Optimal offending in view of the offender's criminal record [1] [2]

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Abstract

Gary S. Becker's economic approach to crime and punishment (BECKER, 1968) was essentially static. In the sequel several authors adressed the question of optimal law enforcement by including intertemporal aspects. The present paper considers a rational offender who optimizes his/her criminal career by taking into consideration that the severity of the punishment for a given crime depends on the offender's prior criminal record. A life cycle model for describing the behaviour of a recidivous criminal is formulated in terms of optimal control theory, the criminal's record being the state variable.

In particular, we study the impact of an offender's prior criminal record on the punishment of a given crime. It should be stressed that such an analysis may yield important insight into the design of optimal law enforcement policies.

A numerical phase-diagram analysis and, moreover, a sensitivity analysis are provided in order to show which strategies are expedient in reducing the amount of criminal activities and which make it even worse.

1 Introduction

Violation of the law and law enforcement are as old as the history of mankind. A big difference between the earnings from legal and illegal work, respectively, lies in the fact that the fruits of criminal activities can be consumed before the costs of their acquisition must be paid. In all epoches there were people

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who took advantage of the fact that sometimes it's easier to steal goods instead of working to purchase these goods.

As a result of those shortcomings of legal work the authorities are spending efforts to prevent illegal activities. In an attempt to make the society better, many questions arise. How much money should the government devote for detection of crime? What kind of punishment should be applied? How should the severity of the punishment depend on the gravity of the crime?

We think that mathematical models can provide many new insights into this complex field and give us some helpful hints for setting up the optimal punishment policy for a given kind of offence.

The present paper studies the impact on the offence rate of various penalty levels an individual commiting any crime expects if arrested. In particular, punishment frequently depends on the offender's prior criminal record. We try to assess how past accumulated convictions might influence the offender's present and future criminal behaviour. The offender's reactions to various punishment policies may provide interesting insight for the design of optimal law enforcement strategies.

The inclusion of an offender's prior criminal record into the penalty function was already suggested by Caulkins (CAULKINS, 1993). Although his analysis is static, this idea opens a new intertemporal approach of law enforcement, whatever kind of crime might be considered. In contrast to that several authors, for instance Leung (LEUNG, 1991), Kort et al. (KORT et al., 1996), Antoci and Sacco (ANTOCI and SACCO, 1995), Davis (DAVIS, 1988), Gerchak and Parlar (GERCHAK and PARLAR, 1985), took dynamic aspects into consideration. An example of a descriptive continuous time model was examined by Gragnani et al. (GRAGNANI et al., 1994) who, among others, investigated the reaction of a local drug market to a crack-down policy.

In this paper we will analyse the reactions of an offender in a dynamic setup. He/she faces a given punishment policy that satisfies some realistic assumptions. Despite of the harm that the punishment causes and the expenditures for commiting a crime, there is a possibility of an advantage by breaking the law.

In section 2 we formulate an intertemporal model with one decision variable and one control variable. Since our approach to this peculiar problem is dynamic we will consider the punishment an offender experiences when being arrested as a function of the offence level as well as of the record of previous detected offences. Later, in section 3, we derive and analyze the canonical system following from the neccessary optimality conditions. Subsequently, in section 4, we collect some remarkable observations and, finally, in chapter 5 we draw some conclusions and suggest a few interesting extensions.

In the appendix we present a graph to illustrate our reflections.

2 The model

In this section we introduce the basic ingredients of the intertemporal optimization problem of a utility maximizing offender. The key variable is the record of prior criminal offences measuring the stock of accumulated convictions.

Dynamics of the criminal record

As mentioned above, the record of prior criminal offences is the state variable which, however, can increase and decrease depending on the level and amount of punishable actions.

Using a continuous time scale the dynamics of the record may be written as

$$\begin{aligned} \dot{R}(t) &= f(R, u) \\ &= p\, u(t) - \delta R(t), \\ R(0) &= R_0 \quad \geq \quad 0\,, \end{aligned} \tag{1}$$

where

t	is the time argument,
$u(t)$	offence-rate at time t
$R(t)$	denotes the record of previous offences at time t,
p	the probability of being convicted, and
δ	the rate for discounting because of limitation.

As already used before, here and in what follows the time argument is omitted to simplify the notation. The first term in the system dynamics $p\,u$ describes the expected increase of the state variable. It only depends on the conviction probability and the actual offence rate.

For example, in the case of California's *Three Strikes Law* (GREENWOOD et al., 1994), someone who was twice convicted of robbery and in both instances given probation would be subject to the same sentencing enhancements as someone who was twice convicted of robbery and imprisoned each time.

In turn, a high offence rate leads to an increase of the criminal record. The multiple δR is the forgetting term which enables the decision maker to reduce his record by stopping illegal behaviour and being a kind person.

In reality each item of the criminal record remains for a certain period and then it vanishes completely. Thus, the graph of the criminal record plotted against time should be a step-function. For mathematical convenience we approximate the dynamics by means of exponential increase and decrease.

Law enforcement policy

The penalty of the convicted offender should be increasing in u and R, respectively. For getting somewhat easy expressions we assume that it might be of the shape of a logarithmic function. Therefore, it can be written as:

$$c_a(u, R) = \Big(b + \beta \ln(1 + \gamma\, R)\Big) u \tag{2}$$

where b, β, and γ are positive constants.

According to Becker((BECKER, 1968) p.179) *'the cost of different punishments to an offender can be made comparable by converting them into their monetary equivalent or worth'*. This suggests to measure both u and R in monetary units.

The multiple $b\,u$ is the level of punishment an offender suffers when he gets detected for the very first time, β is a factor for the increase of the severity caused by the record, and γ is a parameter for describing the slope of the influence of R, the record. The punishment policy suggested here is a function of the level of offence as well as of the criminal record, as remarked by Caulkins (CAULKINS, 1993). However, we were looking for a function that meets some, from our point of view reasonable, conditions.

1. It is strictly increasing in both of its arguments.

2. People who do not offend at all should not be punished:

   $$c_a(0, R) = 0 \text{ for all } R. \tag{3}$$

   That can be ascertained by using u as a multiplier in (2).

3. Any detected positive level of offence should lead to a positive amount of punishment:

   $$c_a(u, R) > 0 \quad \forall u > 0,\ \forall R. \tag{4}$$

   To achieve this we use the 'fixed punishment' term b in (2).

The logarithmic function fulfills the desired properties and, moreover, the second partial derivate with respect to R is negative:

$$\frac{\partial c_a}{\partial R} > 0, \quad \frac{\partial c_a}{\partial u} > 0. \frac{\partial^2 c_a}{\partial R^2} < 0, \tag{5}$$

The first two inequalities ensure that the level of punishment is strictly increasing with R and u. The third condition guarantees that the slope is declining with an increasing criminal record. Otherwise the severity of punishment would rise sharply for very experienced criminals. Moreover, for simplicity we assume

$$\frac{\partial^2 c_a}{\partial u^2} = 0. \tag{6}$$

If there is a restriction in the decision variable u (let's say $\bar{u}$ for instance), this leads to an upper bound for R. It holds that

$$R \leq \bar{R} = \frac{p}{\delta}\,\bar{u}. \tag{7}$$

However, from our point of view the existence of a maximum offence level seems to be doubtful. If we are thinking of a worldwide operating company there might be no identifiable upper bound. Moreover, if we have a look at the appropriate fine, there is no reason for establishing a maximum punishment in case of a huge criminal company.

Utility-function

For mathematical convenience we use the logarithm to derive the utility an offender experiences from his actions, i.e:

$$U(u) = \alpha\ ln(1 + au). \tag{8}$$

This is a rather simple function that fulfills the assumptions that the marginal utility is always positive and strictly declining. It is evident that for each positive level of offence the utility is also positive.

Costs

People who commit crimes are charged several types of costs. Not all of these are actually costs in terms of monetary units but we always consider a monetary equivalent of each term that has a negative impact from the viewpoint of the offender.

Costs of the offending intensity
Committing a crime is accompanied by several expenses. For instance burglars have to purchase some implements for following their profession. Another example are the costs of a pettifogger required for white-collar-crimes like bribery and tax evasion.

In most cases it might be difficult to estimate the accurate offending costs c_o. If we focus, for instamce, on consumption of illicit drugs we can assume that the costs for buying them will be a multiple of the desired amount. However, drug-consumption might presumably not be the only crime with costs that increase linear with the offence level. In turn, given the price for one unit we can calculate the total costs very easily:

$$c_o = c_p \; u. \tag{9}$$

In the above formula c_p denotes the price for one unit and u is the chosen quantity. In the following we will always deal with this type of cost function.

Costs of being punished
The costs of being punished can either be the actual expectation value of a fine that has to be paid, or the monetary equivalent of being imprisoned for a certain period. They can be defined as the multiple of the probability of being punished times the level of punishment:

$$\begin{aligned} expected\ penalty \quad &= \quad p \; c_a(u, R)\,, \\ & \quad\;\; p \in [0, 1]. \end{aligned} \tag{10}$$

Optimization problem

The decision maker faces the following intertemporal decision problem:

$$max J := \int_0^\infty e^{-rt} \Big[U\left(u(t)\right) - c_p u(t) - p \; c_a\Big(u(t), R(t)\Big)\Big] dt. \tag{11}$$

The parameter r is the discount rate ensuring that a certain amount of money possessed now has a higher utility than the same amount of money possessed in the future. The objective functional J is the discounted utility stream gained from engaging in illegal activities.

Using a model with an infinite planning horizon will not cause any problems for analyzing a firm with transferable property rights. In case of one individual offender it might be questionable because her/his lifetime is limited. However, if the discount rate is large enough, the profit stream close to the end of the planning horizon discounted to present value will become very small and, therefore, the impact on the objective functional will be quite small.

For a comprehensive description of the used methods for solving this optimization problem see Leonard and Long (LEONARD and LONG, 1995) or Feichtinger and Hartl (FEICHTINGER and HARTL, 1986).

3 Analysis

Having made the assumptions of the previous chapter we define the current value Hamiltonian:

$$H(R,u,\lambda) \;=\; U(u) - c_p u - p\; c_a(u,R) + \lambda\; \dot{R} \tag{12}$$

In this particular model, the current value Hamiltonian is

$$\begin{aligned} H(R,u,\lambda) \;=\; & \alpha \ln(1+au) - c_p\, u - p\Big(b + \beta \ln(1+\gamma R)\Big)u \\ & + \lambda\,(p\,u - \delta R)\,, \end{aligned} \tag{13}$$

where λ is the costate variable representing the shadow price of the recorded crimes. The Hamiltonian is well-defined and differentiable for all nonnegative values of R and u.

Utility maximization

We derive the dynamical system for the optimal control u from the following necessary optimality conditions:

$$u = \arg\max_u H \tag{14}$$

and

$$\dot{\lambda} = r\lambda - H_R. \tag{15}$$

The Hamiltonian (13) is strictly concave with respect to u. In a first step supposing that there are no restrictions for the control u we can conclude that the maximum-condition (14) is equivalent to $H_u = 0$. It leads to

$$u = \frac{\alpha}{c_p + p\,\Big(\,b + \beta\,\ln\,(\,1 + \gamma\,R\,) - \lambda\Big)} - \frac{1}{a}, \tag{16}$$

$$\lambda = b + \beta\,\ln\,(\,1 + \gamma\,R\,) + \frac{c_p}{p} - \frac{\alpha\,a}{p\,(1 + a\,u)}. \tag{17}$$

Another interesting aspect occurs when we have a look at the maximum condition (14). The decision maker will always have the desire to increase the severity if the first partial derivative of the Hamiltonian with respect to u is positive, i.e.:

$$U_u > c_p + p\, c_{a_u} - \lambda\, p\ . \tag{18}$$

If this condition is not fulfilled for any nonnegative value of u, then the optimal u is negative. This solution is not feasible in this model and, in turn, the offender has to choose $u = 0$ in such a situation.

In economic terms the above inequality means that the marginal utility U_u has to exceed the marginal costs. The marginal costs consist of the investment c_p required for offending with an intensity of one, the expectation value $p\, c_{a_u}$ of the marginal penalty, and the marginal indirect effect of offending, expressed by the multiple $-\lambda\, p$ - the shadow price times the conviction probability. To bring it to the point, the marginal utility must be greater than the marginal costs. Otherwise it is not favourable to choose a positive offence rate.

The state control dynamics

Differentiation of (17) with respect to time gives

$$\dot{\lambda} = \frac{\beta\,\gamma}{1 + \gamma\, R}\,\dot{R} + \frac{\alpha\, a^2}{p\,(1 + a\, u)^2}\,\dot{u}. \tag{19}$$

For the second condition (15), we derive from (13)

$$\dot{\lambda} = r\lambda - H_R = \lambda\,(r + \delta) + \frac{p\,\beta\,\gamma\, u}{1 + \gamma\, R}. \tag{20}$$

Next we set expression (19) equal to (20) to obtain

$$\frac{\beta\,\gamma}{1 + \gamma\, R}\,\dot{R} + \frac{\alpha\, a^2}{p\,(1 + a\, u)^2}\,\dot{u} = \lambda\,(r + \delta) + \frac{p\,\beta\,\gamma\, u}{1 + \gamma\, R}. \tag{21}$$

Finally, substitution of (1) and (17) into (21) yields the differential equation for the decision u in the unrestricted case:

$$\begin{aligned} \dot{u} \;\; = \;\; & \frac{p}{\alpha}\left(\frac{1 + a\, u}{a}\right)^2 \Bigg[\frac{\beta\,\gamma\,\delta\, R}{1 + \gamma\, R} + \\ & \left(b + \beta\, \ln(1 + \gamma\, R) + \frac{c_p}{p} - \frac{\alpha\, a}{p\,(1 + a\, u)}\right)\,(\, r + \delta\,)\Bigg]. \end{aligned} \tag{22}$$

The $\dot{\mathbf{R}} = 0$ isocline

The state dynamics are

$$\dot{R} = p\,u - \delta\,R, \tag{23}$$

and hence, $\dot{R} = 0$ implies

$$u = \psi(R) = \frac{\delta}{p}\,R. \tag{24}$$

This isocline is a straight line through the origin with positive slope $\frac{\delta}{p}$. As a result, it is positive for all positive values of R and it goes to infinity when $R \to \infty$.

The $\dot{\mathbf{u}} = 0$ isocline

From (22) we can conclude that $\dot{u} = 0$ if and only if

$$u = \varphi(R) = \frac{(r+\delta)\,\frac{\alpha}{p}}{\left(b + \beta\,\ln(1+\gamma\,R) + \frac{c_p}{p}\right)(r+\delta) + \frac{\beta\gamma\,\delta\,R}{1+\gamma\,R}} - \frac{1}{a}\,. \tag{25}$$

Existence of stationary points

Since we already know the exact shape of the $\dot{R} = 0$ locus we are now going to collect some information about the $\dot{u} = 0$ nullcline. First we are interested in what happens if the stock variable R takes its extreme values:

$$\varphi(0) = \frac{\alpha}{p\,b + c_p} - \frac{1}{a} \tag{26}$$

$$\lim_{R\to\infty} \varphi(R) = -\frac{1}{a} < 0 \tag{27}$$

Further, we have a look at the slope of this curve:

$$\begin{aligned}\varphi'(R) \;=\; & -(\,r+\delta\,)\,\frac{\alpha}{p}\left[\left(b + \beta\,\ln(1+\gamma\,R\,) + \frac{c_p}{p}\right)(\,r+\delta\,) + \frac{\beta\,\gamma\,\delta\,R}{1+\gamma\,R}\right]^{-2} \\ & \left[\frac{\beta\,\gamma}{1+\gamma\,R}\,(\,r+\delta\,) + \frac{\beta\,\gamma\,\delta}{(\,1+\gamma\,R\,)^2}\right] < 0\;. \end{aligned} \tag{28}$$

It turns out that $\varphi(R)$ is strictly decreasing. Therefore, a necessary and sufficient condition for the existence of a stationary point in the positive orthant is given by

$$\frac{\alpha}{p\,b + c_p} > \frac{1}{a}. \tag{29}$$

For $a\alpha \leq c_p + pb$, there is no intersection of the isoclines because then the $\dot{u} = 0$-line is negative (just considering nonnegative values of R). But once $\varphi(0) > 0$, an equilibrium exists, since $\varphi(R)$ decreases to a negative constant and $\psi(R)$ increases to infinity. As they are strictly monotonic functions, the equilibrium is even unique.

Stability analysis

Starting with the system dynamics in the state-control space, which is given by (1) and (22), we can calculate the Jacobian matrix

$$\mathcal{J} = \begin{pmatrix} \dot{R}_R & \dot{R}_u \\ \dot{u}_R & \dot{u}_u \end{pmatrix} . \tag{30}$$

If it's determinat turns out to be positive, the stationary point (if it exists) is an unstable equilibrium. In contrast, if the determinat takes a negative value, it follows that we face a saddle-point equilibrium. Obviously, the determinant has to be evaluated in the considered equilibrium.

The partial derivatives

$$\dot{R}_R = -\delta < 0, \text{ and} \tag{31}$$

$$\dot{R}_u = p > 0 \tag{32}$$

are given by quite simple formulas. Computing the second row of the Jacobian matrix leads us to

$$\dot{u}_R = \frac{p}{\alpha}\left(\frac{1+a\,u}{a}\right)^2 \frac{\beta\,\gamma}{1+\gamma\,R}\left(r+\delta+\frac{\delta}{1+\gamma\,R}\right) > 0, \text{ and} \tag{33}$$

$$\dot{u}_u = 2\,\frac{p}{\alpha}\left(\frac{1+a\,u}{a}\right)\left[\left(b+\beta\,\ln(1+\gamma\,R)+\frac{c_p}{p}\right)(\,r+\delta\,)+\frac{\beta\,\gamma\,\delta\,R}{1+\gamma\,R}\right] -(\,r+\delta\,)\,. \tag{34}$$

The sign of the last expression cannot be determined in general because it depends on the particular numerical values. However, the determinant

$$\det \mathcal{J} = \dot{R}_R\,\dot{u}_u - \dot{R}_u\,\dot{u}_R \tag{35}$$

is negative if and only if

$$\dot{u}_u > \frac{\dot{R}_u\,\dot{u}_R}{\dot{R}_R} . \tag{36}$$

Remark: The right-hand side of the above inequality is negative.

With some simplifications and estimations it follows that

$$\frac{2}{\alpha}\,(\,p\,b + c_p\,) > 1 \tag{37}$$

is a sufficient but not necessary condition for a saddle-point equilibrium. Together with (29) a sufficient condition for the existence of a saddle-point equilibrium can be expressed as a pretty simple chain of inequalities:

$$\alpha\,a > p\,b + c_p > \frac{\alpha}{2}\,. \tag{38}$$

Remark: This also implies $a > \frac{1}{2}$.

The special case $\delta = 0$

In an extreme scenario with absolutely no limitation all the crimes remain in the record forever. As a result the derivative of the criminal record with respect to time can only take nonnegative values, and the state-control dynamics become

$$\begin{aligned}
\dot{R} &= p\,u \geq 0\,, && (39)\\
\dot{u} &= \frac{p}{\alpha}\left(\frac{1}{a} + u\right)^2 \left(b + \beta\,\ln(1 + \gamma\,R) + \frac{c_p}{p} - \frac{\alpha\,a}{p\,(1 + a\,u)}\right) r\,. && (40)
\end{aligned}$$

From the above formulas we can conclude that the $\dot{R} = 0$ locus is just the R-axis and the $\dot{u} = 0$ curve is given by

$$u = \frac{\alpha}{p\,(b + \beta\,\ln(1 + \gamma\,R)) + c_p} - \frac{1}{a}. \tag{41}$$

In that case the equilibrium obviously lies on the R-axis. This means a criminal starts his career with a positive level of offence until the record reaches a peak where the costs of additional crimes are higher than the utility. As a result, the individuum stops any criminal activity from that moment and remains in the saddle point where $u = 0$ and

$$R = \frac{\exp\left(\frac{\alpha\,a - c_p}{p\,\beta} - \frac{b}{\beta}\right) - 1}{\gamma}\,. \tag{42}$$

4 Discussion

Economic interpretation

First of all, we shall give a more detailed description of the equilibrium of the dynamic system and the behaviour of the optimal paths. The existence of an equilibrium turned out to be connected to the inequality (29).

Any existing positive equilibrium that also fulfills (37) must be a saddle point. Interpretation of the parameters figuring in these inequalities shows that these conditions are not purely technical but even reasonable in economic terms.

$\alpha\, a$: These are the parameters determining the utility function $U(u)$ in (8). More precisely,

$$\alpha\, a = \left(\frac{\partial}{\partial\, u} U(u) \right)\Bigg|_{u=0}, \tag{43}$$

the marginal utility at offence rate zero. Since $U(u)$ is strictly concave, $U'(0)$ is also the maximal marginal utility one can expect.

c_p: The criminal's costs arising when he commits the crime increase linearly in his level of offence u with slope c_p. Therefore, c_p are the marginal offending costs, independent of the severity of the crime. Similarily to the previous paragraph we can state

$$c_p = \frac{\partial}{\partial\, u} c_p\, u\,. \tag{44}$$

pb: Also punishment increases linearly in u, but with a rate that depends on the prior criminal record R. Clearly this rate is minimal for an empty record $R = 0$, where it amounts to b. By multiplying it with the probability p of being detected we pass to the expectation value of the punishment. That means we can write down

$$p\, b = \left(\frac{\partial}{\partial\, u} p\, c_a(u, R) \right)\Bigg|_{R=0}. \tag{45}$$

Thus in the above conditions the highest possible marginal utility is compared to the minimal expected overall costs of committing the crime. Only if the former is greater than the latter, the criminal will - at least for a while - find it profitable to break the law.

Phase diagram analysis

Now we take a look at the orientation of the phase plane according to equations (1) and (22). Along the R-axis, that is for $u = 0$, we have $\dot{R} \leq 0$. And along the u-axis we find $\dot{R} \geq 0$, so $\dot{R}$ is positive above $\psi(R)$ and negative below it. For $\dot{u}$, we see that it is positive if and only if $u > \varphi(R)$.

The $\dot{R} = 0$-isocline $\psi(R)$ is nonnegative for nonnegative values of R, but for the other one, the $\dot{u} = 0$-isocline $\varphi(R)$, we can distinguish two cases:

- The $\dot{u} = 0$-isocline always remains nonpositive. This holds when

$$\alpha\, a \leq p\, b + c_p\,. \tag{46}$$

- The $\dot{u} = 0$-isocline has an intersection with the positive R-axis. This situation occurs when

$$\alpha\, a > p\, b + c_p\,. \tag{47}$$

With this particular orientation it is clear, that the saddle point path remains below the $\dot{u} = 0$-isocline $\varphi(R)$, whenever we start with an R on the left side of the equilibrium value, and above it in the opposite case. Otherwise this path wouldn't be able to converge to the saddle point.

Having a look at the phase portrait we see that along the u-axis between the two intersections with the isoclines all the crossing trajectories go from the left to the right. If we consider the set of points surrounded by the y-axis and the part of the nullclines that lies between the y-axis and the saddle point we see that all the trajectories crossing the isoclines leave this set when crossing the isocline. Therefore we can conclude (see (FEICHTINGER and HARTL, 1986), Lemma 4.6 on p. 115) that there will always exist a saddle point path starting with $R = 0$.

The case of starting with $R_0 > 0$ is relevant, too. In that case the criminal will face a new optimization problem with a positive initial criminal record.

Sensitivity analysis

In the case of a saddle point equilibrium we want to know in which direction the equilibrium moves if the set of parameters changes. The obtained results will hopefully provide hints to choose the optimal enforcement policy to minimize the offence rate.

As a result of the neccessary optimality conditions in the equilibrium it holds that

$$\begin{aligned} f(\hat{u}, \hat{R}, \gamma) &= 0 \\ r\,\hat{\lambda} - H_R(\hat{u}, \hat{R}, \hat{\lambda}, \gamma) &= 0 \\ H_u(\hat{R}, \hat{u}, \hat{\lambda}, \gamma) &= 0\,. \end{aligned} \tag{48}$$

Computing the total derivative of the above equations with respect to γ we get

$$\begin{pmatrix} 0 & f_R & f_u \\ f_R - r & H_{RR} & H_{Ru} \\ f_u & H_{uR} & H_{uu} \end{pmatrix} \begin{pmatrix} \frac{\partial\hat{\lambda}}{\partial\gamma} \\ \frac{\partial\hat{R}}{\partial\gamma} \\ \frac{\partial\hat{u}}{\partial\gamma} \end{pmatrix} = \begin{pmatrix} -f_\gamma \\ -H_{R\gamma} \\ -H_{u\gamma} \end{pmatrix} \tag{49}$$

Remark: From the system dynamics (1) it follows that $f_\gamma = 0$. The next step is solving this system of linear equations with Cramer's rule. Since we are only interested in the signs of $\frac{\partial \hat{\lambda}}{\partial \gamma}$, $\frac{\partial \hat{R}}{\partial \gamma}$, and $\frac{\partial \hat{u}}{\partial \gamma}$, we will first observe the sign of the determiant $\triangle$ of the matrix of the coefficients:

$$\triangle = \underbrace{-f_R\Big((f_R - r)H_{uu} - f_u H_{Ru}\Big)}_{>0} + \underbrace{f_u(f_R - r)H_{uR}}_{>0} - \underbrace{f_u^2 H_{RR}}_{<0}. \tag{50}$$

This expression is positive if

$$f_u^2 H_{RR} = \frac{p^3\,\beta\,\gamma^2\,u}{(1+\gamma\,R)^2} \tag{51}$$

is sufficiently small. This will hold for instance in case of a small conviction probability.

The sign of $\frac{\partial \hat{\lambda}}{\partial \gamma}$, can now be derived easyly by building the fraction $\frac{\triangle_\lambda}{\triangle}$, with

$$\begin{aligned}
\triangle_\lambda &:= \det \begin{pmatrix} 0 & f_R & f_u \\ -H_{R\gamma} & -H_{RR} & -H_{Ru} \\ -H_{u\gamma} & H_{uR} & H_{uu} \end{pmatrix} \\
&= \underbrace{H_R\gamma\Big(f_R H_{uu} - f_u H_{uR}\Big)}_{<0} - \underbrace{H_{u\gamma} f_R H_{Ru}}_{<0} + \underbrace{H_{u\gamma} f_u H_{RR}}_{<0}.
\end{aligned} \tag{52}$$

The above expression is negative for small values of the discount rate δ. Thus, in case of $\triangle > 0$ and a small discount rate it turns out that the shadow price in the stationary point decreases when the slope of the punishment function rises.

The criminal record is something bad from the viewpoint of the offender. As a result, the costate (i.e. the shadow price of one unit) is negative and, furthermore, a decrease means that the absolute value of the negative utility rises.

In the same way we can derive the sign of $\frac{\partial \hat{R}}{\partial \gamma}$, which we get through computing

$$\begin{aligned}
\triangle_R &:= \det \begin{pmatrix} 0 & 0 & f_u \\ f_R - r & -H_{R\gamma} & H_{Ru} \\ f_u & -H_{u\gamma} & H_{uu} \end{pmatrix} \\
&= \underbrace{f_u\Big(-(f_R - r)H_{u\gamma} + f_u H_{R\gamma}\Big)}_{<0}.
\end{aligned} \tag{53}$$

This determinant is always negative. Assuming $f_u H_{RR}$ is small, that means if the change of severity depending on the criminal record is high, the offenders try to reduce their record. On the other hand, if the influence is only very small they increase it, because it doesn't cause such a big harm and, moreover, the additional utility of a higher offence level exceeds the negative impact of the increased punishment.

Finally, we have a look at the change of the offence level caused by a change in the slope of the punishment. That means we are now interested in the sign of $\frac{\partial \hat{u}}{\partial \gamma}$.

Like before we compute the determinant

$$\begin{aligned}\triangle_u &:= \det \begin{pmatrix} 0 & f_R & 0 \\ f_R - r & H_{RR} & -H_{R\gamma} \\ f_u & H_{uR} & -H_{u\gamma} \end{pmatrix} \\ &= \underbrace{-f_R\Big(-(f_R - r)H_{u\gamma} + f_u H_{R\gamma}\Big)}_{<0} . \end{aligned} \tag{54}$$

This determinant is also negative. As a consequence the partial derivative $\frac{\partial \hat{u}}{\partial \gamma}$ is negative if $\triangle > 0$. Like in the previous paragraph the offender reduces his offence level when the influence of the record on the severity of the punishment increases and vice versa.

5 Conclusions and extensions

In this paper we studied the impact of an offender's prior criminal record on her/his offence rate. The underlying assumption is that (some) criminals behave rationally.

The natural framework of such a dynamic scenario of crime and punishment is a life cycle model which can be stated and solved in terms of optimal control theory. A phase-portrait analysis yields valuable insight into the qualitative structure of optimal offence paths of a representative utility maximizing offender.

The influence of various punishment-policies, in particular policies that change the severity of punishment depending on the criminal record, on the offence rate is of central importance for law enforcement agencies. It might help them in the design of enforcement and appropriate enforcement policies. As Caulkins (CAULKINS, 1993) has exposed in a study of illicit drug consumption, the analysis may yield surprising results: under reasonable conditions some plausible enforcement policies may run counter to their good intentions.

The main idea of the model considered is the inclusion of the offender's criminal record into the punishment-function. It changes the costs of the decision-maker and thus makes him vary his actions over time. The criminal record therefore allows the authorities to learn about the criminal: The more often he acts against the law, the more severe is the punishment he has to expect.

There are many mathematical ways to simulate learning, but we just want to mention the most simple one: replacing a constant by a deterministic function. The first step was done in this paper, focussing on the level of punishment per 'offence unit'. We imagine that the same idea can be applied to the detection probability p next. It seems plausible that this parameter may vary due to learning effects, too.

To model its behaviour, the relevant variables have to be specified that p depends on. Often big crimes hit the eye, so this probability can be an increasing function of the level u of offending, as in the case of a shop-lifter hiding his haul. But when the physical amount stands in no relation to its monetary value, this argument fails. A drug addict can make one gram of heroine disappear into his pocket as well as twenty without anybody noticing the difference. For this question we thus have to specialize on one particular crime.

In most cases we can think of, the probability p is determined by the frequency by which the crime is committed, because the police learns about the possible suspects and their favourite methods. Conversely, the criminals can also get to know the authorities' habits, but we assume (and hope!) that the police can use this knowledge in a more organized and powerful way. If the criminals are organized as well (mafia), also this argumentation may lose its significance. Furthermore in terms of our models, we have difficulties how to measure how much has been learned already. Using the prior criminal record is no satisfying solution. If a criminal is well-known to the police, it is because of many crimes and so his criminal record R is very likely to be high. In contrast, a person with a high criminal record could have committed only one crime, but a very severe one, and a low record may also be due to stopping illegal actions for a while or simply to incapability of the police. By the way, there is no reason why he shall have habits at all or never change them during his criminal career.

Finally, one can include dependency on time, reflecting new technical evolutions. But once again it is not clear who can take more profit from those improvements, police or criminals.

As mentioned before, we can return to the basic model of Caulkins (CAULKINS, 1993) and introduce the offending frequency v as a second control variable. There, one can better distinguish the cases leading to a high criminal record level: frequent but less serious crimes and rare but severe ones. Compared to the former case, the latter hardly gives any information about the offender and therefore implies a lower conviction probability. The punishment will raise with increasing frequency, but perhaps still be more sensible to the severity of the crime. To gain hints about effective punishment policies it would be interesting to analyse the sensitivity of the equilibrium with respect to those parameters that can be influenced by the authorities.

Another simplification we made was about the 'offending costs' (see equation (9)). We assumed that there existed a kind of constant 'price per offending unit'. This might be valid for a drug market, but there are other crimes that need some equipment before being able to commit it. We can for instance think of a safe-cracker who can not work without adequate tools. And even when we agree with a formula 'price times amount' for the current costs, the price still may vary. It is a general rule in economics that risk-averse agents demand additional premiums for risky businesses. This applies for insurance or finance as well as for drug dealers or tax consultants who help for tax evasion. Consequently, the price c_p is an increasing function of the risk, expressed in terms of the conviction probability p. Moreover, the offender often has to face a sort of 'social loss'; the loss of confidence of clients or just of the own familiy or any other, which is always very difficult to measure.

The last essential extension is to switch from a micro to a macro model. In our micro model, the behavior of one single offender won't influence the market's price c_p for offending. But if we can establish a global tendency in the market to 'buy' (that is breaking the law), prices will increase. Here again it can be helpful to consider the second decision variable v. Another idea from general markets in economy is to add as a state variable the size of the market, in other words the number of active criminals. An increasing number of criminals means an increasing number of rivals, thus a lower profit of the illegal action and a negative impact on the derived utility. A growing market can also influence the prices, though again it is not easy to say in which direction, so both utility function U and offending costs c_0 depend on the number of participants.

Appendix A

The $\dot{\mathbf{R}} = \mathbf{0}$ isocline

The state dynamics are

$$\dot{R} = p\,u - \delta\,R, \tag{55}$$

and hence, $\dot{R} = 0$ implies

$$u = \psi(R) = \frac{\delta}{p} R. \tag{56}$$

This isocline is a straight line through the origin with positive slope $\frac{\delta}{p}$. As a result, it is positive for all positive values of R and it goes to infinity when $R \to \infty$.

The $\dot{\mathrm{u}} = 0$ isocline

From (22) we can conclude that $\dot{u} = 0$ if and only if

$$u = \varphi(R) = \frac{(r+\delta)\,\frac{\alpha}{p}}{\left(b + \beta\,\ln(1+\gamma\,R) + \frac{c_p}{p}\right)(r+\delta) + \frac{\beta\,\gamma\,\delta\,R}{1+\gamma\,R}} - \frac{1}{a}\,. \tag{57}$$

Existence of stationary points

Since we already know the exact shape of the $\dot{R} = 0$ locus we are now going to collect some information about the $\dot{u} = 0$ nullcline. First we are interested in what happens if the stock variable R takes its extreme values:

$$\varphi(0) = \frac{\alpha}{p\,b + c_p} - \frac{1}{a} \tag{58}$$

$$\lim_{R\to\infty} \varphi(R) = -\frac{1}{a} < 0 \tag{59}$$

Further, we have a look at the slope of this curve:

$$\begin{aligned}\varphi'(R) \;\; &= \;\; -(\,r+\delta\,)\,\frac{\alpha}{p}\left[\left(b + \beta\,\ln(1+\gamma\,R\,) + \frac{c_p}{p}\right)(\,r+\delta\,) + \frac{\beta\,\gamma\,\delta\,R}{1+\gamma\,R}\right]^{-2} \\ &\left[\frac{\beta\,\gamma}{1+\gamma\,R}\,(\,r+\delta\,) + \frac{\beta\,\gamma\,\delta}{(\,1+\gamma\,R\,)^2}\right] < 0\ .\end{aligned} \tag{60}$$

It turns out that $\varphi(R)$ is strictly decreasing. Therefore, a necessary and sufficient condition for the existence of a stationary point in the positive orthant is given by

$$\frac{\alpha}{p\,b + c_p} > \frac{1}{a}. \tag{61}$$

For $a\alpha \leq c_p + pb$, there is no intersection of the isoclines because then the $\dot{u} = 0$-line is negative (just considering nonnegative values of R). But once $\varphi(0) > 0$, an equilibrium exists, since $\varphi(R)$ decreases to a negative constant and $\psi(R)$ increases to infinity. As they are strictly monotonic functions, the equilibrium is even unique.

Appendix B

Stability analysis

Starting with the system dynamics in the state-control space, which is given by (1) and (22), we can calculate the Jacobian matrix

$$\mathcal{J} = \begin{pmatrix} \dot{R}_R & \dot{R}_u \\ \dot{u}_R & \dot{u}_u \end{pmatrix} . \tag{62}$$

If it's determinat turns out to be positive, the stationary point (if it exists) is an unstable equilibrium. In contrast, if the determinat takes a negative value, it follows that we face a saddle-point equilibrium. Obviously, the determinant has to be evaluated in the considered equilibrium.

The partial derivatives

$$\dot{R}_R = -\delta < 0, \text{ and} \tag{63}$$

$$\dot{R}_u = p > 0 \tag{64}$$

are given by quite simple formulas. Computing the second row of the Jacobian matrix leads us to

$$\dot{u}_R = \frac{p}{\alpha}\left(\frac{1+a\,u}{a}\right)^2 \frac{\beta\,\gamma}{1+\gamma\,R}\left(r+\delta+\frac{\delta}{1+\gamma\,R}\right) > 0, \text{ and} \tag{65}$$

$$\begin{aligned}\dot{u}_u = {} & 2\,\frac{p}{\alpha}\left(\frac{1+a\,u}{a}\right)\left[\left(b+\beta\,\ln(1+\gamma\,R)+\frac{c_p}{p}\right)(\,r+\delta\,)+\frac{\beta\,\gamma\,\delta\,R}{1+\gamma\,R}\right] \\ & -(\,r+\delta\,) .\end{aligned} \tag{66}$$

The sign of the last expression cannot be determined in general because it depends on the particular numerical values. However, the determinant

$$\det \mathcal{J} = \dot{R}_R\,\dot{u}_u - \dot{R}_u\,\dot{u}_R \tag{67}$$

is negative if and only if

$$\dot{u}_u > \frac{\dot{R}_u\,\dot{u}_R}{\dot{R}_R} . \tag{68}$$

Remark: The right-hand side of the above inequality is negative.

With some simplifications and estimations it follows that

$$\frac{2}{\alpha}\,(\,p\,b + c_p\,) > 1 \tag{69}$$

is a sufficient but not necessary condition for a saddle-point equilibrium. Together with (29) a sufficient condition for the existence of a saddle-point equilibrium can be expressed as a pretty simple chain of inequalities:

$$\alpha\, a > p\, b + c_p > \frac{\alpha}{2}\,. \tag{70}$$

Remark: This also implies $a > \frac{1}{2}$.

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Figure Captions

Figure 1 Phase-portrait of the state-control space in case $\delta = 0.1$

Figure 2 Phase-portrait of the state-control space in case $\delta = 0$

Figure 1

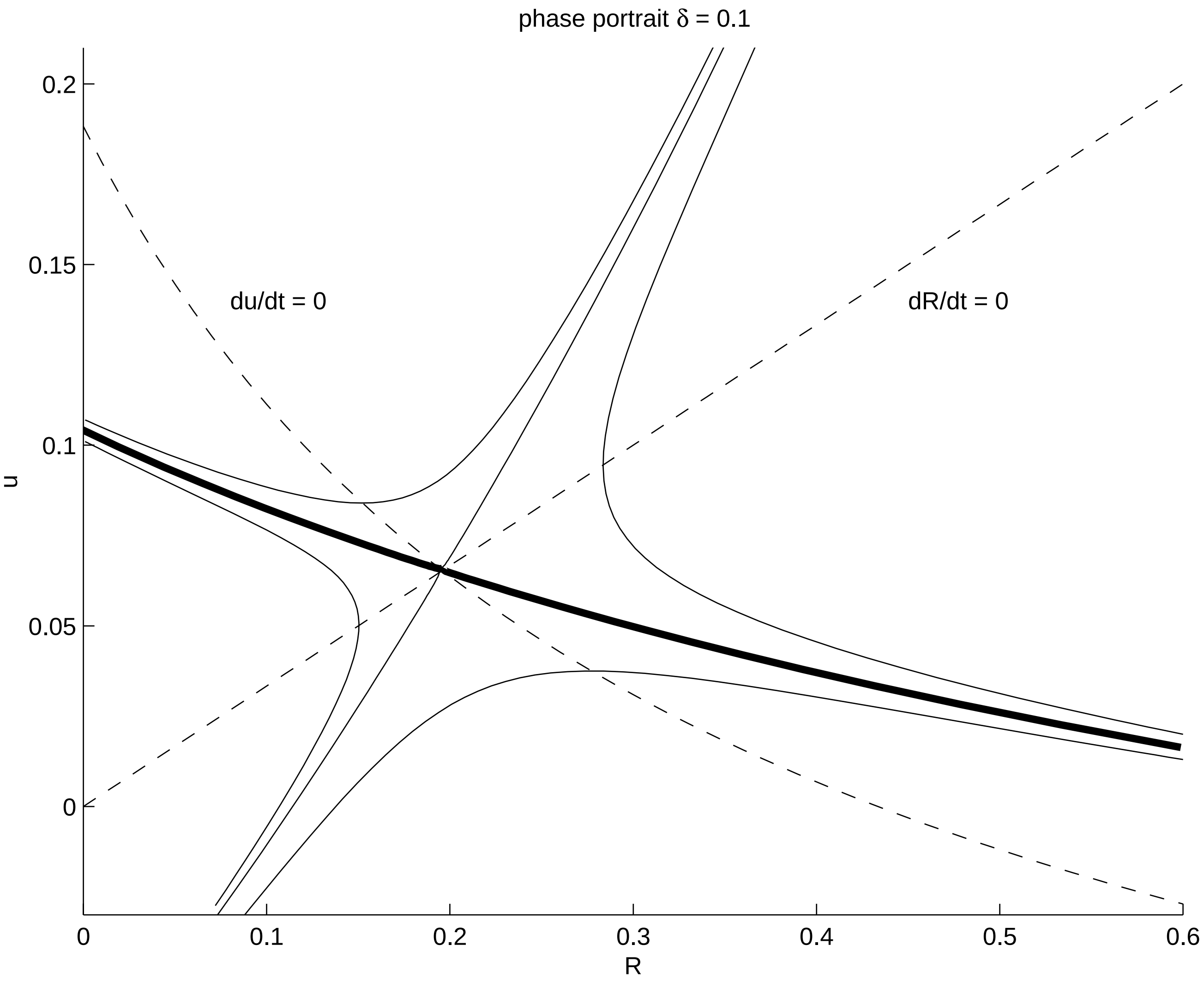
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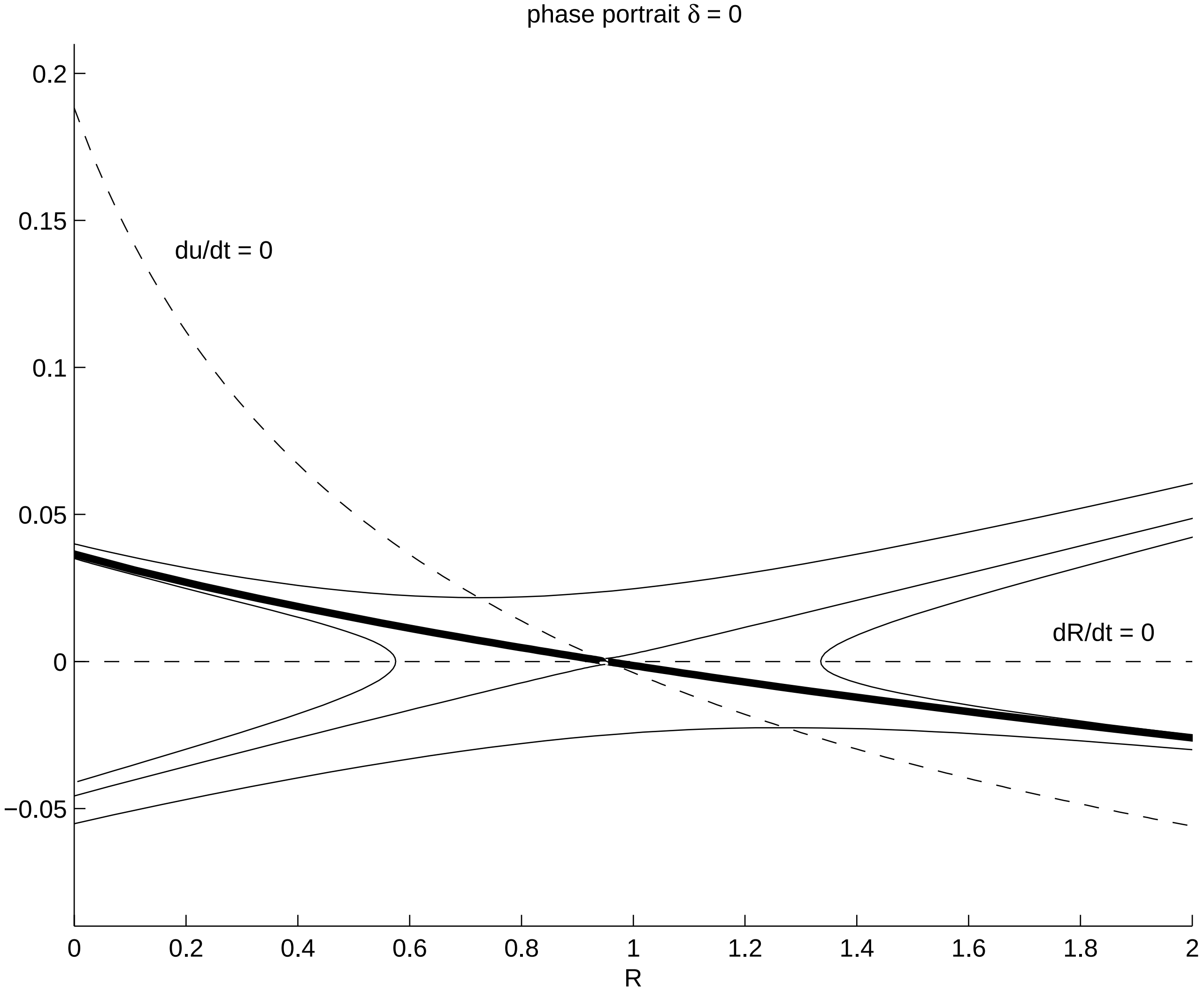
Figure 2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