there exists $\varepsilon_0 > 0$ such that $\mathcal{J}(\bar{u}) < \mathcal{J}(u)$ for all $u \in \mathcal{U}$ with $u \neq \bar{u}$ and $|u - \bar{u}|_{L^1(\Omega)} \leq \varepsilon_0$.

Under Assumption 1, problem (1.1)–(1.2) has at least one global solution. The proof is routine and can be obtained by standard arguments; namely, taking a minimizing sequence and using the weak compactness of $\mathcal{U}$ in $L^2(\Omega)$.

Lemma 2.9. *Problem (1.1)–(1.2) has at least one global solution.*

In order to make notation simpler, from now on we fix a local solution $\bar{u} \in \mathcal{U}$ of problem (1.1)–(1.2). We call the function $H : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by

$$H(x, y, p, u) := g(x, y, u) + pf(x, y, u),$$

the Hamiltonian of problem (1.1)–(1.2). Given $u \in \mathcal{U}$, we say that $p_u \in D(\mathcal{L})$ is the costate associated with $u \in \mathcal{U}$ if

$$\mathcal{L}p_u = H_y(\cdot, y_u, p_u, u).$$

The following proposition shows that the mapping $u \to p_u$ from $\mathcal{U}$ to $D(\mathcal{L})$ is well defined. We give the proof of this elementary result because it seems not to be explicitly stated in the literature.

Proposition 2.10. *For each $u \in \mathcal{U}$ there exists a unique costate $p_u \in D(\mathcal{L})$ associated with $u \in \mathcal{U}$. Moreover, $\{p_u : u \in \mathcal{U}\}$ is a bounded subset of $H^1(\Omega) \cap C(\bar{\Omega})$ and for each $r > n/2$ there exist $c > 0$ such that*

$$|p_{u_2} - p_{u_1}|_{H^1(\Omega) \cap C(\bar{\Omega})} \leq c|u_2 - u_1|_{L^r(\Omega)}$$

for all $u_1, u_2 \in \mathcal{U}$.

Proof. The existence and uniqueness follows from Lemma 2.3. Given $u \in \mathcal{U}$, the function p_u satisfies

$$\mathcal{L}p_u + d_y(\cdot, y_u)p_u = g_y(\cdot, y_u, u).$$

By *(ii)*, *(iii)* and *(iv)* of Assumption 1, for each $u \in \mathcal{U}$, the function $d_y(\cdot, y_u)$ is nonnegative and belongs to $L^\infty(\Omega)$. By *(ii)* and *(iii)* of Assumption 1, for each $u \in \mathcal{U}$ the function $g_y(\cdot, y_u, u)$ belongs to $L^\infty(\Omega)$. Furthermore, since by Proposition 2.7 the set $\{y_u : u \in \mathcal{U}\}$ is bounded in $C(\bar{\Omega})$, there exists $M_1 > 0$ such that

$$|g_y(\cdot, y_u, u)|_{L^\infty(\Omega)} \leq M_1$$

for all $u \in \mathcal{U}$. By Lemma 2.3, there exists a positive number c_1 such that for all $u \in \mathcal{U}$

$$|p_u|_{H^1(\Omega) \cap C(\bar{\Omega})} \leq c_1 |g_y(\cdot, y_u, u)|_{L^\infty(\Omega)}.$$

Thus, $M_2 := c_1 M_1$ is a bound for the set $\{p_u : u \in \mathcal{U}\}$ in $H^1(\Omega) \cap C(\bar{\Omega})$. Let $u_1, u_2 \in \mathcal{U}$ and $r > n/2$. We have then

$$\mathcal{L}(p_{u_2} - p_{u_1}) + d_y(\cdot, y_{u_2})(p_{u_2} - p_{u_1}) = H_y(\cdot, y_{u_2}, p_{u_1}, u_2) - H_y(\cdot, y_{u_1}, p_{u_1}, u_1).$$

By Lemma 2.3, there exists a positive number c_2 (independent of u_1 and u_2) such that

$$|p_{u_2} - p_{u_1}|_{H^1(\Omega)\cap C(\bar\Omega)} \le c_2 |H_y(\cdot, y_{u_2}, p_{u_1}, u_2) - H_y(\cdot, y_{u_1}, p_{u_1}, u_1)|_{L^r(\Omega)}.$$

By (ii) of Assumption 1 and the boundedness of the set $\{p_u : u \in \mathcal{U}\}$ in $C(\bar\Omega)$, there exists $L > 0$ such that

$$|H_y(\cdot, y_{u_2}, p_{u_1}, u_2) - H_y(\cdot, y_{u_1}, p_{u_1}, u_1)| \le L\Big(|y_{u_2} - y_{u_1}| + |u_2 - u_1|\Big) \quad \text{a.e. in } \Omega.$$

Consequently,

$$\begin{aligned} |p_{u_2} - p_{u_1}|_{H^1(\Omega)\cap C(\bar\Omega)} &\le c_2 L\big(|y_{u_1} - y_{u_2}|_{L^r(\Omega)} + |u_1 - u_2|_{L^r(\Omega)}\big) \\ &\le c_2 L\Big((\operatorname{meas}\Omega)^{\frac{1}{r}} |y_{u_2} - y_{u_1}|_{L^\infty(\Omega)} + |u_2 - u_1|_{L^r(\Omega)}\Big). \end{aligned}$$

By Proposition 2.7, there exists a constant $c_3 > 0$ (independent of u_1 and u_2) such that

$$|y_{u_2} - y_{u_1}|_{C(\bar\Omega)} \le c_3 |u_2 - u_1|_{L^r(\Omega)}.$$

Thus,

$$|p_{u_2} - p_{u_1}|_{H^1(\Omega)\cap C(\bar\Omega)} \le c_2 L\big(1 + c_3 (\operatorname{meas}\Omega)^{\frac{1}{r}}\big)|u_2 - u_1|_{L^r(\Omega)}.$$

The estimate follows defining $c := c_2 L\big(1 + c_3(\operatorname{meas}\Omega)^{\frac{1}{r}}\big)$. □

We call the function $\mathcal{S} : \mathcal{U} \to H^1(\Omega) \cap C(\bar\Omega)$ given by $\mathcal{S}(u) := p_u$ the control-to-adjoint mapping. The following proposition gives us another useful estimate; it can be easily proved employing Lemma 2.4 and the argument in the proof of [39, Theorem 4.16].

Proposition 2.11. *There exists $c > 0$ such that*

$$|y_{u_2} - y_{u_1}|_{L^2(\Omega)} + |p_{u_2} - p_{u_1}|_{L^2(\Omega)} \le c|u_2 - u_1|_{L^1(\Omega)}$$

for all $u_1, u_2 \in \mathcal{U}$.

We close this subsection with the following result.

Proposition 2.12. *Let $\{u_m\}_{m=1}^\infty$ be a sequence in $\mathcal{U}$ and $u \in \mathcal{U}$. If $u_m \rightharpoonup u$ weakly in $L^2(\Omega)$, then $y_{u_m} \to y_u$ and $p_{u_m} \to p_u$ in $C(\bar\Omega)$.*

Proof. We prove only the convergence $p_{u_m} \to p_u$ in $C(\bar\Omega)$, the convergence $y_{u_m} \to y_u$ in $C(\bar\Omega)$ is analogous. Let $\{p_{u_{m_k}}\}_{k=1}^\infty$ be an arbitrary subsequence of $\{p_{u_m}\}_{m=1}^\infty$. By the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, there exists a subsequence of $\{p_{u_{m_k}}\}_{k=1}^\infty$, denoted in the same way, and $p \in L^2(\Omega)$ such that $p_{u_{m_k}} \to p$ in $L^2(\Omega)$. Since $y_{u_{m_k}} \to y_u$ in $C(\bar\Omega)$, one can deduce that

$$H_y(\cdot, y_{u_{m_k}}, p_{u_{m_k}}, u_{m_k}) \rightharpoonup H_y(\cdot, y_u, p, u) \quad \text{weakly in } L^2(\Omega).$$

By Lemma 2.5, we have $p_{u_{m_k}} \to p_u$ in $C(\bar\Omega)$. The result follows, since every subsequence of $\{p_{u_m}\}_{m=1}^\infty$ has a further subsequence that converges to p_u in $C(\bar\Omega)$. □

3 Differentiability of the mappings involved

In this section, we prove some preliminary results concerning the differentiability of the control-to-state mapping, the control-to-adjoint mapping and the switching mapping (to be defined later). Some of these properties are well known for the control-to-state mapping; see, e.g., [5, 9, 10, 34, 39]. Nevertheless, we require more specific estimates than the ones in the literature. The differentiability of the control-to-adjoint mapping and the switching mapping has not been studied before in the literature on elliptic control-constrained problems, therefore we devote this section to obtain appropriate estimates needed in the study of stability in the next section. In the sequel, we treat differentiability by means of Gâteaux differentials, as they provide a very natural setting that adjusts in a very versatile way to our purposes.

3.1 The state and adjoint mappings

We begin this subsection recalling the definition of Gâteaux differential, see [17, pp.2-4] or [23, p.171]. Let Y be a Banach space and $\mathcal{F}:\mathcal{U}\to Y$ a mapping. Given $u\in\mathcal{U}$ and $v\in\mathcal{U}-u$, if the limit

$$d\mathcal{F}(u;v):=\lim_{\varepsilon\to 0^+}\frac{\mathcal{F}(u+\varepsilon v)-\mathcal{F}(u)}{\varepsilon}$$

exists in Y, we say that $\mathcal{F}(u;v)$ is the Gâteaux differential of $\mathcal{F}$ at u in the direction v. Note that by convexity of $\mathcal{U}$, $u+\varepsilon v$ belongs to $\mathcal{U}$ for every $u\in\mathcal{U}$, $v\in\mathcal{U}-u$ and $\varepsilon\in[0,1]$.

Recall that $\bar u\in\mathcal{U}$ is a fixed solution of problem (1.1)–(1.2). As it is well-known, the Gâteaux differential of the control-to-state mapping at $\bar u$ is related to the linearization of the system equation around $\bar u$. Bearing this in mind, given $v\in L^2(\Omega)$, we denote by z_v the unique[1] solution of the equation

$$\mathcal{L}z_v=f_y(\cdot,y_{\bar u},\bar u)z_v+f_u(\cdot,y_{\bar u},\bar u)v. \tag{3.1}$$

The proof of the following estimate can be found in the standard literature, see the proof of [39, Theorem 4.17] for the case of a Neumann boundary problem (the proof is the same for Robin or Dirichlet boundary). It can also be deduced by the same arguments given in the proof of Proposition 3.2.

Proposition 3.1. *For each $r>n/2$ there exists $c>0$ such that*

$$|y_u-y_{\bar u}-z_{u-\bar u}|_{H^1(\Omega)\cap C(\bar\Omega)}\le c|u-\bar u|^2_{L^r(\Omega)}\qquad\forall u\in\mathcal{U}.$$

One of the first things that can be deduced from Proposition 3.1 is the differentiability of the control-to-state mapping $\mathcal{G}$. Given $v\in L^2(\Omega)$ satisfying $\bar u+v\in\mathcal{U}$, the Gâteaux differential of the control-to-state mapping $\mathcal{G}$ at $\bar u$ in the direction v exists and is given by $d\mathcal{G}(\bar u;v)=z_v$. Moreover, one can prove that $\mathcal{G}$ is of class C^2. This is a standard application of the Implicit Function Theorem to the function $\mathcal{F}:D(\mathcal{L})\times L^r(\Omega)\to L^r(\Omega)$ given by $\mathcal{F}(y,u):=\mathcal{L}y+d(\cdot,y)-\beta(\cdot)u$, where $r>n/2$; see [7, Theorem 2.12] for details in the Dirichlet boundary case.

In order to study the Gâteaux differential of the control-to-adjoint mapping we introduce the following notations. Given $v\in L^2(\Omega)$, we denote by q_v the unique[2] solution of the equation

$$\mathcal{L}q_v=H_{yy}(\cdot,y_{\bar u},p_{\bar u},\bar u)z_v+H_{yp}(\cdot,y_{\bar u},p_{\bar u},\bar u)q_v+H_{yu}(\cdot,y_{\bar u},p_{\bar u},\bar u)v. \tag{3.2}$$

The following estimate is concerned with the differentiability of the control-to-adjoint mapping. To the best of our knowledge, this result does not appear in the literature; therefore we present its proof, although it is standard.

Proposition 3.2. *For each $r>n/2$ there exists $c>0$ such that*

$$|p_u-p_{\bar u}-q_{u-\bar u}|_{H^1(\Omega)\cap C(\bar\Omega)}\le c|u-\bar u|^2_{L^r(\Omega)}\qquad\forall u\in\mathcal{U}.$$

Proof. Given $u\in\mathcal{U}$, we define $\psi_u:\Omega\to\mathbb{R}^4$ by $\psi_u(x):=(x,y_u(x),p_u(x),u(x))$. For each $u\in\mathcal{U}$, we denote by $\tilde q_{u-\bar u}$ the unique solution of the equation

$$\mathcal{L}\tilde q_{u-\bar u}=H_{yy}(\psi_{\bar u})(y_u-y_{\bar u})+H_{yp}(\psi_{\bar u})\tilde q_{u-\bar u}+H_{yu}(\psi_{\bar u})(u-\bar u).$$

Let $u\in\mathcal{U}$ and $r>n/2$ be arbitrary. Using the Taylor Theorem and (ii)-(iii) of Assumption 1, one can find $\alpha_1,\alpha_2,\alpha_3\in L^\infty(\Omega)$ such that

$$\begin{aligned}H_y(\psi_u)=&H_y(\psi_{\bar u})+H_{yy}(\psi_{\bar u})(y_u-y_{\bar u})+H_{yp}(\psi_{\bar u})(p_u-p_{\bar u})+H_{yu}(\psi_{\bar u})v\\&+\alpha_1(\cdot)(y_u-y_{\bar u})^2+\alpha_2(\cdot)(y_u-y_{\bar u})(p_u-p_{\bar u})+\alpha_3(\cdot)(y_u-y_{\bar u})v,\end{aligned}$$

[1] The uniqueness follows from Lemma 2.3, and the fact that equation (3.1) can be rewritten as

$$\mathcal{L}z_v+d_y(\cdot,y_{\bar u})z_v=\beta(\cdot)v.$$

[2] The uniqueness follows from Lemma 2.3, and the fact that equation (3.2) can be rewritten as

$$\mathcal{L}q_v+d_y(\cdot,y_{\bar u})q_v=H_{yy}(\cdot,y_{\bar u},p_{\bar u},\bar u)z_v+H_{yu}(\cdot,y_{\bar u},p_{\bar u},\bar u)v.$$

where $v = u - \bar{u}$. Hence

$$\mathcal{L}(p_u - p_{\bar{u}} - \tilde{q}_v) = H_{yp}(\psi_{\bar{u}})(p_u - p_{\bar{u}} - \tilde{q}_v) + \Big[\alpha_1(\cdot)(y_u - y_{\bar{u}}) + \alpha_2(\cdot)(p_u - p_{\bar{u}}) + \alpha_3(\cdot)v\Big](y_u - y_{\bar{u}}).$$

By Lemma 2.3, Proposition 2.7 and Proposition 2.10, there exists $c_1 > 0$ such that

$$|p_u - p_{\bar{u}} - \tilde{q}_v|_{H^1(\Omega)\cap C(\bar{\Omega})} \le c_1 |v|^2_{L^r(\Omega)}.$$

Now,

$$\mathcal{L}(\tilde{q}_v - q_v) = H_{yy}(\psi_{\bar{u}})(y_u - y_{\bar{u}} - z_v) + H_{yp}(\psi_{\bar{u}})(\tilde{q}_v - q_v).$$

By Lemma 2.3 and Proposition 3.1, there exists $c_2 > 0$ such that

$$|\tilde{q}_v - q_v|_{H^1(\Omega)\cap C(\bar{\Omega})} \le c_2 |v|^2_{L^r(\Omega)}.$$

Finally, by the triangle inequality

$$|p_u - p_{\bar{u}} - q_v|_{H^1(\Omega)\cap C(\bar{\Omega})} \le |p_u - p_{\bar{u}} - \tilde{q}_v|_{H^1(\Omega)\cap C(\bar{\Omega})} + |\tilde{q}_v - q_v|_{H^1(\Omega)\cap C(\bar{\Omega})}.$$

The result follows taking $c := c_1 + c_2$. □

Given $v \in L^\infty(\Omega)$ satisfying $\bar{u} + v \in \mathcal{U}$, the Gâteaux differential of the control-to-adjoint mapping $\mathcal{S}$ at $\bar{u}$ in the direction v exists and is given by $d\mathcal{S}(\bar{u}; v) = q_v$. It is worth mentioning that the map $\mathcal{S}$ is of class C^2, this can be seen applying the Implicit Function Theorem to the function $\mathcal{H} : D(\mathcal{L}) \times L^r(\Omega) \to L^r(\Omega)$ given by $\mathcal{H}(p, u) := \mathcal{L}p - H_y(\cdot, y_u, p, u)$, where $r > n/2$.

We now state further properties concerning the mappings $v \to z_v$ and $v \to q_v$.

Proposition 3.3. *The following statements hold.*

(i) For each $r > n/2$ there exists a positive number c such that

$$|z_v|_{H^1(\Omega)\cap C(\bar{\Omega})} + |q_v|_{H^1(\Omega)\cap C(\bar{\Omega})} \le c|v|_{L^r(\Omega)} \quad \forall v \in L^2(\Omega) \cap L^r(\Omega).$$

(ii) There exists a positive number c such that

$$|z_v|_{L^2\Omega} + |q_v|_{L^2\Omega} \le c|v|_{L^1(\Omega)} \quad \forall v \in L^2(\Omega).$$

(iii) Let $\{v_k\}_{k=1}^\infty$ be a sequence in $L^2(\Omega)$ and $v \in L^2(\Omega)$. If $v_k \rightharpoonup v$ weakly in $L^2(\Omega)$, then $z_{v_k} \to z_v$ and $q_{v_k} \to q_v$ in $C(\bar{\Omega})$.

Proof. Items (i) and (ii) follow from Lemma 2.3 and 2.4, respectively. Item (iii) follows from Lemma 2.5. □

3.2 The switching mapping

Let us begin this subsection recalling the first order necessary condition (Pontryagin principle in integral form) for problem (1.1)–(1.2). If $u \in \mathcal{U}$ is a local solution of problem (1.1)–(1.2), then

$$\int_\Omega \Big[s(x, y_u) + \beta(x)p_u\Big](w - u)\,dx \ge 0 \quad \forall w \in \mathcal{U}. \tag{3.3}$$

The variational inequality (3.3) motivates the following definition. For each $u \in \mathcal{U}$, define

$$\sigma_u := s(\cdot, y_u) + \beta(\cdot)p_u.$$

The mapping $\mathcal{Q} : \mathcal{U} \to L^\infty(\Omega)$ given by $\mathcal{Q}(u) := \sigma_u$ is called the switching mapping. Given $v \in L^2(\Omega)$, we define the linearization

$$\pi_v := H_{uy}(\cdot, y_{\bar{u}}, p_{\bar{u}})z_v + H_{up}(\cdot, y_{\bar{u}}, p_{\bar{u}})q_v.$$

This definition is justified by the following estimate.

Proposition 3.4. *For each $r > n/2$ there exists $c > 0$ such that*

$$|\sigma_u - \sigma_{\bar{u}} - \pi_{u-\bar{u}}|_{L^\infty(\Omega)} \le c|u - \bar{u}|^2_{L^r(\Omega)} \qquad \forall u \in \mathcal{U}.$$

Proof. Given $u \in \mathcal{U}$, we define $\psi_u : \Omega \to \mathbb{R}^3$ by $\psi_u(x) := (x, y_u(x), p_u(x))$. For each $u \in \mathcal{U}$, we denote

$$\tilde{\pi}_{u-\bar{u}} := H_{uy}(\psi_{\bar{u}})(y_u - y_{\bar{u}}) + H_{up}(\psi_{\bar{u}})(p_u - p_{\bar{u}}).$$

Let $u \in \mathcal{U}$ and $r > n/2$ be arbitrary, and abbreviate $v = u - \bar{u}$. Using the Taylor Theorem and (*ii*)-(*iii*) of Assumption 1, one can find $\alpha_1, \alpha_2 \in L^\infty(\Omega)$ such that

$$\begin{aligned} H_u(\psi_u) =& H_u(\psi_{\bar{u}}) + H_{uy}(\psi_{\bar{u}})(y_u - y_{\bar{u}}) + H_{up}(\psi_{\bar{u}})(p_u - p_{\bar{u}}) \\ &+ \alpha_1(\cdot)(y_u - y_{\bar{u}})^2 + \alpha_2(\cdot)(y_u - y_{\bar{u}})(p_u - p_{\bar{u}}). \end{aligned}$$

Therefore, by Proposition 2.7 and 2.10, there exists $c_1 > 0$ such that

$$|\sigma_u - \sigma_{\bar{u}} - \tilde{\pi}_v|_{L^\infty(\Omega)} \le c_1 |v|^2_{L^r(\Omega)}.$$

Now,

$$|\tilde{\pi}_v - \pi_v|_{L^\infty(\Omega)} \le |H_{uy}(\cdot, y_{\bar{u}}, p_{\bar{u}})(y_u - y_{\bar{u}} - z_v) + H_{up}(\cdot, y_{\bar{u}}, p_{\bar{u}})(q_u - q_{\bar{u}} - q_v)|_{L^\infty(\Omega)}.$$

Hence, by Proposition 3.1 and 3.2, there exists $c_2 > 0$ such that

$$|\tilde{\pi}_v - \pi_v|_{L^\infty(\Omega)} \le c_2 |v|^2_{L^r(\Omega)}.$$

Finally, by the triangle inequality,

$$|\sigma_u - \sigma_{\bar{u}} - \pi_v|_{L^\infty(\Omega)} \le |\sigma_u - \sigma_{\bar{u}} - \tilde{\pi}_v|_{L^\infty(\Omega)} + |\tilde{\pi}_v - \pi_v|_{L^\infty(\Omega)}.$$

The result follows defining $c := c_1 + c_2$. □

Proposition 3.4 yields immediately that the Gâteaux differential of the switching mapping $\mathcal{Q}$ at $\bar{u}$ in any direction $v \in \mathcal{U} - \bar{u}$ exists and is given by $d\mathcal{Q}(\bar{u}; v) = z_v$.

One of the important features of the mapping $v \to \pi_v$ is the following.

Proposition 3.5. *For all $v \in L^2(\Omega)$, we have*

$$\int_\Omega \pi_v v \, dx = \int_\Omega \Big[H_{yy}(x, y_{\bar{u}}, p_{\bar{u}}, \bar{u}) z_v^2 + 2H_{uy}(x, y_{\bar{u}}, p_{\bar{u}}, \bar{u}) z_v v \Big] \, dx.$$

Proof. In order to simplify notation, we write $\psi_{\bar{u}}(x) := (x, y_{\bar{u}}(x), p_{\bar{u}}(x), \bar{u}(x))$ for each $x \in \Omega$. Let $v \in L^2(\Omega)$ be arbitrary. By the integration by parts formula (2.2), we get

$$\begin{aligned} \int_\Omega H_{up}(\psi_{\bar{u}}) q_v v \, dx &= \int_\Omega \big(\mathcal{L} z_v + d_y(x, y_{\bar{u}}) z_v\big) q_v \, dx = \int_\Omega \big(\mathcal{L} q_v + d_y(x, y_{\bar{u}}) q_v\big) z_v \, dx \\ &= \int_\Omega \big(H_{yy}(\psi_{\bar{u}}) z_v + H_{uy}(\psi_{\bar{u}}) v\big) z_v = \int_\Omega \Big[H_{yy}(\psi_{\bar{u}}) z_v^2 + H_{uy}(\psi_{\bar{u}}) z_v v \Big] \, dx. \end{aligned}$$

The result follows since

$$\int_\Omega \pi_v v \, dx = \int_\Omega H_{uy}(\psi_{\bar{u}}) z_v v \, dx + \int_\Omega H_{up}(\psi_{\bar{u}}) q_v v \, dx.$$

□

We give further properties of the mapping $v \to \pi_v$ in the next proposition, its proof follows trivially from Proposition 3.3.

Proposition 3.6. *The following statements hold.*

(i) For each $r > n/2$ there exists a positive number c such that

$$|\pi_v|_{L^\infty(\Omega)} \le c|v|_{L^r(\Omega)} \quad \forall v \in L^2(\Omega) \cap L^r(\Omega).$$

(ii) There exists a positive number c such that

$$|\pi_v|_{L^2(\Omega)} \le c|v|_{L^1(\Omega)} \quad \forall v \in L^2(\Omega).$$

(iii) Let $\{v_k\}_{k=1}^\infty$ be a sequence in $L^2(\Omega)$ and $v \in L^2(\Omega)$. If $v_k \rightharpoonup v$ weakly in $L^2(\Omega)$, then $\pi_{v_k} \to \pi_v$ in $L^\infty(\Omega)$.

Proposition 3.5 motivates the following definition. For each $v \in L^2(\Omega)$, define

$$\Lambda(v) := \int_\Omega \Big[H_{yy}(x, y_{\bar u}, p_{\bar u}, \bar u) z_v^2 + 2H_{uy}(x, y_{\bar u}, p_{\bar u}, \bar u) z_v v \Big] \, dx. \tag{3.4}$$

Remark 3.7. We mention that the quadratic form $\Lambda : L^2(\Omega) \to \mathbb{R}$ is the second variation of the objective functional $\mathcal{J} : \mathcal{U} \to \mathbb{R}$ at $\bar u$. By Proposition 3.5, we also have the following representation

$$\Lambda(v) = \int_\Omega \pi_v v \, dx \quad \forall v \in L^2(\Omega).$$

We close this section with a result concerning the quadratic form (3.4).

Proposition 3.8. *Let $\{v_k\}_{k=1}^\infty \subset L^2(\Omega)$ and $v \in L^2(\Omega)$. If $v_k \rightharpoonup v$ weakly in $L^2(\Omega)$, then $\Lambda(v_k) \to \Lambda(v)$.*

Proof. By Proposition 3.6, $\pi_{v_k} \to \pi_v$ in $L^\infty(\Omega)$, therefore

$$\Lambda(v_k) = \int_\Omega (\pi_{v_k} - \pi_v) v_k \, dx + \int_\Omega \pi_v v_k \, dx \to \int_\Omega \pi_v v \, dx.$$

□

4 Stability

In this section, we study the stability of the optimal solution of problem (1.1)–(1.2) with respect to perturbations. As usual in optimization, the stability of the solution is derived from stability of the system of necessary optimality conditions. The investigated stability property of the latter is the so-called strong metric Hölder subregularity (SMHSr), see e.g., [16, Section 3I] or [11, Section 4]. After introducing the assumptions we study the SMHSr property of the variational inequality (9). Then the result is used to obtain this property for the whole system of necessary optimality conditions

4.1 The main assumption

We begin the section recalling that $\bar u \in \mathcal{U}$ is a local minimizer of problem (1.1)–(1.2), and the definition of the quadratic form $\Lambda : L^2(\Omega) \to \mathbb{R}$ in (3.4).

Assumption 2. *There exist positive numbers α_0, γ_0 and $k^* \in [1, 4/n)$ such that*

$$\int_\Omega \sigma_{\bar u}(u - \bar u) \, dx + \Lambda(u - \bar u) \ge \gamma_0 |u - \bar u|_{L^1(\Omega)}^{k^*+1}, \tag{4.1}$$

for all $u \in \mathcal{U}$ with $|u - \bar u|_{L^1(\Omega)} \le \alpha_0$.

Assumption 2 resembles the well-known L^2-coercivity condition in optimal control, with two substantial differences: (i) the left-hand side of (4.1) involves a linear term (not only the quadratic form in the L^2-coercivity condition); (ii) the L^1-norm appears in the right-hand side of (4.1). Assumption 2 in the particular case $k^* = 1$ has been used before in the literature on optimal control problems constrained by ordinary differential equations, see [28, Assumption A2'] or [29, Assumption A2]. A similar assumption was used in [14, Assumption 2]. We first point out that if $\bar u$ satisfies Assumption 2, then it must be bang-bang. A control $u \in \mathcal{U}$ is bang-bang if $u(x) \in \{b_1(x), b_2(x)\}$ for a.e. x in Ω. The proof of this result follows the arguments given in the proof of [10, Theorem 2.1].

Proposition 4.1. *If $\bar{u} \in \mathcal{U}$ satisfies Assumption 2, then $\bar{u}$ is bang-bang.*

Proof. Let α_0 and γ_0 be the positive numbers in Assumption 2. Suppose that there exists $\varepsilon > 0$ and a measurable set $E \subset \Omega$ of positive measure such that

$$\bar{u}(x) \in [b_1(x) + \varepsilon, b_2(x) - \varepsilon] \quad \text{for a.e. } x \in E.$$

Define $\varepsilon^* := \min\{\alpha_0(\operatorname{meas} E)^{-1}, \varepsilon\}$. Let $\{v_m\}_{m=1}^{\infty} \subset L^2(\Omega)$ be a sequence converging to zero weakly in $L^2(\Omega)$ such that for each $m \in \mathbb{N}$, $v_m(x) \in \{-\varepsilon^*, \varepsilon^*\}$ for a.e. $x \in \Omega$. For each $m \in \mathbb{N}$, define

$$u_m(x) := \begin{cases} \bar{u}(x) & if \quad x \notin E \\ \\ \bar{u}(x) + v_m(x) & if \quad x \in E. \end{cases}$$

Clearly, for each $m \in \mathbb{N}$, u_m belongs to $\mathcal{U}$ and

$$|u_m - \bar{u}|_{L^1(\Omega)} = \varepsilon^* \operatorname{meas} E.$$

Hence, by Assumption 2

$$\int_\Omega \sigma_{\bar{u}}(u_m - \bar{u})\, dx + \Lambda(u_m - \bar{u}) \geq \gamma_0 \Big(\varepsilon^* \operatorname{meas} E\Big)^{k^*+1} \tag{4.2}$$

for all $m \in \mathbb{N}$. Since $u_m \rightharpoonup \bar{u}$ weakly in $L^2(\Omega)$, we have by Proposition 3.8 that the left hand side of (4.2) converges to 0; a contradiction. □

Proposition 4.1 makes the following lemma relevant.

Lemma 4.2. *Let $u \in \mathcal{U}$ be bang-bang, and $\{u_k\}_{k=1}^{\infty} \subset \mathcal{U}$ be a sequence. If $u_k \rightharpoonup u$ weakly in $L^1(\Omega)$, then $u_k \to u$ in $L^1(\Omega)$.*

Proof. Let $\Omega_i := \{x \in \Omega : u(x) = b_i(x)\}$, $i = 1, 2$. Let $\chi_{\Omega_i} : \Omega \to \{0, 1\}$ denote the characteristic function of the set Ω_i, $i = 1, 2$. Now, by definition of weak convergence

$$\int_\Omega |u_k - u|\, dx = \int_\Omega \chi_{\Omega_1}(u_n - \bar{u})\, dx - \int_\Omega \chi_{\Omega_2}(u_n - \bar{u})\, dx \to 0.$$

□

The next proposition shows that the switching mapping satisfies a growth condition. The proof consists of two steps. The first one is to show that Assumption 2 implies this growth condition for the linearization of the switching mapping. The second step is to adequately use the linearization as an approximation of the switching mapping.

Proposition 4.3. *Let Assumption 2 be fulfilled. Then there exist positive numbers α and γ such that*

$$\int_\Omega \sigma_u(u - \bar{u})\, dx \geq \gamma |u - \bar{u}|_{L^1(\Omega)}^{k^*+1}$$

for all $u \in \mathcal{U}$ with $|u - \bar{u}|_{L^1(\Omega)} \leq \alpha$.

Proof. Let α_0, γ_0 and k^* be the positive numbers in Assumption 2. Fix $r \in (n/2, 2/k^*)$. Using Proposition 3.4, a constant $c > 0$ can be found such that

$$|\sigma_u - \sigma_{\bar{u}} - \pi_{u-\bar{u}}|_{L^\infty(\Omega)} \leq c|u - \bar{u}|_{L^1(\Omega)}^{2/r} \qquad \forall u \in \mathcal{U}. \tag{4.3}$$

From Proposition 3.5 and Assumption 2, we have

$$\int_\Omega \Big[\sigma_{\bar{u}} + \pi_{u-\bar{u}}\Big](u - \bar{u})\, dx = \int_\Omega \sigma_{\bar{u}}(u - \bar{u})\, dx + \Lambda(u - \bar{u}) \geq \gamma_0 |u - \bar{u}|_{L^1(\Omega)}^{k^*+1} \tag{4.4}$$

for all $u \in \mathcal{U}$ with $|u-\bar{u}|_{L^1(\Omega)} \leq \alpha_0$. Define $\gamma := \gamma_0/2$ and

$$\alpha := \min\left\{\alpha_0, \gamma^{\frac{r}{2-k^*r}} c^{-\frac{r}{2-k^*r}}\right\}.$$

Then, by (4.3)

$$|\sigma_u - \sigma_{\bar{u}} - \pi_{u-\bar{u}}|_{L^\infty(\Omega)} \leq c|u-\bar{u}|_{L^1(\Omega)}^{\frac{2}{r}} = c|u-\bar{u}|_{L^1(\Omega)}^{\frac{2}{r}-k^*}|u-\bar{u}|_{L^1(\Omega)}^{k^*} \leq \gamma|u-\bar{u}|_{L^1(\Omega)}^{k^*} \tag{4.5}$$

for all $u \in \mathcal{U}$ with $|u-\bar{u}|_{L^1(\Omega)} \leq \alpha$. We have for all $u \in \mathcal{U}$

$$\int_\Omega \sigma_u(u-\bar{u})\,dx = \int_\Omega \Big[\sigma_{\bar{u}} + \pi_{u-\bar{u}}\Big](u-\bar{u})\,dx + \int_\Omega \Big[\sigma_u - \sigma_{\bar{u}} - \pi_{u-\bar{u}}\Big](u-\bar{u})\,dx.$$

Consequently, by (4.4) and (4.5),

$$\begin{aligned}\int_\Omega \sigma_u(u-\bar{u})\,dx &\geq \gamma_0|u-\bar{u}|_{L^1(\Omega)}^{k^*+1} - |\sigma_u - \sigma_{\bar{u}} - \pi_{u-\bar{u}}|_{L^\infty(\Omega)}|u-\bar{u}|_{L^1(\Omega)} \\ &= (\gamma_0-\gamma)|u-\bar{u}|_{L^1(\Omega)}^{k^*+1} = \gamma|u-\bar{u}|_{L^1(\Omega)}^{k^*+1}\end{aligned}$$

for all $u \in \mathcal{U}$ with $|u-\bar{u}|_{L^1(\Omega)} \leq \alpha$. □

4.2 Some existence and stability results

We now pass to some preparatory lemmas concerning the existence of solutions of inclusions (also called generalized equations, see [36]) related to the first order necessary condition of problem (1.1)–(1.2). Given $r \in [1,\infty]$, we denote by $\mathbb{B}_{L^r}(c;\alpha)$ the closed ball in $L^r(\Omega)$ with center $c \in L^r(\Omega)$ and radius $\alpha > 0$.

The variational inequality (3.3) can be written as the inclusion

$$0 \in \sigma_u + N_{\mathcal{U}}(u),$$

where the normal cone at u to the set $\mathcal{U}$ is given by

$$N_{\mathcal{U}}(u) = \left\{\sigma \in L^\infty(\Omega) : \int_\Omega \sigma(w-u)\,dx \leq 0 \quad \forall w \in \mathcal{U}\right\}$$

Lemma 4.4. *For all $\rho \in L^\infty(\Omega)$ and $\varepsilon > 0$ there exists $u \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon)$ satisfying*

$$\rho \in \sigma_u + N_{\mathcal{U}\cap\mathbb{B}_{L^1}(\bar{u};\varepsilon)}(u).$$

Proof. Let $\rho \in L^\infty(\Omega)$ and $\varepsilon > 0$. Consider the functional $\mathcal{J}_\rho : \mathcal{U} \to \mathbb{R}$

$$\mathcal{J}_\rho(u) := \int_\Omega \big[g(y_u,u) - \rho u\big]\,dx = \mathcal{J}(u) - \int_\Omega \rho u\,dx.$$

The functional $\mathcal{J}_\rho$ has at least one global minimizer $u_\rho \in \mathcal{U}$ since $\mathcal{U} \cap \mathbb{B}_{L_1}(\bar{u};\varepsilon)$ is a weakly sequentially compact subset of $L^2(\Omega)$ and $\mathcal{J}_\rho$ is weakly sequentially continuous. By the Pontryagin principle,

$$\int_\Omega \big[\sigma_{u_\rho} - \rho\big](u-u_\rho)\,dx \geq 0 \quad \forall u \in \mathcal{U}.$$

We have then that u_ρ satisfies $\rho \in \sigma_{u_\rho} + N_{\mathcal{U}\cap\mathbb{B}_{L^1}(\bar{u};\varepsilon)}(u_\rho)$. □

Lemma 4.5. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be closed and convex subsets of $L^1(\Omega)$ such that $\mathcal{V}_1 \cap int\,\mathcal{V}_2 \neq \emptyset$. Then*

$$N_{\mathcal{V}_1\cap\mathcal{V}_2}(u) = N_{\mathcal{V}_1}(u) + N_{\mathcal{V}_2}(u) \tag{4.6}$$

for all $u \in \mathcal{V}_1 \cap \mathcal{V}_2$.

Proof. Given a set $\mathcal{W} \subset L^1(\Omega)$, let $s_{\mathcal{W}} : L^\infty(\Omega) \to \mathbb{R} \cup \{\infty\}$ denote the support function to $\mathcal{W}$, that is

$$s_{\mathcal{W}}(h) := \sup_{w \in \mathcal{W}} \int_\Omega hw \, dx.$$

By [3, Proposition 3.1], the set $\operatorname{Epi} s_{\mathcal{V}_1} + \operatorname{Epi} s_{\mathcal{V}_2}$ is weak* closed in $L^\infty(\Omega)$. Then the representation (4.6) holds according to [3, Theorem 3.1]. □

We can now prove existence of solutions of the inclusion $\rho \in \sigma_u + N_{\mathcal{U}}(u)$ that are close (in the L^1-norm) to $\bar{u}$ whenever ρ is close to zero (in the norm L^∞-norm). The proof follows the arguments in [13, p. 1127].

Lemma 4.6. *Let Assumption 2 hold. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\rho \in \mathbb{B}_{L^\infty}(0;\delta)$ there exists $u \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon)$ satisfying $\rho \in \sigma_u + N_{\mathcal{U}}(u)$.*

Proof. Let α and γ be the numbers in Proposition 4.3. Define $\varepsilon_0 := \min\{\varepsilon, \alpha\}$ and $\delta := \varepsilon_0^{k^*}\gamma/2$. Let $\rho \in L^\infty(\Omega)$ with $|\rho|_{L^\infty(\Omega)} \leq \delta$. By Lemma 4.4, there exists $u \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon_0)$ such that

$$\rho \in \sigma_u + N_{\mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u).$$

Since trivially $\bar{u} \in \mathcal{U} \cap \operatorname{int} \mathbb{B}_{L^1}(\bar{u}, \varepsilon_0)$, by Lemma 4.5 we have

$$N_{\mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u) = N_{\mathcal{U}}(u) + N_{\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u). \tag{4.7}$$

Thus there exists $\nu \in N_{\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u)$ such that

$$\rho - \sigma_u - \nu \in N_{\mathcal{U}}(u).$$

By definition of the normal cone,

$$0 \geq \int_\Omega (\rho - \sigma_u)(\bar{u} - u) \, dx - \int_\Omega \nu(\bar{u} - u) \, dx. \tag{4.8}$$

As $\bar{u} \in \mathbb{B}_{L^1}(\bar{u};\varepsilon_0)$ and $\nu \in N_{\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u)$, we have

$$\int_\Omega \nu(\bar{u} - u) \, dx \leq 0.$$

Consequently, by (4.8) and Proposition 4.3

$$0 \geq \int_\Omega (\rho - \sigma_u)(\bar{u} - u) \, dx \geq -|\rho|_{L^\infty(\Omega)} |u - \bar{u}|_{L^1(\Omega)} + \gamma |u - \bar{u}|_{L^1(\Omega)}^{k^*+1},$$

which implies

$$|u - \bar{u}|_{L^1(\Omega)} \leq \gamma^{-\frac{1}{k^*}} |\rho|_{L^\infty(\Omega)}^{\frac{1}{k^*}} \leq 2^{-\frac{1}{k^*}} \varepsilon_0 < \varepsilon_0.$$

As $u \in \operatorname{int} \mathbb{B}_{L^1}(\bar{u};\varepsilon_0)$, we have $N_{\mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u) = \{0\}$. Thus by (4.7),

$$\rho \in \sigma_u + N_{\mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u};\varepsilon_0)}(u) = \sigma_u + N_{\mathcal{U}}(u). \tag{4.9}$$

□

The following lemma shows how Proposition 4.3 (and consequently Assumption 2) is related to Hölder-stability.

Lemma 4.7. *Let Assumption 2 hold. There exist positive numbers α and c such that*

$$|u - \bar{u}|_{L^1(\Omega)} \leq c |\rho|_{L^\infty(\Omega)}^{\frac{1}{k^*}} \tag{4.10}$$

for all $\rho \in L^\infty(\Omega)$ and $u \in \mathbb{B}_{L^1}(\bar{u};\alpha)$ satisfying $\rho \in \sigma_u + N_{\mathcal{U}}(u)$.

Proof. Let α and γ be the positive numbers in Proposition 4.3. Since $\rho - \sigma_u \in N_{\mathcal{U}}(u)$, we have

$$\int_\Omega (\rho - \sigma_u)(\bar{u} - u)\, dx \le 0.$$

By Proposition 4.3,

$$\begin{aligned} 0 \ge \int_\Omega (\rho - \sigma_u)(\bar{u} - u)\, dx &= \int_\Omega \sigma_u(u - \bar{u})\, dx + \int_\Omega \rho(\bar{u} - u)\, dx \\ &\ge \gamma \left(\int_\Omega |u - \bar{u}|\, dx \right)^{k^*+1} - |\rho|_{L^\infty(\Omega)} \int_\Omega |u - \bar{u}|\, dx. \end{aligned}$$

Hence

$$\int_\Omega |u - \bar{u}|\, dx \le \left(\frac{1}{\gamma} |\rho|_{L^\infty(\Omega)} \right)^{1/k^*} = \gamma^{-\frac{1}{k^*}} |\rho|_{L^\infty(\Omega)}^{\frac{1}{k^*}}.$$

The result follows defining $c = \gamma^{-\frac{1}{k^*}}$. □

Lemma 4.7 requires that the controls are close (in the L^1-norm) a priori for the inequality (4.10) to hold. This assumption can be removed if the solution of the inclusion $0 \in \sigma_u + N_{\mathcal{U}}(u)$ is unique.

Lemma 4.8. *Let Assumption 2 hold, and suppose additionally that $\bar{u} \in \mathcal{U}$ is the unique solution of $0 \in \sigma_u + N_{\mathcal{U}}(u)$. There exist positive numbers δ and c such that*

$$|u - \bar{u}|_{L^1(\Omega)} \le c |\rho|_{L^\infty(\Omega)}^{\frac{1}{k^*}}.$$

for all $\rho \in \mathbb{B}_{L^\infty}(0;\delta)$ and $u \in \mathcal{U}$ satisfying $\rho \in \sigma_u + N_{\mathcal{U}}(u)$.

Proof. Let α and c be the positive numbers in Lemma 4.7. First we prove that there exists $\delta > 0$ such that if $u \in \mathcal{U}$ and $\rho \in L^\infty(\Omega)$ satisfy $\rho \in \sigma_u + N_{\mathcal{U}}(u)$ and $|\rho|_{L^\infty(\Omega)} \le \delta$, then $u \in \mathbb{B}_{L^1}(\bar{u};\alpha)$. Suppose not, then there exist sequences $\{\rho_k\}_{k=1}^\infty \subset L^\infty(\Omega)$ and $\{u_k\}_{k=1}^\infty \subset \mathcal{U}$ such that $\rho_k \in \sigma_{u_k} + N_{\mathcal{U}}(u_k)$ and $|u_k - \bar{u}|_{L^1(\Omega)} > \alpha$. Since $\mathcal{U}$ is weakly sequentially compact in $L^2(\Omega)$, there exists a subsequence of $\{u_k\}_{k=1}^\infty$, denoted in the same way, and $u^* \in \mathcal{U}$ such that $u_k \rightharpoonup u^*$ weakly in $L^2(\Omega)$. Using Proposition 2.12, one can see that $\rho_k - \sigma_{u_k} \to \sigma_{u^*}$ in $L^\infty(\Omega)$. Consequently, as $\rho_k \in \sigma_{u_k} + N_{\mathcal{U}}(u_k)$ for all $n \in \mathbb{N}$, we obtain $0 \in \sigma_{u^*} + N_{\mathcal{U}}(u^*)$. Then, by assumption, $u^* = \bar{u}$, so u^* is bang-bang. By Lemma 4.2, we have $u_k \to u^*$ in $L^1(\Omega)$; a contradiction. The result follows from Lemma 4.7. □

4.3 Strong metric subregularity

Let us begin considering the following system representing the necessary optimality conditions (Pontryagin principle) for problem (1.1)–(1.2):

$$\left\{ \begin{array}{lcl} 0 &=& \mathcal{L}y - f(\cdot, y, u), \\ 0 &=& \mathcal{L}p - H_y(\cdot, y, p, u), \\ 0 &\in& H_u(\cdot, y, p) + N_{\mathcal{U}}(u), \end{array} \right. \tag{4.11}$$

If $u \in \mathcal{U}$ is a local solution of problem (1.1)–(1.2), then the triple (y_u, p_u, u) is a solution of (4.11). Therefore, the mapping that defines the right-hand side is referred to as the *optimality mapping*. In order to give a strict definition and recast system (4.11) in a functional frame, we introduce the metric spaces

$$\mathcal{Y} := D(\mathcal{L}) \times D(\mathcal{L}) \times \mathcal{U} \quad \text{and} \quad \mathcal{Z} := L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega),$$

endowed with the following metrics. For $\psi_i = (y_i, p_i, u_i) \in \mathcal{Y}$ and $\zeta_i = (\xi_i, \eta_i, \rho_i) \in \mathcal{Z}$, $i \in \{1, 2\}$,

$$\begin{aligned} d_{\mathcal{Y}}(\psi_1, \psi_2) &:= |y_1 - y_2|_{L^2(\Omega)} + |p_1 - p_2|_{L^2(\Omega)} + |u_1 - u_2|_{L^1(\Omega)}, \\ d_{\mathcal{Z}}(\zeta_1, \zeta_2) &:= |\xi_1 - \xi_2|_{L^2(\Omega)} + |\eta_1 - \eta_2|_{L^2(\Omega)} + |\rho_1 - \rho_2|_{L^\infty(\Omega)}. \end{aligned}$$

Both metrics are shift-invariant. We denote by $\mathbb{B}_{\mathcal{Y}}(\psi;\alpha)$ the closed ball in $\mathcal{Y}$, centered at ψ and with radius α. The notation for the ball $\mathbb{B}_{\mathcal{Z}}(\zeta;\alpha)$ is identical. Then the optimality mapping is defined as the set-valued mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ given by

$$\Phi(y,p,u) = \begin{pmatrix} \mathcal{L}y - f(\cdot,y,u) \\ \mathcal{L}p - H_y(\cdot,y,p,u) \\ H_u(\cdot,y,p,u) + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{4.12}$$

Then the optimality system (4.11) can be recast as the inclusion

$$0 \in \Phi(y,p,u). \tag{4.13}$$

Our purpose is to study the stability of system (4.11), or equivalently of inclusion (4.13), with respect to perturbations in the right-hand side. From now on, we denote $\bar{\psi} := (\bar{y},\bar{p},\bar{u}) = (y_{\bar{u}}, p_{\bar{u}}, \bar{u})$, where $\bar{u}$ is the fixed local solution of problem (1.1)–(1.2).

Definition 4.9. The optimality mapping $\Phi : \mathcal{Y} \twoheadrightarrow \mathcal{Z}$ is called strongly Hölder subregular with exponent $\lambda > 0$ at $(\bar{\psi}, 0)$ if there exist positive numbers α_1, α_2 and κ such that

$$d_{\mathcal{Y}}(\psi,\bar{\psi}) \le \kappa d_{\mathcal{Z}}(\zeta, 0)^{\lambda} \tag{4.14}$$

for all $\psi \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi};\alpha_1)$ and $\zeta \in \mathbb{B}_{\mathcal{Z}}(0;\alpha_2)$ satisfying $\zeta \in \Phi(\psi)$.

More explicitly, the inequality (4.14) reads as

$$|y - y_{\bar{u}}|_{L^2(\Omega)} + |p - p_{\bar{u}}|_{L^2(\Omega)} + |u - \bar{u}|_{L^1(\Omega)} \le \kappa\Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)} + |\rho|_{L^\infty(\Omega)}\Big)^{\lambda}. \tag{4.15}$$

Hence, if the optimality mapping is strongly Hölder subregular, all solutions of the system

$$\left\{ \begin{array}{lcl} \xi & = & \mathcal{L}y - f(\cdot,y,u), \\ \eta & = & \mathcal{L}p - H_y(\cdot,y,p,u), \\ \rho & \in & H_u(\cdot,y,p) + N_{\mathcal{U}}(u). \end{array} \right. \tag{4.16}$$

that are near $(y_{\bar{u}}, p_{\bar{u}}, \bar{u})$ satisfy the Hölder estimate (4.15) with respect to the perturbations $\zeta = (\xi,\eta,\rho)$, provided they are small enough.

Remark 4.10. If Φ is strongly Hölder subregular at $(\bar{\psi}, 0)$, then from (4.14) applied with $\zeta = 0$ we obtain that $\bar{\psi}$ is the unique solution of (4.13) in $B_{\mathcal{Y}}(\bar{\psi};\alpha_1)$, hence $\bar{u}$ is the unique local solution of problem (1.1)–(1.2) in this ball. In particular, $\bar{u}$ is a strict local minimizer.

We are now ready to state our main result.

Theorem 4.11. *Let Assumption 2 hold. Then the optimality mapping Φ is strongly Hölder subregular at $(\bar{\psi}, 0)$ with exponent $\lambda = 1/k^*$.*

Proof. Let α and c be the positive numbers in Lemma 4.7. Let $\zeta = (\xi,\eta,\rho) \in B_{\mathcal{Z}}(0;1)$ and $\psi = (y,p,u) \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi};\alpha)$ such that $\zeta \in \Phi(\psi)$. By a standard argument, there exists $c_1 > 0$ (independent of ψ and ζ) such that

$$|y - y_u|_{L^\infty(\Omega)} + |p - p_u|_{L^\infty(\Omega)} \le c_1\Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)}\Big). \tag{4.17}$$

Since H_u is locally Lipschitz uniformly in the first variable, and the sets $\{y_u : u \in \mathcal{U}\}$, $\{p_u : u \in \mathcal{U}\}$ are bounded in $C(\bar{\Omega})$, there exists $c_2 > 0$ (independent of ψ) such that

$$|H_u(\cdot,y,p) - H_u(\cdot,y_u,p_u)|_{L^\infty(\Omega)} \le c_2\Big(|y - y_u|_{L^\infty(\Omega)} + |p - p_u|_{L^\infty(\Omega)}\Big) \tag{4.18}$$

Define $\nu := \rho + H_u(\cdot,y_u,p_u) - H_u(\cdot,y,p)$. By (4.17) and (4.18), there exists $c_3 > 0$ (independent of ψ and ζ) such that

$$|\nu|_{L^\infty(\Omega)} \le c_3\Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)} + |\rho|_{L^\infty(\Omega)}\Big) = c_3|\zeta|_{\mathcal{Z}}.$$

As $\rho \in H_u(\cdot, y, p) + N_{\mathcal{U}}(u)$, we have $\nu \in H_u(\cdot, y_u, p_u) + N_{\mathcal{U}}(u)$. Then by Lemma 4.7,

$$|u - \bar{u}|_{L^1(\Omega)} \le c|\nu|_{L^\infty(\Omega)}^{\frac{1}{k^*}} \le c c_3^{\frac{1}{k^*}} |\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}} := c_4 |\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}}. \tag{4.19}$$

Now, by Proposition 2.11, there exists $c_5 > 0$ (independent of ψ) such that $|y_u - y_{\bar{u}}|_{L^2(\Omega)} \le c_5 |u - \bar{u}|_{L^1(\Omega)}$. Consequently, by (4.19)

$$\begin{aligned}
|y - y_{\bar{u}}|_{L^2(\Omega)} &\le |y - y_u|_{L^2(\Omega)} + |y_u - y_{\bar{u}}|_{L^2(\Omega)} \\
&\le c_1 \text{meas}\, \Omega^{\frac{1}{2}} \Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)} \Big) + c_5 |u - \bar{u}|_{L^1(\Omega)} \\
&\le (c_1 \text{meas}\, \Omega^{\frac{1}{2}} + c_5 c_4) |\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}} =: c_6 |\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}}.
\end{aligned}$$

Analogously, there exists $c_7 > 0$ (independent of ψ and ζ) such that

$$|p - p_u|_{L^2(\Omega)} \le c_7 |\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}}.$$

Putting all together,

$$|y - y_{\bar{u}}|_{L^2(\Omega)} + |p - p_u|_{L^2(\Omega)} + |u - \bar{u}|_{L^1(\Omega)} \le (c_4 + c_6 + c_7) |\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}}.$$

Finally, let $\alpha_1 := \alpha$, $\alpha_2 := 1$ and $\kappa := c_4 + c_6 + c_7$. Since the constants c_4, c_6 and c_7 are independent of ψ and ζ, so is κ. Thus we have (4.14) for all $\psi \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi}; \alpha_1)$ and $\zeta \in \mathbb{B}_{\mathcal{Z}}(0; \alpha_2)$ satisfying $\zeta \in \Phi(\psi)$. □

The strong subregularity property defined above does not require existence of solutions of the perturbed inclusion (4.16) in a neighborhood of the reference solution $\bar{\psi}$. The next theorem answers the existence question.

Theorem 4.12. *Let Assumption 2 hold. For each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\zeta \in \mathbb{B}_{\mathcal{Z}}(0; \delta)$ there exists $\psi \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi}; \varepsilon)$ satisfying the inclusion $\zeta \in \Phi(\psi)$.*

Proof. For each $u \in \mathcal{U}$ and $\zeta = (\xi, \eta, \rho) \in \mathcal{Z}$, define $\nu_{u,\zeta} := \rho + H_u(\cdot, y_u, p_u) - H_u(\cdot, y_{u,\zeta}, p_{u,\zeta})$, where $y_{u,\zeta}$ and $p_{u,\zeta}$ are the unique solutions of

$$\left\{ \begin{array}{rcl} \mathcal{L}y &=& f(\cdot, y, u) + \xi, \\ \mathcal{L}p &=& H_y(\cdot, y, p, u) + \eta. \end{array} \right. \tag{4.20}$$

By a standard argument, one can find positive numbers c_1 and c_2 such that

$$|y_{u,\zeta} - y_u|_{L^2(\Omega)} + |p_{u,\zeta} - p_u|_{L^2(\Omega)} \le c_1 \Big(|\xi|_{L^2(\Omega)} + |\eta|_{L^2(\Omega)} \Big), \tag{4.21}$$

and $|\nu_{u,\zeta}|_{L^\infty(\Omega)} \le c_2 |\zeta|_{\mathcal{Z}}$ for all $u \in \mathcal{U}$ and $\zeta \in \mathcal{Z}$. Let $\varepsilon > 0$ be arbitrary. By Lemma 4.6, the exists $\delta_0 > 0$ such that for each $\nu \in \mathbb{B}_{L^\infty}(0; \delta_0)$ there exists $u \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \varepsilon/2)$ satisfying $\nu \in \sigma_u + N_{\mathcal{U}}(u)$. Define $\delta := \min\{c_2^{-1}\delta_0, (2c_1)^{-1}\varepsilon\}$ and let $\zeta^* \in \mathbb{B}_{\mathcal{Z}}(0; \delta)$ be arbitrary; we will prove that there exists $u^* \in \mathcal{U} \cap \mathbb{B}_{L^1}(\bar{u}; \varepsilon/2)$ such that $\nu_{u^*,\zeta^*} \in \sigma_{u^*} + N_{\mathcal{U}}(u^*)$. First, observe that

$$|\nu_{u,\zeta^*}|_{L^\infty(\Omega)} \le c_2 |\zeta^*|_{\mathcal{Z}} \le \delta_0 \quad \forall u \in \mathcal{U}.$$

Therefore, by Lemma 4.6, we can inductively define a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{U}$ such that $\nu_{u_k,\zeta^*} \in \sigma_{u_{k+1}} + N_{\mathcal{U}}(u_{k+1})$ and $|u_k - \bar{u}|_{L^1(\Omega)} \le \varepsilon/2$ for all $k \in \mathbb{N}$. Since $\mathcal{U}$ is weakly compact in $L^2(\Omega)$, we may assume that $u_k \rightharpoonup u^*$ weakly in $L^2(\Omega)$ for some $u^* \in \mathcal{U}$. Weak convergence in $L^2(\Omega)$ implies weak convergence in $L^1(\Omega)$ and $\mathbb{B}_{L^1}(\bar{u}; \varepsilon/2)$ is weakly sequentially closed in $L^1(\Omega)$, therefore $u^* \in \mathbb{B}_{L^1}(\bar{u}; \varepsilon/2)$. Using Proposition 2.12, one can see that $\nu_{u_k,\zeta^*} - \sigma_{u_{k+1}} \to \nu_{u^*,\zeta^*} - \sigma_{u^*}$ in $L^\infty(\Omega)$, and consequently that $\nu_{u^*,\zeta^*} \in \sigma_{u^*} + N_{\mathcal{U}}(u^*)$. We conclude then that $\zeta^* \in \Phi(\psi^*)$, where $\psi^* := (y_{u^*,\zeta^*}, p_{u^*,\zeta^*}, u^*)$. Finally, by definition of δ and (4.21)

$$|\psi^* - \bar{\psi}|_{\mathcal{Y}} \le c_1 |\zeta|_{\mathcal{Z}} + \varepsilon/2 \le \varepsilon.$$

Thus, $\zeta^* \in \Phi(\psi^*)$ and $\psi^* \in \mathbb{B}_{\mathcal{Y}}(\bar{\psi}; \varepsilon)$, which completes the proof. □

The next theorem claims that *all* solutions of the perturbed optimality system (4.16) are arbitrarily close to the solution of the unperturbed optimality system, provided that the solution of the latter is globally unique, Assumption 2 holds, and the perturbation is sufficiently small.

Theorem 4.13. *Let Assumption 2 hold and suppose additionally that $\bar\psi$ is the unique element of $\mathcal{Y}$ that satisfies $0 \in \Phi(\bar\psi)$. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\zeta \in \mathbb{B}_{\mathcal{Z}}(0;\delta)$ and $\psi \in \mathcal{Y}$ satisfy $\zeta \in \Phi(\psi)$, then $\psi \in \mathbb{B}_{\mathcal{Y}}(\bar\psi;\varepsilon)$.*

Proof. Let δ_0 and c_0 be the positive numbers in Lemma 4.8. Let $\zeta = (\xi,\eta,\rho) \in \mathcal{Z}$ and $\psi = (y,p,u) \in \mathcal{Y}$ be such that $\zeta \in \Phi(\psi)$. Define $\nu := \rho + H_u(\cdot,y_u,p_u) - H_u(\cdot,y,p)$. Arguing as in the proof of Theorem 4.11, we can find positive numbers c_1 and c_2 (independent of ψ and ζ) such that $|\nu|_{L^\infty(\Omega)} \le c_1|\zeta|_{\mathcal{Z}}$ and

$$|y - y_{\bar u}|_{L^2(\Omega)} + |p - p_{\bar u}|_{L^2(\Omega)} \le c_2\Big(|\zeta|_{\mathcal{Z}} + |u - \bar u|_{L^1(\Omega)}\Big).$$

Let $\delta := \min\{c_1^{-1}\delta_0, (2c_0c_2)^{-k^*}c_1^{-1}\varepsilon^{k^*}, (2c_2)^{-1}\varepsilon\}$ and suppose that $\zeta \in \mathbb{B}_{\mathcal{Z}}(0;\delta)$. As $\rho \in H_u(\cdot,y,p) + N_{\mathcal{U}}(u)$, we have $\nu \in H_u(\cdot,y_u,p_u) + N_{\mathcal{U}}(u)$. By Lemma 4.8,

$$|u - \bar u|_{L^1(\Omega)} \le c_0|\nu|_{L^\infty(\Omega)}^{\frac{1}{k^*}} \le c_0 c_1^{\frac{1}{k^*}}|\zeta|_{\mathcal{Z}}^{\frac{1}{k^*}} \le c_2^{-1}\varepsilon/2.$$

Thus,

$$|y - y_{\bar u}|_{L^2(\Omega)} + |p - p_{\bar u}|_{L^2(\Omega)} + |u - \bar u|_{L^1(\Omega)} \le c_2\Big(\delta + c_2^{-1}\varepsilon/2\Big) \le \varepsilon.$$

□

5 Nonlinear Perturbations

In this section we apply the subregularity results in Section 4 for studying the effect of certain nonlinear perturbations on the optimal solution. We consider the following family of problems

$$\min_{u\in\mathcal{U}} \left\{ \int_\Omega \Big[g(x,y,u) + \eta(x,y,u)\Big]\, dx \right\}, \tag{5.1}$$

subject to

$$\left\{\begin{array}{rcll} -\operatorname{div}\big(A(x)\nabla y\big) + d(x,y) + \xi(x,y) &=& \beta(x)u & \text{in} \quad \Omega \\ A(x)\nabla y\cdot\nu + b(x)y &=& 0 & \text{on} \quad \partial\Omega. \end{array}\right. \tag{5.2}$$

In order to specify the perturbations under consideration and their topology, we begin the section recalling some elementary notions of functional analysis.

As usual, $C(\mathbb{R}^s)$ denotes the space of all continuous functions $\omega : \mathbb{R}^s \to \mathbb{R}$. For each $m \in \mathbb{N}$, let K_m denote the closed ball in $\mathbb{R}^s$ centered at zero with radius m. Consider the metric on $C(\mathbb{R}^s)$ given by

$$d_C(\omega_1,\omega_2) := \sum_{m=1}^{\infty} \frac{1}{2^m}\frac{|\omega_1 - \omega_2|_{L^\infty(K_m)}}{1 + |\omega_1 - \omega_2|_{L^\infty(K_m)}}.$$

This metric induces the compact-convergence topology on $C(\mathbb{R}^s)$. In this topology, a sequence $\{\omega_m\}_{m=1}^\infty \subset C(\mathbb{R}^s)$ converges to $\omega \in C(\mathbb{R}^s)$ if and only if $|\omega - \omega_m|_{L^\infty(K)} \to 0$ for every compact set $K \subset \mathbb{R}^s$. This topology is also known as the compact-open topology, see [26, Chapter 7].

Lemma 5.1. *For each compact set $K \subset \mathbb{R}^s$ there exists $m \in \mathbb{N}$ such that*

$$|\omega_1 - \omega_2|_{L^\infty(K)} \le 2^m d_C(\omega_1,\omega_2)$$

for all $\omega_1,\omega_2 \in C(\mathbb{R}^s)$ such that $d_C(\omega_1,\omega_2) \le 2^{-m}$.

5.1 The perturbations

We begin describing the space of perturbations appearing in equation (5.2). Let Υ_s be the set of all continuously differentiable functions $\xi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $d_y(x, y) + \xi_y(x, y) \geq 0$ for all $x \in \Omega$ and $y \in \mathbb{R}$. The set Υ_s does not constitute a linear space, but it allows to have well-defined states for each perturbation.

Proposition 5.2. *For each $u \in \mathcal{U}$ and $\xi \in \Upsilon_s$ there exists a unique function $y_u^\xi \in D(\mathcal{L})$ satisfying*

$$\mathcal{L} y_u^\xi + d(\cdot, y_u^\xi) + \xi(\cdot, y_u^\xi) = \beta(\cdot) u.$$

Moreover, there exist positive numbers M and δ such that $|y_u^\xi|_{L^\infty(\Omega)} \leq M$ for all $u \in \mathcal{U}$ and $\xi \in \Upsilon_s$ with $d_C(\xi, 0) \leq \delta$.

Proof. The existence follows from [39, Theorem 4.8]. Moreover, also from this theorem, there exists $c > 0$ such that

$$|y_u^\xi|_{L^\infty(\Omega)} \leq c \big|\beta(\cdot) u - d(\cdot, 0) - \xi(\cdot, 0)\big|_{L^\infty(\Omega)}$$

for all $u \in \mathcal{U}$ and $\xi \in \Upsilon_s$. Let $K := \bar{\Omega} \times \{0\}$, then by Lemma 5.1 there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} |y_u^\xi|_{L^\infty(\Omega)} &\leq c\Big(|\beta|_{L^\infty(\Omega)} |u|_{L^\infty(\Omega)} + |d(\cdot, 0)|_{L^\infty(\Omega)} + |\xi|_{L^\infty(K)}\Big) \\ &\leq c\Big(|\beta|_{L^\infty(\Omega)} \sup_{u \in \mathcal{U}} |u|_{L^\infty(\Omega)} + |d(\cdot, 0)|_{L^\infty(\Omega)} + 2^m d_C(\xi, 0)\Big) \\ &\leq c\Big(|\beta|_{L^\infty(\Omega)} \sup_{u \in \mathcal{U}} |u|_{L^\infty(\Omega)} + |d(\cdot, 0)|_{L^\infty(\Omega)} + 1\Big) \end{aligned}$$

for all $u \in \mathcal{U}$ and $\xi \in \Upsilon_s$ with $d_C(\xi, 0) \leq 2^{-m}$. The result follows defining $\delta := 2^{-m}$ and

$$M := c\Big(|\beta|_{L^\infty(\Omega)} \sup_{u \in \mathcal{U}} |u|_{L^\infty(\Omega)} + |d(\cdot, 0)|_{L^\infty(\Omega)} + 1\Big).$$

□

We now proceed to describe the perturbations appearing in the cost functional (5.1). Consider the set Υ_c of all continuously differentiable functions $\eta : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\eta(x, y, \cdot)$ is convex for all $x \in \Omega$ and $y \in \mathbb{R}$. We have the following result concerning the adjoint variable of the perturbed problem. Its proof is similar to the one of Proposition 5.2.

Proposition 5.3. *For each $u \in \mathcal{U}$, $\xi \in \Upsilon_s$ and $\eta \in \Upsilon_c$ there exists a unique function $p_u^{\xi,\eta} \in D(\mathcal{L})$ satisfying*

$$\mathcal{L} p_u^{\xi,\eta} + \big[d_y(\cdot, y_u^\xi) + \xi_y(\cdot, y_u^\xi)\big] p_u^{\xi,\eta} = g_y(\cdot, y_u^\xi, u) + \eta_y(\cdot, y_u^\xi, u).$$

Moreover, there exist positive numbers M and δ such that $|p_u^{\xi,\eta}|_{L^\infty(\Omega)} \leq M$ for all $u \in \mathcal{U}$, $\xi \in \Upsilon_s$ and $\eta \in \Upsilon_c$ with $d_C(\xi, 0) + d_C(\xi_y, 0) + d_C(\eta_y, 0) \leq \delta$.

We denote $\Upsilon := \Upsilon_s \times \Upsilon_c$, and write $\zeta := (\xi, \eta)$ for a generic element of Υ. We endow Υ with the pseudometric $d_\Upsilon : \Upsilon \times \Upsilon \to [0, \infty)$ given by

$$d_\Upsilon(\zeta, \zeta') := d_C(\xi, \xi') + d_C(\xi_y, \xi'_y) + d_C(\eta_y, \eta'_y) + d_C(\eta_u, \eta'_u).$$

5.2 The stability result

We are now ready to state problem (5.1)-(5.2) in a precise way. Given $\zeta \in \Upsilon$, problem $\mathcal{P}_\zeta$ is given by

$$\min_{u \in \mathcal{U}} \left\{ \mathcal{J}_\zeta(u) := \int_\Omega \Big[g(x, y_u^\xi, u) + \eta(x, y_u^\xi, u)\Big]\, dx \right\}. \tag{5.3}$$

Due to the convexity of the cost in the control variable, each problem $\mathcal{P}_\zeta$ has at least one local solution. For each $\zeta \in \Upsilon$, we fix a local minimizer $\hat{u}_\zeta \in \mathcal{U}$ of problem $\mathcal{P}_\zeta$. By the Pontryagin principle, for each $\zeta = (\xi, \eta) \in \Upsilon$, the triple $(\hat{y}_\zeta, \hat{p}_\zeta, \hat{u}_\zeta) := (y_{\hat{u}_\zeta}^\xi, p_{\hat{u}_\zeta}^{\xi,\eta}, \hat{u}_\zeta)$ satisfies the system

$$\left\{ \begin{array}{lcl} 0 &=& \mathcal{L} y - f(\cdot, y, u) - \xi(\cdot, y), \\ 0 &=& \mathcal{L} p - H_y(\cdot, y, p, u) + \eta_y(\cdot, y, u) - \xi_y(\cdot, y) p, \\ 0 &\in& H_u(\cdot, y, p) + \eta_u(\cdot, y, u) + N_{\mathcal{U}}(u). \end{array} \right. \tag{5.4}$$

As a consequence of Theorem 4.11, we have the following result.

Theorem 5.4. *Let Assumption 2 hold. There exist positive numbers α, α' and c such that*

$$|\hat{y}_\zeta - y_{\bar{u}}|_{L^2(\Omega)} + |\hat{p}_\zeta - p_{\bar{u}}|_{L^2(\Omega)} + |\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \le c d_\Upsilon(\zeta, 0)^{1/k^*}$$

for all $\zeta \in \Upsilon$ such that $|\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \le \alpha$ and $d_\Upsilon(\zeta, 0) \le \alpha'$.

Proof. By Theorem 4.11, the mapping Φ is strongly Hölder subregular at $(\bar{\psi}, 0)$ with exponent $1/k^*$. Let α_1, α_2 and κ be the positive numbers in the definition of strong subregularity. By Proposition 5.2 and 5.3 there exist positive numbers M and δ_0 such that

$$|y_u^\xi|_{L^\infty(\Omega)} + |p_u^{\xi,\eta}|_{L^\infty(\Omega)} \le M$$

for all $u \in \mathcal{U}$ and $\zeta \in \Upsilon$ with $d_\Upsilon(\zeta, 0) \le \delta_0$. Let $K := \bar{\Omega} \times [-M, M]$. By Lemma 5.1, there exists $m \in \mathbb{N}$ such that

$$|\xi(\cdot, y_u^\xi)|_{L^2(\Omega)} \le \text{meas}\, \Omega^{\frac{1}{2}} |\xi|_{L^\infty(K)} \le 2^m \text{meas}\, \Omega^{\frac{1}{2}} d_C(\xi, 0) \le 2^m \text{meas}\, \Omega^{\frac{1}{2}} d_\Upsilon(\zeta, 0)$$

for all $u \in \mathcal{U}$ and $\zeta \in \Upsilon$ with $d_\Upsilon(\zeta, 0) \le \min\{2^{-m}, \delta_0\}$. Repeating this argument, we can find positive numbers δ and c_0 such that

$$|\xi(\cdot, y_u^\xi)|_{L^2(\Omega)} + |\xi_y(\cdot, y_u^\xi) p_u^{\xi,\eta}|_{L^2(\Omega)} + |\eta_y(\cdot, y_u^\xi, u)|_{L^2\Omega} + |\eta_u(\cdot, y_u^\xi, u)|_{L^\infty} \le c_0 d_\Upsilon(\zeta, 0) \tag{5.5}$$

for all $u \in \mathcal{U}$ and $\zeta \in \Upsilon$ with $d_\Upsilon(\zeta, 0) \le \delta$. Using Proposition 2.11 and Lemma 5.1, one can find positive numbers α and δ' such that

$$|\hat{y}_\zeta - y_{\bar{u}}|_{L^2(\Omega)} + |\hat{p}_\zeta - p_{\bar{u}}|_{L^2(\Omega)} + |\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \le \alpha_1$$

for all $\zeta \in \Upsilon$ with $|\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \le \alpha$ and $d_\Upsilon(\zeta, 0) \le \delta'$. Observe that by (5.4), we have

$$\begin{pmatrix} \xi(\cdot, \hat{y}_\zeta) \\ -\eta_y(\cdot, \hat{y}_\zeta, \hat{u}_\zeta) + \xi_y(\cdot, \hat{y}_\zeta)\hat{p}_\zeta \\ -\eta_u(\cdot, \hat{y}_\zeta, \hat{u}_\zeta) \end{pmatrix} \in \Phi(\hat{y}_\zeta, \hat{p}_\zeta, \hat{u}_\zeta)$$

for all $\zeta \in \Upsilon$. Let $\alpha' := \min\{c_0^{-1}\alpha_2, \delta, \delta'\}$. Then by Hölder subregularity of Φ and (5.5),

$$|\hat{y}_\zeta - y_{\bar{u}}|_{L^2(\Omega)} + |\hat{p}_\zeta - p_{\bar{u}}|_{L^2(\Omega)} + |u_\zeta - \bar{u}|_{L^1(\Omega)} \le \kappa c_0^{\frac{1}{k^*}} d_\Upsilon(\zeta, 0)^{\frac{1}{k^*}}$$

for all $\zeta \in \Upsilon$ such that $|\hat{u}_\zeta - \bar{u}|_{L^1(\Omega)} \le \alpha$ and $d_\Upsilon(\zeta, 0) \le \alpha'$. The result follows defining $c := \kappa c_0^{\frac{1}{k^*}}$. □

5.3 An application: Tikhonov regularization

In what follows we present an application of the theory derived in the previous chapters, namely the so-called Tikhonov regularization. For a more detailed description and an account of the state of art, the reader is referred to [32, 41, 40]. We derive estimates on the convergence rate of the solution of the regularized problem when the regularization parameter tends to zero. The results that appear in the literature require the so-called structural assumption and positive-definiteness (in some sense) of the second derivative of the objective functional. Using Theorem 4.11, we can obtain this results under weaker assumptions than used in the literature so far. One can compare this results with [32, Theorem 4.4] (where a tracking problem with semilinear elliptic equation is considered) when it comes to stability of the controls. In Section 6, we give more details on how the assumptions in the literature interplay with Assumption 2.

We consider the following family of problems $\{\mathcal{P}_\varepsilon\}_{\varepsilon \ge 0}$.

$$\min_{u \in \mathcal{U}} \left\{ \int_\Omega g(x, y, u)\, dx + \frac{\varepsilon}{2} \int_\Omega u^2\, dx \right\}, \tag{5.6}$$

subject to

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) + d(x, y) = \beta(x) u & \text{in} \quad \Omega \\ A(x)\nabla y \cdot \nu + b(x) y = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.7}$$

For each $\varepsilon > 0$ we fix a local solution $\hat{u}_\varepsilon \in \mathcal{U}$ of problem $\mathcal{P}_\varepsilon$.

Theorem 5.5. *Let Assumption 2 be fulfilled. Then there exist positive constants α and κ such that*

$$|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq \kappa \varepsilon^{1/k^*} \tag{5.8}$$

for every $\varepsilon > 0$ such that $|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq \alpha$. If in addition, each $\hat{u}_\varepsilon$ is a global solution of problem (5.6)–(5.7) then the last claim holds with $\alpha = +\infty$, i.e., for every $\varepsilon > 0$.

Proof. Let $\alpha, \alpha'$and c be the positive numbers in Theorem 5.4. Define $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ by $\eta_\varepsilon(u) := \varepsilon u^2/2$ and $\zeta_\varepsilon := (0, \eta_\varepsilon) \in \Upsilon$ for each $\varepsilon > 0$. Note that

$$d_C(\eta_\varepsilon, 0) := \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\varepsilon m^2/2}{1+\varepsilon m^2/2} = \varepsilon \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{m^2}{2+\varepsilon m^2} \leq \varepsilon \sum_{m=1}^{\infty} \frac{m^2}{2^{m+1}} = 3\varepsilon$$

for all $\varepsilon > 0$. Analogously,

$$d_C(\frac{\partial \eta_\varepsilon}{\partial u}, 0) := \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\varepsilon m}{1+\varepsilon m} \leq \varepsilon \sum_{m=1}^{\infty} \frac{m}{2^m} = 2\varepsilon$$

for all $\varepsilon > 0$. We conclude that $d_\Upsilon(\zeta_\varepsilon, 0) \leq 5\varepsilon \leq \alpha'$ for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := \alpha'/5$. By Theorem 5.4,

$$|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq 5^{\frac{1}{k^*}} c \varepsilon^{\frac{1}{k^*}}$$

for all $\varepsilon \in (0, \varepsilon_0)$ such that $|\hat{u}_\varepsilon - \bar{u}| \leq \alpha$. Let $M > 0$ be a bound for $\mathcal{U}$ in $L^\infty(\Omega)$. We also have we have

$$|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq 2M \leq 2M\varepsilon^{-\frac{1}{k^*}} \varepsilon^{\frac{1}{k^*}} \leq 2M\varepsilon_0^{-\frac{1}{k^*}} \varepsilon^{\frac{1}{k^*}}$$

for all $\varepsilon \geq \varepsilon_0$. Hence, defining

$$\kappa := \max\left\{5^{\frac{1}{k^*}} c,\, 2M\varepsilon_0^{-1/k^*}\right\},$$

we obtain the first claim.

Let us prove the second claim of the theorem. First we prove that there exists $\varepsilon^* > 0$ such that $|\hat{u}_\varepsilon - \bar{u}|_{L^1(\Omega)} \leq \alpha$ for all $\varepsilon \in (0, \varepsilon^*)$. Suppose the opposite. Then there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ converging to zero such that $|\hat{u}_{\varepsilon_k} - \bar{u}|_{L^1(\Omega)} > \alpha$ for all $k \in \mathbb{N}$. Since $\mathcal{U}$ is weakly compact in $L^2(\Omega)$, we may assume without loss of generality that $u_{\varepsilon_k} \to u^*$ for some $u^* \in \mathcal{U}$. Since $y_{u_{\varepsilon_k}} \to y_{u^*}$ in $C(\bar{\Omega})$, we obtain that

$$J(u^*) \leq \liminf_{k\to\infty} \left[J(u_{\varepsilon_k}) + \frac{\varepsilon_k}{2}|u_{\varepsilon_k}|_{L^2(\Omega)}\right] \leq \liminf_{k\to\infty} \left[J(\bar{u}) + \frac{\varepsilon_k}{2}|\bar{u}|_{L^2(\Omega)}\right] = J(\bar{u}).$$

By Remark 4.10, $\bar{u}$ is a strict local solution, therefore $u^* = \bar{u}$. By Proposition 4.1, $u^* = \bar{u}$ is bang-bang. Weak convergence in $L^2(\Omega)$ implies that in $L^1(\Omega)$; consequently, by Lemma 4.2, $u_{\varepsilon_n} \to u^*$ in $L^1(\Omega)$, which is a contradiction. Then the first claim of the theorem implies (5.8) for all $\varepsilon \in (0, \varepsilon^*)$. For $\varepsilon \geq \varepsilon^*$, (5.8) remains true if we increase the constant c (if needed) so that $c \geq 2M(\varepsilon^*)^{-1/k^*}$. □

6 Assumptions related to subregularity

In this section, we gather some results concerning Assumption 2, in order to provide sufficient conditions under which it is fulfilled. Furthermore, we analyze related assumptions and their relation between themselves. Recall that $\bar{u} \in \mathcal{U}$ is a local solution of problem (1.1)–(1.2). Since $\bar{u} \in \mathcal{U}$ satisfies the variational inequality (3.3), we have

$$\bar{u}(x) = \begin{cases} b_1(x) & if \quad \sigma_{\bar{u}}(x) > 0 \\ \\ b_2(x) & if \quad \sigma_{\bar{u}}(x) < 0. \end{cases}$$

We introduce the following extended cone suggested in [5]. For a fixed $\tau > 0$ define

$$C_{\bar{u}}^{\tau} = \left\{ v \in L^2(\Omega) : v(x) \begin{cases} = 0 & \text{if} \quad |\sigma_{\bar{u}}(x)| > \tau \text{ or } \bar{u}(x) \in (b_1(x), b_2(x)) \\ \geq 0 & \text{if} \quad |\sigma_{\bar{u}}(x)| \leq \tau \text{ and } \bar{u}(x) = b_1(x) \\ \leq 0 & \text{if} \quad |\sigma_{\bar{u}}(x)| \leq \tau \text{ and } \bar{u}(x) = b_2(x) \end{cases} \right\}.$$

We introduce the following modification of Assumption 2.

Assumption 2′. *There exist positive numbers α_0 and γ_0 such that*

$$\int_\Omega \sigma_{\bar u}(u-\bar u)\,dx + \Lambda(u-\bar u) \geq \gamma_0 |u-\bar u|_{L^1(\Omega)}^{k^*+1},$$

for all $u \in \mathcal{U}$ with $u - \bar u \in C_{\bar u}^\tau \cap \mathbb{B}_{L^1(\Omega)}(\bar u;\alpha_0)$.

This assumption is seemingly weaker than Assumption 2. However, we will prove that the two assumptions are equivalent. Before that, for technical purposes, we introduce the bilinear form $\Gamma : L^2(\Omega)\times L^2(\Omega)\to\mathbb{R}$ given by

$$\Gamma(v_1,v_2) := \frac{1}{2}\int_\Omega \Big[\pi_{v_1}v_2 + \pi_{v_2}v_1\Big]\,dx. \tag{6.1}$$

The bilinear form is particularly useful because of the following property.

$$\Lambda(v_1+v_2) = \Gamma(v_1,v_1) + 2\Gamma(v_1,v_2) + \Gamma(v_2,v_2) \quad \forall v_1,v_2 \in L^2(\Omega). \tag{6.2}$$

We will require the following technical lemma.

Lemma 6.1. *For every positive number M, there exits a positive number c such that*

$$|\Gamma(v_1,v_2)| \leq c|v_1|_{L^1(\Omega)}^{1/2}|v_2|_{L^1(\Omega)}$$

for all $v_1,v_2 \in \mathbb{B}_{L^\infty}(0;M)$.

Proof. By Proposition 3.6, there exist $c_1,c_2>0$ such that $|\pi_v|_{L^\infty(\Omega)} \leq c_1|v|_{L^2(\Omega)}$ and $|\pi_v|_{L^2(\Omega)} \leq c_2|v|_{L^1(\Omega)}$ for all $v\in L^2(\Omega)$. Let $M>0$ be arbitrary. Observe that

$$\Big|\int_\Omega \pi_{v_1}v_2\,dx\Big| \leq |\pi_{v_1}|_{L^\infty(\Omega)}|v_2|_{L^1(\Omega)} \leq c_1 M^{\frac12}|v_1|_{L^1(\Omega)}^{\frac12}|v_2|_{L^1(\Omega)},$$

and that

$$\Big|\int_\Omega \pi_{v_2}v_1\,dx\Big| \leq |\pi_{v_2}|_{L^2(\Omega)}|v_1|_{L^2(\Omega)} \leq c_2 M^{\frac12}|v_1|_{L^1(\Omega)}^{\frac12}|v_2|_{L^1(\Omega)}$$

for all $v_1,v_2\in\mathbb{B}_{L^\infty}(0;M)$. There result follows defining $c := 2^{-1}(c_1+c_2)M^{\frac12}$. □

Proposition 6.2. *Assumptions 2 and 2′ are equivalent.*

Proof. Let α_0 and γ_0 be the numbers in Assumption 2′. Let $u\in\mathcal{U}$ and define

$$v_1(x) := \begin{cases} u(x)-\bar u(x) & if \quad |\sigma_{\bar u}(x)|\leq\tau \\ 0 & if \quad |\sigma_{\bar u}(x)|>\tau, \end{cases}$$

and

$$v_2(x) := \begin{cases} 0 & if \quad |\sigma_{\bar u}(x)|\leq\tau \\ u(x)-\bar u(x) & if \quad |\sigma_{\bar u}(x)|>\tau. \end{cases}$$

Clearly $v_1\in C_{\bar u}^\tau$ and $v_1+v_2 = u-\bar u$. Let M be a bound for $\mathcal{U}$ in $L^\infty(\Omega)$, and let c be the positive number in Lemma 6.1 corresponding to $2M$. By Assumption 2′,

$$\begin{aligned}\int_\Omega \sigma_{\bar u}(u-\bar u)\,dx &= \int_\Omega \sigma_{\bar u}v_1\,dx + \int_{|\sigma_{\bar u}|>\tau}\sigma_{\bar u}v_2\,dx\\ &= \int_\Omega \sigma_{\bar u}v_1\,dx + \Lambda(v_1) - \Lambda(v_1) + \int_{|\sigma_{\bar u}|>\tau}\sigma_{\bar u}v_2\,dx\\ &\geq \gamma_0|v_1|^{k+1} + \tau|v_2|_{L^1(\Omega)} - \Lambda(v_1),\end{aligned}$$

and

$$\begin{aligned}
\Lambda(u-\bar{u}) &= \Lambda(v_1) + 2\Gamma(v_1, v_2) + \Lambda(v_2) \\
&\geq \Lambda(v_1) - 2c|v_1|^{1/2}_{L^1(\Omega)}|v_2|_{L^1(\Omega)} - c|v_2|^{1/2}_{L^1(\Omega)}|v_2|_{L^1(\Omega)} \\
&\geq \Lambda(v_1) - 3c|v_2|_{L^1(\Omega)}|u-\bar{u}|^{1/2}_{L^1(\Omega)}
\end{aligned}$$

for $u \in \mathcal{U}$ with $|u-\bar{u}|_{L^1(\Omega)} \leq \alpha_0$. Thus

$$\begin{aligned}
\int_\Omega \sigma_{\bar{u}}(u-\bar{u})\,dx + \Lambda(u-\bar{u}) &\geq \gamma_0|v_1|^{k+1} + \tau|v_2|_{L^1(\Omega)} - 3c|v_2|_{L^1(\Omega)}|u-\bar{u}|^{1/2}_{L^1(\Omega)} \\
&= \gamma_0|v_1|^{k+1} + |v_2|_{L^1(\Omega)}\Big(\tau - 3c|u-\bar{u}|^{1/2}_{L^1(\Omega)}\Big)
\end{aligned}$$

for $u \in \mathcal{U}$ with $|u-\bar{u}|_{L^1(\Omega)} \leq \alpha_0$. Now, by the reverse triangle inequality and Bernoulli's inequality (consider without loss of generality $u \neq \bar{u}$)

$$\begin{aligned}
|v_1|^{k+1}_{L^1(\Omega)} &= |(u-\bar{u}) - v_2|^{k+1}_{L^1(\Omega)} \geq \Big(|u-\bar{u}|_{L^1(\Omega)} - |v_2|_{L^1(\Omega)}\Big)^{k+1} \\
&= |u-\bar{u}|^{k+1}_{L^1(\Omega)}\Big(1 - \frac{|v_2|_{L^1(\Omega)}}{|u-\bar{u}|_{L^1(\Omega)}}\Big)^{k+1} \geq |u-\bar{u}|^{k+1}_{L^1(\Omega)}\Big(1-(k+1)\frac{|v_2|_{L^1(\Omega)}}{|u-\bar{u}|_{L^1(\Omega)}}\Big) \\
&= |u-\bar{u}|^{k+1}_{L^1(\Omega)} - (k+1)|u-\bar{u}|^{k}_{L^1(\Omega)}|v_2|_{L^1(\Omega)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_\Omega \sigma_{\bar{u}}(u-\bar{u})\,dx + \Lambda(u-\bar{u}) &\geq \gamma_0|v_1|^{k+1} + |v_2|_{L^1(\Omega)}\Big(\tau - 3c|u-\bar{u}|^{1/2}_{L^1(\Omega)}\Big) \\
&\geq \gamma_0|u-\bar{u}|^{k+1}_{L^1(\Omega)} - \gamma_0(k+1)|u-\bar{u}|^{k}_{L^1(\Omega)}|v_2|_{L^1(\Omega)} + |v_2|_{L^1(\Omega)}\Big(\tau - 3c|u-\bar{u}|^{1/2}_{L^1(\Omega)}\Big) \\
&\geq \gamma_0|u-\bar{u}|^{k+1}_{L^1(\Omega)} + |v_2|_{L^1(\Omega)}\Big(\tau - \gamma_0(k+1)|u-\bar{u}|^{k}_{L^1(\Omega)} - 3c|u-\bar{u}|^{1/2}_{L^1(\Omega)}\Big).
\end{aligned}$$

Choosing α small enough, one can ensure

$$\begin{aligned}
\int_\Omega \sigma_{\bar{u}}(u-\bar{u})\,dx + \Lambda(u-\bar{u}) &\geq \gamma_0|u-\bar{u}|^{k+1}_{L^1(\Omega)} + |v_2|_{L^1(\Omega)}\Big(\tau - \gamma_0(k+1)|u-\bar{u}|^{k}_{L^1(\Omega)} - 3c|u-\bar{u}|^{1/2}_{L^1(\Omega)}\Big) \\
&\geq \gamma_0|u-\bar{u}|^{k+1}_{L^1(\Omega)} + \frac{\tau}{2}|v_2|_{L^1(\Omega)} \geq \gamma_0|u-\bar{u}|^{k+1}_{L^1(\Omega)}
\end{aligned}$$

for all $u \in \mathcal{U}$ with $|u-\bar{u}|_{L^1(\Omega)} \leq \alpha$.

□

Proposition 6.2 allows to split Assumption 2 in two parts, as it follows in the next theorem.

Theorem 6.3. *Let there exist numbers $\mu_1, \mu_2 \in \mathbb{R}$ and $\alpha > 0$ such that*

$$\int_\Omega \sigma_{\bar{u}} v\,dx \geq \mu_1|v|^{k^*+1}_{L^1(\Omega)} \tag{6.3}$$

and

$$\Lambda(v) \geq \mu_2|v|^{k^*+1}_{L^1(\Omega)} \tag{6.4}$$

for every $v \in (\mathcal{U}-\bar{u}) \cap C^\tau_{\bar{u}} \cap \mathbb{B}_{L^1(\Omega)}(\bar{u};\alpha)$. If $\mu_1+\mu_2 > 0$, then Assumption 2 is fulfilled, hence the optimality mapping Φ (see (4.12)) of problem (1.1)–(1.2) is strongly Hölder subregular with exponent $\lambda = 1/k^*$ at the reference point $(\bar{y}, \bar{p}, \bar{u})$.

The proof consists of summation of (6.3) and (6.4) and utilization of Proposition 6.2 and Theorem 4.11.

The splitting of Assumption 2 has the advantage that the inequalities in (6.3) and (6.4) can be analyzed separately. The next proposition is related to (6.3).

The following assumption has become standard in the literature on PDE optimal control problems with bang-bang controls, see, e.g., [9, 12, 34, 42].

Assumption 3. *There exists a positive number* μ_0 *such that*

$$meas\,\{x \in \Omega : |\sigma_{\bar{u}}(x)| \le \varepsilon\} \le \mu_0 \varepsilon^{\frac{1}{k^*}} \quad \forall \varepsilon > 0.$$

Proposition 6.4. *The following statements hold.*

(i) If Assumption 3 is fulfilled then there exists $\mu_1 > 0$ *such that (6.3) holds for every* $v \in \mathcal{U} - \bar{u}$.

(ii) Suppose there exists $\nu > 0$ *such that* $b_2(x) - b_1(x) \ge \nu$ *for a.e.* $x \in \Omega$. *If (6.3) holds for every* $v \in \mathcal{U} - \bar{u}$ *then Assumption 3 is fulfilled.*

Proof. The proof of the first claim follows [34, Proposition 3.1], see also [9, Proposition 2.7]. It has been also proved several times in the literature on ordinary differential equations in a somewhat stronger form; see, e.g., [1, 28, 33, 37].

Let us prove the second claim. For each $\varepsilon > 0$, define

$$u_\varepsilon(x) := \begin{cases} \bar{u}(x) & \text{if} \qquad\qquad\qquad\qquad |\sigma_{\bar{u}}(x)| > \varepsilon \\ b_1(x) & \text{if} \quad |\sigma_{\bar{u}}(x)| \le \varepsilon \quad \text{and} \quad \bar{u}(x) \in \Big[\dfrac{b_1(x) + b_2(x)}{2}, b_2(x)\Big] \\ b_2(x) & \text{if} \quad |\sigma_{\bar{u}}(x)| \le \varepsilon \quad \text{and} \quad \bar{u}(x) \in \Big[b_1(x), \dfrac{b_1(x) + b_2(x)}{2}\Big). \end{cases}$$

Clearly each u_ε belongs to $\mathcal{U}$, and

$$|u_\varepsilon(x) - \bar{u}(x)| \ge \frac{1}{2}|b_2(x) - b_1(x)| \tag{6.5}$$

for a.e $x \in \{s \in \Omega : |\sigma_{\bar{u}}(s)| \le \epsilon\}$. From (6.3) we have

$$\mu_1\Big(\int_{|\sigma_{\bar{u}}| \le \varepsilon} |u_\varepsilon - \bar{u}|\,dx\Big)^{k+1} \le \int_{|\sigma_{\bar{u}}| \le \varepsilon} \sigma_{\bar{u}}(u_\varepsilon - \bar{u})\,dx \le \varepsilon \int_{|\sigma_{\bar{u}}| \le \varepsilon} |u_\varepsilon - \bar{u}|\,dx.$$

This implies

$$\int_{|\sigma_{\bar{u}}| \le \varepsilon} |u_\varepsilon - \bar{u}|\,dx \le \mu_1^{-\frac{1}{k}} \varepsilon^{\frac{1}{k}}. \tag{6.6}$$

Using (6.5) and (6.6) we obtain that

$$\begin{aligned} \text{meas}\,\{x \in \Omega : |\sigma_{\bar{u}}(x)| \le \varepsilon\} &= \frac{1}{\nu}\int_{|\sigma_{\bar{u}}| \le \varepsilon} \nu\,dx \le \frac{1}{\nu}\int_{|\sigma_{\bar{u}}| \le \varepsilon} |b_2 - b_1|\,dx \le \frac{2}{\nu}\int_{|\sigma_{\bar{u}}| \le \varepsilon} |u_\varepsilon - \bar{u}|\,dx \\ &\le 2(\mu_1)^{-\frac{1}{k}} \nu^{-1} \varepsilon^{\frac{1}{k}}. \end{aligned}$$

Thus Assumption 3 is fulfilled with $\mu_0 := 2(\mu_1)^{-\frac{1}{k}} \nu^{-1}$. □

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