A Dynamic Game of Offending and Law Enforcement *†

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Abstract

In this paper we analyse a differential game describing the interactions between a potential offender and the law enforcement agency. We assume that both players want to maximize their welfare expressed in monetary units, and compare the results obtained by applying the Nash equilibrium concept under symmetric with that under asymmetric information. The comparison reveals that under asymmetric information the offence rate is lower, due to the deterrence caused by the activities of the law enforcement agency. Both players' controls start at a steady state value and stick to it until close to the end of the planning horizon, which is when they leave the steady state to take into account the scrap value; this can be interpreted as a turnpike property of optimal Nash solutions. Furthermore, a sensitivity analysis is carried out. Among others, it turns out that a myopic offender tends to a higher offence level.

1 Introduction

Crime is a well known phenomenon that has occurred everywhere in the world during all epoches of history. Moreover, every organized society spends some

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efforts – e.g., crime investigation and punishment – to motivate people to submit to the law. In a path-breaking paper, Becker (1968) created an economic approach to crime and punishment. He asked how many resources and how much punishment should be assigned to enforce legislation. In particular, he stressed that there is an optimal trade-off between expenditures for law enforcement and the social loss due to offences. This loss contains the damages, the costs of investigation, apprehension, and conviction, and the costs of executing the imposed punishments. Some misleading results of Becker (1968), caused by ignoring intertemporal aspects, were brought into consideration by Leung (1991).

A static game between a law enforcement agency and a consumer of illicit drugs was analysed by Caulkins (1993). He found out that a punishment policy being a linear function of the amount of illicit drugs is more effective with respect to reduction of consumption of illicit drugs than a policy assigning the maximum punishment to every consumer. Moreover, he suggested that the punishment should be modeled as a function depending not only on the intensity but also on the offender's prior criminal record. This idea was adopted in Fent et al. (1999) to discover the optimal intertemporal strategy of a profit maximizing offender under a given, static punishment policy. A dynamic game of corruption in discrete time was studied by Bicchieri and Rovelli (1995), who posed that a stable equilibrium of corruption can vanish suddenly if a small population of honest individuals appears.

The present paper goes beyond most of the approaches mentioned in the previous paragraphs. While they examined static models (Becker (1968) and Caulkins (1993)), or optimal control models with only one agent (Fent et al. (1999)), we follow here an intertemporal approach of utility maximization, considering two players with conflicting objectives. The authorities attempt to minimize the social loss caused by criminal offences, whereas the offending individual wants to maximize the profit gained from offending or submitting to the law. This leads to a differential game, which makes it possible to study the competitive interactions in a dynamic framework. The criminal record takes the role of a state variable. A high record increases the punishment an offender expects in case of being convicted. The state variable increases with every detected offence and, depending on the national law, it may decrease due to a statute of limitation. In such a case, any entry in the record vanishes after a certain period (e.g., ten years). Another possible explanation of the decay is that judges, and perhaps even prosecutors, would be more likely to count less heavily convictions that are in the distant past.

This paper is organized as follows. In section 2 we set up the model describing the intertemporal game played between a potential offender and the central planner. Since the model fulfills the restrictions of linear quadratic games, it is possible to compute analytical solutions. This is done in section 3, where we compare two different equilibrium concepts. In section 4 we interpret the obtained findings. Finally, in section 5 we provide a brief summary of our results and outline possible extensions and generalizations of our framework.

2 The Model

We consider the behaviour of two rational players.[1] The first player is the potential offender who tries to maximize her profit gained from illegal activities while, on the other hand, the second player is the law enforcement agency (national authority) that tries to maximize public welfare. The offender's decision variable is u_1, the level of offence, and the legal authority chooses its level of crime investigation u_2.

Certainly, there is some interaction between the two opponent parties involved in crime and crime enforcement. Nevertheless, it might be questionable why the authority concentrates its activities — and therefore its spendings in investigation, prosecution, and execution of sentences — on one particular offender.

In case of an average pickpocket, this argument cannot be neglected; it is just impossible to set up an investigation strategy for each single mugger. However, there are indeed cases, in which the police even establishes a task force only to catch one person or one group of criminals. This occurs, for instance, in case of criminal series, taking of hostages, serial murders, or terrorism. Another situation to which our model applies is a conflict between a huge company and the state – e.g., a software company that integrates internet software tools into an operating system. Other examples are activitist groups with a significant political influence undertaking illegal actions or crimes. In all these situations the legal authorities spend particular efforts and allocate resources to one opponent.

The state x represents the offender's experience in committing crimes or his/her record of prior criminal offences. Since there is a very high correlation among these two variables, the particular interpretation will not make much difference. We will discuss this and present the dynamics of x in the next subsection.

In what follows, we omit the time argument t whenever feasible.

[1] The foundation for applying our game theoretical approach is the assumption that the interacting agents – the potential offender and the law enforcement agency – behave rationally. This requirement also plays an important role in the present paper.

System dynamics

As noted, the state variable has two possible interpretations. First, it can be considered as the record of prior crimes. Following Greenwood et al. (1994), we assume that the increase of this record only depends on the criminal activity and the number of convictions, but not on the punishment. A decline introduced by the term $-\delta x$ describes the limitation, for those cases, in which former crimes are only considered for a limited period.

The second interpretation is to regard x as the offender's level of experience. The experience increases in proportion to the intensity u_1. However, the offender will also forget, and the experience decays with time due to changes of law and technology. Thus, the value of experience will be reduced by a rate δ. Hence, the differential equation

$$\dot{x} = u_1 - \delta x, \quad x(t_0) = x_0 \tag{1}$$

describes the dynamics of the state x. In some cases, the discounting rate δ may be zero, which implies that each offence remains in the record forever and stigmatizes the offender. Interpreting x as the level of experience, this would mean that there are no changes in technology or justice. Obviously, this may be an unrealistic situation.

The offender's utility

The utility that an offender obtains from criminal activities consists of revenues minus costs. The revenues R of a potential offender increase with an increasing intensity of offence, u_1, but with a non-increasing marginal rate. Formally[2], $R = R(u_1)$, $R_{u_1} > 0$, and $R_{u_1 u_1} \leq 0$. The simplest function fulfilling these constraints is the linear function

$$R(u_1) = \gamma u_1, \; \gamma = \text{const.} > 0, \tag{2}$$

which will be used in what follows. Note that it could also make sense to let R depend on the state x. However, as it is not obvious whether x influences R negatively or positively (cf. Leung (1995), p. 71), we neglect the influence of x on R in this paper.

The offender's costs are divided into two terms, S and C, which are described next. The term S represents the costs connected with the sentence. These costs depend on the decision variables u_1 and u_2 and the state x, i.e., $S = S(x, u_1, u_2)$. This means the punishment an offender expects to suffer is a function of his own offending intensity, the rate of crime investigation, and

[2]Variables appearing as a subscript denote partial derivatives.

the criminal record. The offence level u_1 influences the probability of being convicted and the level of punishment, the investigation activities u_2 affect the probability of being convicted and prosecuted, and the criminal record x influences the level of punishment. We assume that the influences of the controls and the state are separable, leading to $S(x, u_1, u_2) = \pi(u_1, u_2)+\sigma(x)$, with non-negative functions π and σ.

It is reasonable to assume that the costs π are increasing with respect to the offending-intensity u_1, as well as with respect to the intensity of crime investigation u_2, with nondecreasing first order derivatives. Furthermore, the costs σ are assumed to be a linear function of criminal experience/record x. Summarizing, the assumptions are $\pi_{u_1} \geq 0$, $\pi_{u_2} \geq 0$, $\pi_{u_1u_1} \geq 0$, $\pi_{u_2u_2} \geq 0$, $\pi_{u_1u_2} \geq 0$, $\sigma_x > 0$, and $\sigma_{xx} = 0$.

The functions $\pi(u_1, u_2) := u_1^\alpha u_2^\beta$ ($\alpha \geq 1$, $\beta \geq 1$) and $\sigma(x) := \varphi x$ ($\varphi \geq 0$) meet all these requirements. Choosing $\alpha = \beta = 1$ yields

$$S(x, u_1, u_2) = u_1 u_2 + \varphi x. \tag{3}$$

With the formulation given in (3), the expression $S(x, u_1, u_2)$ takes positive values in case of a criminal record $x > 0$ even in case $u_1 = 0$. This represents the disadvantages someone might experience in the legitimate sector after having been convicted, for instance in the labour market.

The second cost term C represents costs that are not related to incarceration or conviction. For instance, C might be the cost of tools, consulting, or just the value of the offender's time. These costs are an increasing function of the criminal intensity, i.e. $C = C(u_1)$ and $C_{u_1} > 0$. We assume for simplicity that these costs do not depend on the experience x, and consider two particular cost-functions:

$$C(u_1) = \psi u_1 \quad \text{or} \quad C(u_1) = \psi u_1^2. \tag{4}$$

The linear cost will be analysed in model 1, while we will elaborate on the quadratic one in models 2 and 3.

Finally, for a complete model formulation we need to define the salvage value $Q_1(x(T))$, which describes the value of the state x at the end of the planning period, T. In our case, it represents the damage or harm (or the utility) caused to the offender by the criminal record at time T. This can be the point in the offender's life when she decides to stop her criminal career and seeks a legal employment, or it can just be the end of the planning horizon. We assume that

$$Q_1(x(T)) := -\eta \, x(T). \tag{5}$$

If the criminal record $x(T)$ has a negative impact, η must be positive. In the special case when the offender does not consider the criminal record to be something bad at all, the parameter η can be set less or equal zero.

Summarizing, the offender's objective functional J_1 is described by

$$J_1 = \int_{t_0}^{T} e^{-r_1 t}\Big[R(u_1) - S(x, u_1, u_2) - C(u_1)\Big]\, dt + e^{-r_1 T} Q_1(x(T)) \mapsto \max_{u_1},$$

where $r_1 \geq 0$ denotes the discount rate. Under the assumptions (2), (3), (4), (5) this objective functional becomes

$$J_1 = \int_{t_0}^{T} e^{-r_1 t}\Big[\gamma u_1 - u_1 u_2 - \varphi x - \psi u_1\Big]\, dt - e^{-r_1 T}\eta\, x(T) \mapsto \max_{u_1}$$

or

$$J_1 = \int_{t_0}^{T} e^{-r_1 t}\Big[\gamma u_1 - u_1 u_2 - \varphi x - \psi u_1^2\Big]\, dt - e^{-r_1 T}\eta\, x(T) \mapsto \max_{u_1},$$

respectively.

The law enforcement agency

So far, the model is very similar to the one in Fent et al. (1999), where an optimal control model is applied to derive the optimal offending strategy of an individual reacting to a given punishment policy. In this paper, we go one step further by considering the authority to be another utility maximizing agent, which results in a two player game. The objective functional of the law enforcement agency is of the type

$$J_2 = -\int_{t_0}^{T} e^{-r_2 t}\Big[D(u_1) + K(x, u_2) + L(u_1, u_2)\Big]\, dt \mapsto \max_{u_2}, \tag{6}$$

where $r_2 \geq 0$ denotes the authority's discount rate, and all terms are costs for the state. In this objective functional there is no scrap value for $x(T)$, since we assume that the criminal record and the experience of a previously convicted offender at the end of her career do not influence public welfare.

The damage D caused by illegal activities increases with the offence-rate, and we assume it is convex, i.e., the marginal damage is nondecreasing. Thus, $D = D(u_1)$, $D_{u_1} > 0$, $D_{u_1 u_1} \geq 0$. For the analysis in section 3 we will use

$$D(u_1) := \zeta u_1^2. \tag{7}$$

The costs of law-enforcement, K, increase and have nondecreasing marginal costs, i.e., $K_{u_2} > 0$ and $K_{u_2 u_2} \geq 0$. Furthermore, a very experienced offender might be more difficult to arrest; on the other hand, the higher the level of criminal experience already is, the smaller the advantage of one additional unit of experience, i.e., $K_x > 0$ and $K_{xx} \leq 0$. In what follows we will use

$$K(x, u_2) = \vartheta u_2^2 + \nu x. \tag{8}$$

The last term $L(u_1, u_2)$ in the law enforcement agency's objective functional reflects the costs of imposing a certain punishment. For instance, the costs implied by maintaining prisons might be included here. In case of a fine, these costs may be zero, otherwise we assume they are linear in both decision variables, i.e.,

$$L(u_1, u_2) = \omega u_1 u_2. \tag{9}$$

Inserting (7), (8), (9) into (6) yields

$$J_2 = -\int_{t_0}^{T} e^{-r_2 t} \left[\zeta u_1^2 + \vartheta u_2^2 + \nu x + \omega u_1 u_2 \right] dt \mapsto \max_{u_2}$$

as the objective functional of the state.

3 Analysis

We will elaborate on three different models in this paper. In the first one, the cost function of the offender is linear in u_1. This leads to a bang-bang optimal control. As we will see, the solution seems plausible and allows meaningful economic interpretations. Later on, we use quadratic costs and compare solution concepts with symmetric and asymmetric information, respectively.

For the analysis we define the players' Hamiltonians, given by

$$\begin{aligned} \mathcal{H}_1 &= R(u_1) - S(x, u_1, u_2) - C(u_1) + \mu_1(u_1 - \delta x) \\ &= \gamma u_1 - (u_1 u_2 + \varphi x) - \psi u_1 + \mu_1(u_1 - \delta x), \end{aligned} \tag{10}$$

for the linear case, and

$$= \gamma u_1 - (u_1 u_2 + \varphi x) - \psi u_1^2 + \mu_1(u_1 - \delta x), \tag{11}$$

for the quadratic case.

$$\begin{aligned} \mathcal{H}_2 &= -D(u_1) - K(x, u_2) - L(u_1, u_2) + \mu_2(u_1 - \delta x) \\ &= -\zeta u_1^2 - (\vartheta u_2^2 + \nu x) - \omega u_1 u_2 + \mu_2(u_1 - \delta x). \end{aligned} \tag{12}$$

In (10), (11), and (12), $\mu_1 = \mu_1(t)$ and $\mu_2 = \mu_2(t)$ are the costate variables.

Model 1: Symmetric information, linear cost function

In this subsection, both players simultaneously maximize their Hamiltonians. The authority does not gain directly from crime investigations, only indirectly because of the possible reduction of offences. However, in a game with symmetric information those indirect effects do not influence the authority's Hamiltonian.

Proposition 1 *A Nash equilibrium of the model defined in section 2 with a linear cost function $C(u_1) = \psi u_1$ is given by*

$$u_2^* \equiv 0, \tag{13}$$

and

$$u_1^* = \left\{ \begin{array}{c} \overline{u_1} \\ \textit{undefined} \\ 0 \end{array} \right\} \textit{if } \gamma \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \psi - \mu_1, \tag{14}$$

where $\overline{u_1}$ denotes the upper bound on u_1.
Proof: *see appendix B.*

Thus, it is optimal for the offender to choose a positive offence level u_1 when the marginal profit γ exceeds the sum of the marginal costs ψ and the absolute value of the shadow price $-\mu_1$. The singular case, in which there exists a time interval of positive length where the control u_1 is undefined, only occurs in the special case $\eta = \frac{\varphi}{r_1+\delta} = \gamma - \psi$.

The optimal solution given by (14) can be completely characterized. Solving the costate equations yields[3]

$$\mu_1(t) = \left(\frac{\varphi}{r_1 + \delta} - \eta \right) e^{(r_1+\delta)(t-T)} - \frac{\varphi}{r_1 + \delta} \tag{15}$$

and

$$\mu_2(t) = \frac{\nu}{r_2 + \delta} \left(e^{(r_2+\delta)(t-T)} - 1 \right) \tag{16}$$

As expected, equations (15) and (16) show that $\mu_1, \mu_2 \leq 0 \, \forall \, t \in [t_0, T]$, i.e., an additional unit of the criminal record or experience x is negatively evaluated by both players, throughout the planning period. Thus, having a negative shadow price, x is a bad stock.

[3]The computations can be found in appendix C.

Considering a very long planning horizon T. At the beginning, the costates nearly remain constant, but when time t approaches T, the costate trajectories move towards $-\eta$ and zero, respectively. The costate of the player i with the bigger discount rate r_i remains longer at an "almost" constant level. The intuition behind is that from the viewpoint of the myopic agent the end of the planning period seems to be further away. Hence, the preparations for the time after the planning period start later.

Looking at equation (14), and using the costate trajectory (15), we conclude that for the offence level u_1, four different scenarios are possible.

- u_1 is zero over the whole time interval $[t_0, T]$, if $\gamma < \psi - \mu_1$ for all t;
- u_1 remains constant at its upper bound $\bar{u}_1$, if $\gamma > \psi - \mu_1$ for all t;
- u_1 starts at zero, jumps to the upper bound when $\gamma = \psi - \mu_1$, and remains there until $t = T$ (this can only occur when the per unit scrap value η is small, which implies that μ_1 is strictly increasing); or
- if η is large, the costate μ_1 always decreases, so the control u_1 starts at the upper bound, remains there until $\gamma = \psi - \mu_1$, and equals zero until the end.

The first two patterns are characterized by a constant control u_1 for the whole planning period. This happens when the difference between γ, the parameter determining the utility of the offender, and ψ, determining the costs not related to punishment, is large enough to compensate the changes performed by μ_1 during the planning period. In the third and the fourth pattern this difference is smaller. Thus, the changes in μ_1 influence the sign of $\partial\mathcal{H}/\partial u_1 = \gamma - \psi + \mu_1$. Therefore, the control u_1 jumps from one boundary to the other one.

We will next introduce a quadratic cost function for the offender to get some more complex — and perhaps also more realistic — optimal control trajectories.

Model 2: Symmetric information, quadratic cost function

To obtain more general policies, from now on we will use the quadratic cost function $C(u_1) = \psi u_1^2$. Then the Hamiltonians (11) and (12) become

$$\mathcal{H}_1 = \gamma\, u_1 - u_1\, u_2 - \varphi\, x - \psi\, {u_1}^2 + \mu_1\,(u_1 - \delta\, x) \qquad (17)$$

and

$$\mathcal{H}_2 \;=\; -\zeta\, {u_1}^2 - \vartheta\, {u_2}^2 - \nu\, x - \omega\, u_1\, u_2 + \mu_2\, (u_1 - \delta\, x). \tag{18}$$

Clearly, as in the previous model, the authority decides not to investigate at all. The offender's optimal control now becomes

$$u_1^* = \frac{\gamma + \mu_1}{2\,\psi}\,. \tag{19}$$

The dynamics of the costates μ_1 and μ_2 are given by (15) and (16)). Substituting (15) into (19) yields the optimal offence rate as a function of time, and the resulting state dynamics can be derived.

Proposition 2 *A Nash equilibrium under symmetric information of the model defined in section 2 with a quadratic cost function $C(u_1) = \psi u_1^2$ is given by*[4]

$$\begin{aligned} u_1^*(t) &= \frac{1}{2\,\psi}\left[\gamma + \left(\frac{\varphi}{r_1+\delta} - \eta\right)\, e^{(r_1+\delta)(t-T)} - \frac{\varphi}{r_1+\delta}\right], \\ u_2^*(t) &\equiv 0 \end{aligned} \tag{20}$$

and

$$\begin{aligned} x(t) \;=\; & \frac{1}{2\psi(r_1+\delta)}\left[\left(\frac{\gamma\, r_1 - \varphi}{\delta} + \gamma\right)(1 - e^{\delta(t_0-t)}) + \frac{\varphi - \eta\,(r_1+\delta)}{r_1 + 2\,\delta}\right. \\ & \left.\cdot\left(e^{(r_1+\delta)(t-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-t)}\right)\right] + x_0\, e^{\delta(t_0-t)}. \end{aligned} \tag{21}$$

This solution is time consistent.
Proof: *see appendices D and E.*

Note that the controls u_1 and u_2 do not depend on the initial state x_0, because the costate equations are decoupled from the state equation along an optimal pair (u_1, u_2).

As will be outlined in detail in section 4, the controls and the state tend towards an almost constant level. We will call these levels $\hat{u}_1$, $\hat{u}_2$, and $\hat{x}$,

[4] Obviously, we restrict ourselves to solutions fulfilling $u_1 \in [0, \infty)$. Since $t - T \leq 0$, we can conclude that $e^{(r_1+\delta)(t-T)} \in [0, 1]$. In order to make sure that equation (20) delivers feasible solutions it must hold that $\gamma \geq max\{\frac{\varphi}{r_1+\delta}, \eta\}$. Otherwise, we have to deal explicitly with the constraint $u_1 \geq 0$.

respectively. They can be derived analytically by computing the limits for $T \to \infty$ and $\dot{x} = 0$, which leads to

$$\begin{aligned} \hat{u}_1 &= \frac{1}{2\psi}\left[\gamma - \frac{\varphi}{r_1 + \delta}\right], \\ \hat{u}_2 &= 0\,, \text{and} \\ \hat{x} &= \frac{\hat{u}_1}{\delta}\,. \end{aligned} \tag{22}$$

Both models analysed so far have the feature that the optimal decision of the legal authority is having no activities. Clearly, this stems from the objective functional (6), which only contains terms expressing costs. The Nash equilibrium does not take into account any indirect effects. In the following subsection, we will introduce asymmetric information to overcome this deficiency.

Model 3: Asymmetric information, quadratic cost function

In contrast to the previous subsections, now the government has the possibility to foresee the offender's reactions. This means that now the optimization is done hierarchically. We assume that the offender is the follower, and the authority the leader, in a Stackelberg differential game. In particular, the decision of the follower (player 1) is substituted into the leader's (player 2's) Hamiltonian before the leader's optimal decision is computed (for details, see appendix F).

With this asymmetric information concept, the offender reduces her offence rate as a consequence of law enforcement, which, in turn, reduces the harm caused by illegal activities. From (17) we can derive the offender's optimal control, assuming $u_1 > 0$:

$$\frac{\partial \mathcal{H}_1}{\partial u_1} = 0 \quad \Rightarrow \quad u_1^* \;=\; \frac{\gamma - u_2 + \mu_1}{2\psi}. \tag{23}$$

Compared to (19), we see that u_1^* now has been reduced by $u_2/2\psi$. As expected, the activities of the authority reduce the offence rate. The costate dynamics are the same as in the previous models. Therefore, if we apply the solutions (15) and (16), we get both players' decision variables as a function of time.

Proposition 3 *A Nash equilibrium with asymmetric information in the model of section 2 with the quadratic cost function $C(u_1) = \psi u_1^2$ is given* *by*[5]

$$
\begin{aligned}
u_1^*(t) &= \frac{1}{2\,(\zeta+4\vartheta\,\psi^2-2\,\omega\,\psi)}\left\{\left[\gamma+\left(\frac{\varphi}{r_1+\delta}-\eta\right)\,e^{(r_1+\delta)(t-T)}-\frac{\varphi}{r_1+\delta}\right]\right.\\
&\quad \left.(4\,\vartheta\,\psi-\omega)+\frac{\nu}{r_2+\delta}\left(e^{(r_2+\delta)(t-T)}-1\right)\right\}, && (24)\\
u_2^*(t) &= \frac{1}{\zeta+4\vartheta\psi^2-2\omega\psi}\left\{(\,\zeta-\omega\,\psi\,)\left[\gamma+\left(\frac{\varphi}{r_1+\delta}-\eta\right)\,e^{(r_1+\delta)(t-T)}-\frac{\varphi}{r_1+\delta}\right]\right.\\
&\quad \left.-\frac{\psi\,\nu}{r_2+\delta}\left(e^{(r_2+\delta)(t-T)}-1\right)\right\}, && (25)
\end{aligned}
$$

and

$$
\begin{aligned}
x(t) &= \frac{B_0}{\delta}\left(1-e^{\delta(t_0-t)}\right)+\frac{B_1}{r_1+2\delta}\left(e^{(r_1+\delta)(t-T)}-e^{(r_1+\delta)(t_0-T)+\delta(t_0-t)}\right)\\
&\quad +\frac{B_2}{r_2+2\delta}\left(e^{(r_2+\delta)(t-T)}-e^{(r_2+\delta)(t_0-T)+\delta(t_0-t)}\right)+x_0\,e^{\delta\,(t_0-t)}\,. \qquad (26)
\end{aligned}
$$

In (26)

$$
\begin{aligned}
B_0 &= \frac{1}{2}\,\frac{(\gamma-\dfrac{\varphi}{r_1+\delta})\,(4\,\vartheta\,\psi-\omega)-\dfrac{\nu}{r_2+\delta}}{\zeta+4\,\vartheta\,\psi^2-2\,\omega\,\psi},\\
B_1 &= \frac{1}{2}\,\frac{(4\,\vartheta\,\psi-\omega)\,\left(\dfrac{\varphi}{r_1+\delta}-\eta\right)}{\zeta+4\,\vartheta\,\psi^2-2\,\omega\,\psi}, \qquad (27)
\end{aligned}
$$

and

$$
B_2 = \frac{1}{2}\,\frac{\nu}{(\zeta+4\,\vartheta\,\psi^2-2\,\omega\,\psi)\,(r_2+\delta)}.
$$

are constants. This solution is also time consistent.
Proof: *see appendices F and G.*

If T is infinite, the steady states of the optimal trajectories presented in Proposition 3 are given by

$$
\hat{u}_1 = \frac{1}{2\,(\zeta+4\vartheta\,\psi^2-2\,\omega\,\psi)}\left\{\left[\gamma-\frac{\varphi}{r_1+\delta}\right](4\,\vartheta\,\psi-\omega)-\frac{\nu}{r_2+\delta}\right\},
$$

[5]The parameters of the model must be chosen to provide non-negative values of u_1^* and u_2^*. A sufficient condition is: $\zeta+4\vartheta\psi^2\geq 2\omega\psi$, **AND** $\gamma\geq\max\{\frac{\varphi}{r_1+\delta},\eta\}+\frac{\psi\nu}{r_2+\delta}\max\{\frac{1}{\zeta-\omega},\frac{1}{4\vartheta\psi^2-\omega\psi}\}$, **AND** $\zeta\geq\omega\psi$.

$$\hat{u}_2 = \frac{1}{\zeta + 4\vartheta\psi^2 - 2\omega\psi} \Big\{ (\zeta - \omega\,\psi) \left[\gamma - \frac{\varphi}{r_1 + \delta}\right] - \frac{\psi\,\nu}{r_2 + \delta} \Big\}, \tag{28}$$

and

$$\hat{x} = \frac{\hat{u}_1}{\delta}.$$

We close this subsection with remarking that by applying a Stackelberg solution concept, we obtained nontrivial solutions for both player's controls. As will be demonstrated in the following section, these solutions are plausible, and also their sensitivity with respect to shifts of the parameters is intuitive.

4 Interpretations

In the numerical calculations that are following, we restrict ourselves to models 2 and 3, because model 1 has a different (i.e., linear) cost function. We start with demonstrating that the trajectories show some kind of *"steady state"* behaviour. That means, we investigate whether the results given in (22) and (28) can also be observed when plotting the trajectories for specific parameter values. For that purpose, we start with a rather long planning horizon of 100 years. The full set of parameter values, which were used for the numerical analysis, is listed in the following table.

r_1	0.04	r_2	0.03	T	100	t_0	0	x_0	0
γ	15	δ	0.1	ζ	1.2	η	1	ϑ	1
ν	1	φ	1	ψ	1	ω	1		

Note that since $r_1 > r_2$, the potential offender is slightly more myopic than the social planner.

The graphs in Figure 1 show the trajectories of the state x, the controls u_1 and u_2, and the costates μ_1 and μ_2, with initial states x_0 varying from 0 to 80. Since the initial state only influences the state x, but not the optimal controls or the costates, we used a surface plot in the upper pictures to illustrate state trajectories for different initial states.

In case of symmetric information (Model 2), the plots of the variables as functions of time are depicted in the graphs on the left-hand side of Figure 1. The plots for the case of asymmetric information, on the other hand, are shown in the graphs on the right-hand side.

Figure 1 about here

In both models, the state x — illustrated by a surface plot — first approaches the steady state value, then remains almost stable for a certain time, and finally, it increases along a convex path just before the end of the planning horizon. The first movement can be explained by the fact that we start with a "criminal stock" that deviates from the steady state, and it takes some time to reach that equilibrium. The later increase, on the other hand, follows from the smaller harm one unit of crime causes after ending the criminal career, which is represented by the scrap value η.

The most remarkable difference between symmetric and asymmetric information is represented by the curve indicating the activity of the law enforcement agency, $u_2(t)$, which is shown by the dotted lines in the lower graphs. As pointed out, it is zero when assuming a symmetric information pattern — due to the negative impact of the enforcement costs on the objective functional of the social planner. In contrast, when assuming assymetric information, it remains at a positive level before it vanishes at the end of the planning period. In the model with asymmetric information, it is favourable for the social planner to investigate and reduce criminal activities. The decline at the end of the planning period is implied by the assumption that the criminal record does not have a salvage value for the authority.

Although the shape of the state-trajectory looks similar in both graphs in Figure 1, there is a difference when we look at the particular values of x. We see that in the scenario of symmetric information, the criminal record reaches higher values. This is clearly due to the fact that the authorities do not establish any law enforcement at all and, therefore, the potential offender can afford a higher crime rate and a higher criminal record as well. In case of asymmetric information, the activities of the law enforcement agency deter potential offenders. The final increase also is more significant with asymmetric information.

Starting with an initial criminal stock higher than the steady state results in an initial decline in x. Nevertheless, the offence rate $u_1(t)$ — illustrated by a dashed line — remains at the same level, irrespective of the initial state x_0. This is a consequence of the fact, that the optimal controls are independent of the (initial) state (see (20) and (24)).

If we increase the scrap value parameter η, the state dynamics change significantly. In Figure 2 the trajectories of both models are illustrated with $\eta = 12$ (which is higher than the base level $\eta = 1$). Instead of an increase of x at the end, there is now a decline, since the offender wants to reduce her criminal record before entering into the legal job market.

Figure 2 about here

Another interesting aspect of the change in the parameter η is the behaviour of the shadow price μ_1 — indicated by the upper dash-dotted line — when t approaches T. In case of $\eta = 1$, the curve increases, reflecting the small negative impact of $x(T)$. When $\eta = 12$, on the other hand, the costate decreases, because the harm of one unit of the state x is bigger and, consequently, the offender tries more actively to reduce the criminal record before the end of the planning period. (This also follows from the transversality condition $\mu_1(T) = -\eta$.)

In Figure 3 we have phase portraits, in which the controls u_1 and u_2 are plotted against the state x. The graphs in the first column illustrate the dynamics under symmetric information. The upper picture indicates that in the beginning, the increase of the state x is much faster than the movement of the offence level u_1. Later, the system remains in a neighbourhood of $(\hat{x}, \hat{u}_1)$ and, finally, the trajectory moves upwards. This is due to the increase of the offence level, which is chosen in case of a small scrap value parameter η.

Figure 3 about here

At first sight, the almost linear movement may be surprising, since in the previous diagrams we had smooth curves. However, if we look at the equations describing the dynamics, we find time functions of the form $e^{\delta(t-T)}$, multiplied by some scaling factors. Thus, in phase portraits which do not show the time explicitly, the nonlinear terms do not influence the paths significantly.

The second column in Figure 3 shows the equivalent phase portraits under asymmetric information. Again, we can see an almost linear movement at the beginning and at the end, and a *steady state* at the intersection of the straight lines.

Concluding, there is a *steady state* in which the system could remain for a long time, depending on the length of the planning horizon. At the beginning, there is a deviation due to the initial value x_0. Just before the end of the planning period, there is another deviation (mostly depending on the scrap value η). This can be interpreted as a *turnpike* property of optimal trajectories.

Next, we look at a scenario, in which the planning horizon is 50 years. The trajectories can be found in Figures 4 and 5. Figure 4 shows that the steady state of x has disappeared because the time interval is too short to remain in the steady state on a time interval of positive length. The increase at the beginning fades smoothly into the increase at the end. Consequently, in the phase portraits in Figure 5 there is no abrupt jump from one straight line to another one, and there is no temporary equilibrium either.

Figure 4 about here

Figure 5 about here

Finally, we look at the sensitivity of the solutions with respect to the interest rates r_1 and r_2. In Figure 6, the discount rate r_1 was fixed at a level of 0.04, while we varied the discount rate r_2 from 0 to 1. In Figure 7, on the other hand, r_2 was fixed at 0.03, while r_1 was changed from 0 to 1.

Figure 6 about here

A change of the social planner's interest rate r_2 obviously has no effects in case of the symmetric information pattern, as shown by the pictures in the first column of Figure 6. The social planner always chooses $u_2 = 0$ and, therefore, it does not make any difference whether the social planner is far-sighted or myopic. When the asymmetric information pattern is used, then a social planner with a long-term focus (small r_2) will choose a high investigation rate (see the second graph in the second line). In the long run, reduced criminal activities (reduced harm of the public economy) have a higher value than the costs caused by the investigation. The increased investigation leads to a lower offence rate, as shown in the upper graph of the second column, and as already known from (23).

Figure 7 about here

Changing the interest rate of the offender, r_1, always has a significant influence, no matter which information structure is applied, as demonstrated in Figure 7. In both cases, a myopic offender chooses a higher offence rate. This behaviour is very plausible, because the punishment of a crime is always received with some delay, while the benefits can be consumed immediately. In a comprehensive study of the effects of the timing of rewards and punishment, Davis (1988) showed, that *the expected income from crime must be discounted at a higher rate than the rate of discount for income from legal endeavours and, more crucially, at a rate that will vary with the crime rate.*

However, this delay is not included in our model. Therefore, the lower offence level only originates from the negative impact of the criminal record x. Individuals who maximize profit in the long term pay more attention to this record than myopic decision makers. Nevertheless, an offender who is interested to maximize her short term payoff has a higher offence rate. In case of asymmetric information, the social planner reduces the investigation when r_1 is low. This is just a response to the low offence rates.

Before we move on to some possible extensions to the approach presented in this paper, we point out that the parameter δ describing the rate of decline of the criminal record always occurs in the form $(r_i + \delta)$. Thus, a change in δ has the same effect as a simultaneous variation of r_1 and r_2.

5 Concluding Remarks and Possible Extensions

In this paper we have analyzed three variations of a differential game taking into account the intertemporal aspects of crime, investigation, and punishment. We have shown under which circumstances the agents (potential offender vs. law enforcement agency) are motivated to engage in crime and crime prevention. Basically, it turned out that a rational potential offender undertakes illegal activities if and only if the discounted stream of expected utility exceeds the discounted stream of punishment. The authority, on the other hand, only takes action against criminal behaviour when it has some information about the reaction of the offender.

So far we have only been looking at open-loop solutions. This means, both players define the trajectory of their decision variables without taking into account possible perturbations that might result in a different policy. Whether or not the conclusions of our analysis remain valid in a closed-loop setup is a topic for future research.

A possible generalization of our model would be to make the offender's cost function depending on the state x as well. This would provide the possibility to observe learning effects in the offending process. Like in industry, a criminal individual can be assumed to become more efficient in her work when accumulating more experience. A control model taking into account this type of decreasing unit "production" costs of the offender can be found in Friedman et al. (1989).

A promising improvement would be to deal with a population of offenders rather than only one individual. We admit that apart from the already mentioned types of offences (which may gain a lot of public attention), it is unrealistic to assume that the law enforcement agency focuses on one individual person. Elaborating such a population based model requires to formulate a distribution of the offenders with respect to their criminal record and/or their age. Then the dynamics can be described by means of a partial differential equation. To avoid such a tricky approach, one could start with considering only two classes of offenders to get some first clues about the development of a heterogenous population of offenders.

A Explanation of the variables and constants

$A_1, \ldots, A_5$	$\ldots$	constants appearing in the solutions of the differential equations
B_1, B_2, B_3	$\ldots$	constants used to simplify the system dynamics in model 3
$C(x, u_1)$	$\ldots$	those costs depending on sentence and conviction
$D(u_1)$	$\ldots$	the public damage caused by criminal activities with intensity u_1
e	$\ldots$	Euler's constant
$\mathcal{H}_i$	$\ldots$	Hamiltonian of player i
J_i	$\ldots$	objective functional of player i
$K(x, u_2)$	$\ldots$	costs of crime investigation with an intensity u_2 against an offender with a level x of experience
$L(u_1, u_2)$	$\ldots$	the public of giving a punishment
$Q_1(x(T))$	$\ldots$	scrap value of a criminal record of $x(T)$ at time T
$R(u_1)$	$\ldots$	utility of an offender gained from criminal activity with intensity u_1
r_i	$\ldots$	discount rate of player i
$S(x, u_1, u_2)$	$\ldots$	the offenders costs that do not depend on sentence and conviction
T	$\ldots$	end of the planning horizon
t	$\ldots$	time
t_0	$\ldots$	beginning of the planning horizon
u_1	$\ldots$	the offenders offence level
u_2	$\ldots$	intensity of law enforcement
u_1	$\ldots$	the offenders offence level
$\underline{u_1}$	$\ldots$	lower bound for u_1
$\overline{u_1}$	$\ldots$	upper bound for u_1
x	$\ldots$	experience or criminal record of the offender
α, β	$\ldots$	parameters of $\pi(u_1, u_2)$
$\alpha_1, \alpha_2, \beta_1, \beta_2$	$\ldots$	parameters specifying player i's decision in the closed-loop models
$\alpha_3, \alpha_4, \beta_3, \beta_4$	$\ldots$	parameters specifying the costates μ_1 and μ_2 in the closed-loop models
γ	$\ldots$	parameter of the offenders utility
δ	$\ldots$	forgetting rate of experience or criminal record
ζ	$\ldots$	parameter of $D(u_1)$
η	$\ldots$	parameter of $Q_1(x(T))$

ϑ	...	parameter of $K(x, u_2)$
μ_i	...	costate variable of player i
ν	...	parameter of $K(x, u_2)$
$\pi(u_1, u_2)$	...	costs not depending on sentence and conviction
$\sigma(x)$	...	costs not depending on sentence and conviction
φ	...	parameter of $\sigma(x)$
ψ	...	parameter of $C(x, u_1)$
ω	...	parameter of $L(u_1, u_2)$

B Proof of Proposition 1

Computing the first order partial derivatives of the Hamiltonians (10) and (12) with respect to each of the players' decision variables, we get

$$\frac{\partial \mathcal{H}_1}{\partial u_1} = \gamma - u_2 - \psi + \mu_1$$

and

$$\frac{\partial \mathcal{H}_2}{\partial u_2} = -2\vartheta u_2 - \omega u_1 \,,$$

where the latter implies

$$u_2^* = \arg\max_{u_2 \geq 0} \mathcal{H}_2(x^*, u_1, u_2, \mu_1, \mu_2) \equiv 0, \quad \text{since } \frac{\partial \mathcal{H}_2}{\partial u_2} \leq 0.$$

The legal planner's optimal control is zero. Consequently, the offender faces an optimal control problem with the objective functional

$$J_1 = \int_0^T e^{-r_1 t}\Big[\gamma u_1 - \varphi x - \psi u_1\Big]\, dt - e^{-r_1 T}\eta\, x(T) \mapsto \max_{u_1}$$

and the Hamiltonian

$$\mathcal{H}_1 = \gamma u_1 - \varphi x - \psi u_1 + \mu_1(u_1 - \delta x).$$

Again, we compute the first order partial derivative with respect to the control variable,

$$\frac{\partial \mathcal{H}_1}{\partial u_1} = \gamma - \psi + \mu_1,$$

which provides the offender's optimal solution

$$\begin{aligned}
u_1^* &= \arg\max_{u_1 \geq 0} \mathcal{H}_1(x^*, u_1, u_2, \mu_1, \mu_2) \\
&= \left\{ \begin{array}{c} \overline{u_1} \\ \text{undefined} \\ 0 \end{array} \right\} \text{ if } \gamma - \psi + \mu_1 \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0. \quad \square
\end{aligned}$$

C The costate dynamics

The adjoint equations are

$$\dot{\mu}_1 = r_1\mu_1 - \frac{\partial \mathcal{H}_1}{\partial x} = \varphi + (r_1 + \delta)\mu_1$$

and

$$\dot{\mu}_2 = r_2\mu_2 - \frac{\partial \mathcal{H}_2}{\partial x} = \nu + (r_2 + \delta)\mu_2 \, .$$

This leads to the solutions

$$\mu_1 = -\frac{\varphi}{r_1 + \delta} + A_1 e^{(r_1+\delta)t} \tag{29}$$

and

$$\mu_2 = -\frac{\nu}{r_2 + \delta} + A_2 e^{(r_2+\delta)t}. \tag{30}$$

The symbols A_1 and A_2 are the integration constants, which can be obtained from the transversality condition

$$\mu_i(T) = \frac{\partial}{\partial x} Q_i(x^*(T)).$$

In the above formula, Q_i means the scrap value of criminal experience at the end of the planning horizon (see equation (5)). From equation(29) it follows that

$$\mu_1(T) = -\frac{\varphi}{r_1 + \delta} + A_1 e^{(r_1+\delta)T}, \tag{31}$$

which must be equal to

$$\frac{\partial}{\partial x} Q_1(x^*(T)) = -\eta$$

resulting in

$$A_1 = \left(\frac{\varphi}{r_1 + \delta} - \eta \right) e^{-(r_1+\delta)\,T}. \tag{32}$$

In case of the law enforcement agency, deriving the integration constant becomes easier since there is no scrap value, thus

$$\mu_2(T) \;=\; -\frac{\nu}{r_2+\delta} + A_2\, e^{(r_2+\delta)\,T} = 0,$$

which leads to

$$A_2 \;=\; \frac{\nu}{r_2+\delta}\, e^{-(r_2+\delta)\,T}\,. \tag{33}$$

Substituting (32) and (33) into equations (29) and (30) results in

$$\begin{aligned}
\mu_1 &= \left(\frac{\varphi}{r_1+\delta} - \eta\right)\, e^{(r_1+\delta)(t-T)} - \frac{\varphi}{r_1+\delta}\,, \\
\mu_2 &= \frac{\nu}{r_2+\delta}\left(e^{(r_2+\delta)(t-T)} - 1\right)\,. \qquad \square
\end{aligned}$$

D Calculations for Model 2

Computing the partial derivatives of the Hamiltonians (17) and (18) with respect to the decision variables u_1 and u_2 we get

$$\begin{aligned}
\frac{\partial \mathcal{H}_1}{\partial u_1} &= \gamma - u_2 - 2\,\psi\, u_1 + \mu_1 \qquad (34)\\
\frac{\partial \mathcal{H}_2}{\partial u_2} &= -2\,\vartheta\, u_2 - \omega\, u_1\,,
\end{aligned}$$

which implies

$$u_2^* \equiv 0$$

and

$$u_1^* = \frac{\gamma+\mu_1}{2\,\psi}\,.$$

Substituting (19) and (15) into the state dynamics (1), we obtain the differential equation

$$\dot{x} = \frac{1}{2\,\psi}\left[\gamma + \left(\frac{\varphi}{r_1+\delta} - \eta\right)\, e^{(r_1+\delta)(t-T)} - \frac{\varphi}{r_1+\delta}\right] - \delta\, x\,, \quad x(t_0) = x_0\,,$$

which has the solution

$$\begin{aligned}
x(t) \;=\; & \frac{1}{2\psi(r_1+\delta)}\left[\frac{\gamma\, r_1 - \varphi}{\delta} + \gamma + \frac{[\varphi - \eta\,(r_1+\delta)]\, e^{(r_1+\delta)(t-T)}}{r_1 + 2\,\delta}\right] \\
& + A_3\, e^{-\,\delta t}.
\end{aligned}$$

The constant A_3 can be obtained from the initial conditon $x(t_0) = x_0$, hence we get

$$\begin{aligned} x(t) &= \frac{1}{2\psi(r_1+\delta)}\left[\left(\frac{\gamma\, r_1 - \varphi}{\delta} + \gamma\right)(1 - e^{\delta(t_0-t)}) + \frac{\varphi - \eta\,(r_1+\delta)}{r_1 + 2\,\delta}\right. \\ &\quad \left.\cdot\left(e^{(r_1+\delta)(t-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-t)}\right)\right] + x_0\, e^{\delta(t_0-t)}. \qquad \square \end{aligned}$$

E Proof of the time-consistence of the solution given in Proposition 2

With appropriate constants K_1 and K_2, equation (21) can be rewritten as

$$\begin{aligned} x(t) &= K_1(1 - e^{\delta(t_0-t)}) \\ &+ K_2\left(e^{(r_1+\delta)(t-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-t)}\right) \\ &+ x_0 e^{\delta(t_0-t)}\,. \end{aligned} \tag{35}$$

Consequently, for some $\hat{t} \in [t_0, T]$ we get

$$\begin{aligned} x(\hat{t}) &= K_1(1 - e^{\delta(t_0-\hat{t})}) \\ &+ K_2\left(e^{(r_1+\delta)(\hat{t}-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-\hat{t})}\right) \\ &+ x_0 e^{\delta(t_0-\hat{t})}\,. \end{aligned} \tag{36}$$

Now, we consider $\hat{t}$ to be the new initial time t_0 and apply $x(\hat{t})$ regarding to (36) as the initial state x_0. If we substitute these expressions into (35) and call the new trajectory $\tilde{x}(t)$, then we have to show that $\tilde{x}(t) = x(t)$.

$$\begin{aligned} \tilde{x}(t) &= K_1\left[(1 - e^{\delta(\hat{t}-t)}) + (1 - e^{\delta(t_0-\hat{t})})e^{\delta(\hat{t}-t)}\right] \\ &+ K_2\left[e^{(r_1+\delta)(t-T)} - e^{(r_1+\delta)(\hat{t}-T)+\delta(\hat{t}-t)}\right. \\ &+ \left.\left(e^{(r_1+\delta)(\hat{t}-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-\hat{t})}\right)e^{\delta(\hat{t}-t)}\right] \\ &+ x_0 e^{\delta(t_0-\hat{t})} e^{\delta(\hat{t}-t)} \\ &= K_1\left[1 - e^{\delta(\hat{t}-t)} + e^{\delta(\hat{t}-t)} - e^{\delta(t_0-\hat{t}+\hat{t}-t)}\right] \\ &+ K_2\left[e^{(r_1+\delta)(t-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-\hat{t}+\hat{t}-t)}\right] \\ &+ x_0 e^{\delta(t_0-\hat{t}+\hat{t}-t)} \\ &= x(t) \qquad \square \end{aligned}$$

F Calculations for Model 3

If we substitute the offender's optimal control (23) into the Hamiltonian of the legal authority (18), we get

$$\begin{aligned}\mathcal{H}_2 &= -\zeta\left(\frac{\gamma-u_2+\mu_1}{2\psi}\right)^2-(\vartheta u_2^2+\nu x)\\ &\quad -\omega\,\frac{\gamma-u_2+\mu_1}{2\psi}\,u_2+\,\mu_2\left(\frac{\gamma-u_2+\mu_1}{2\psi}-\delta x\right)\end{aligned} \tag{37}$$

and

$$\frac{\partial\mathcal{H}_2}{\partial u_2} = \frac{\zeta}{2\psi^2}(\gamma-u_2+\mu_1)-2\vartheta u_2-\frac{\omega\,(\gamma-2\,u_2+\mu_1)}{2\,\psi}-\frac{\mu_2}{2\psi}. \tag{38}$$

This implies that the optimal control u_2^* must be[6]

$$u_2^*=\frac{(\,\zeta-\omega\,\psi\,)(\gamma+\mu_1)-\psi\,\mu_2}{\zeta+4\vartheta\psi^2-2\omega\psi}\,, \tag{39}$$

which, using (23), yields

$$u_1^*=\frac{(\gamma+\mu_1)(4\,\vartheta\,\psi-\omega)+\mu_2}{2\,(\vartheta+4\vartheta\,\psi^2-2\,\omega\,\psi)}. \tag{40}$$

Next, we substitute from (39) and (24) into the differential equation (1) describing the dynamics of the state to obtain

$$\begin{aligned}\dot{x} &= \frac{1}{2\,(\zeta+4\vartheta\,\psi^2-2\,\omega\,\psi)}\Bigg\{\left[\gamma+\left(\frac{\varphi}{r_1+\delta}-\eta\right)\,e^{(r_1+\delta)(t-T)}-\frac{\varphi}{r_1+\delta}\right]\\ &\quad (4\,\vartheta\,\psi-\omega)+\frac{\nu}{r_2+\delta}\left(e^{(r_2+\delta)(t-T)}-1\right)\Bigg\}-\delta\,x,\quad x(t_0)=x_0.\end{aligned} \tag{41}$$

Using the constants defined in (27), (41) can be rewritten as

$$\dot{x}=B_0+B_1\,e^{(r_1+\delta)(t-T)}+B_2\,e^{(r_2+\delta)(t-T)}-\delta\,x,\quad x(t_0)=x_0, \tag{42}$$

which has the solution

$$\mathrm{x}(t)=\frac{B_0}{\delta}+\frac{B_1\,e^{(r_1+\delta)(t-T)}}{r_1+2\,\delta}+\frac{B_2\,e^{(r_2+\delta)(t-T)}}{r_2+2\,\delta}+A_4\,e^{-\delta\,t}\,. \tag{43}$$

[6] To make sure that (39) *maximizes* the Hamiltonian, the second partial derivative with respect to u_2 must be negative, i.e. $\omega<\frac{\zeta}{2\psi}+2\vartheta\psi$.

The integration constant A_4 can be obtained from the initial condition $x(t_0) = x_0$, which results in

$$\begin{aligned} x(t) &= \frac{B_0}{\delta}\left(1 - e^{\delta(t_0-t)}\right) + \frac{B_1}{r_1+2\delta}\left(e^{(r_1+\delta)(t-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-t)}\right) \\ &\quad + \frac{B_2}{r_2+2\delta}\left(e^{(r_2+\delta)(t-T)} - e^{(r_2+\delta)(t_0-T)+\delta(t_0-t)}\right) + x_0\, e^{\delta\,(t_0-t)}\,. \end{aligned} \tag{44}$$

$\square$

G Proof of the time-consistence of the solution given in Proposition 3

The idea for the proof is the same as in appendix E. We use the constants K_1, K_2, and K_3 to rewrite equation (26) as

$$\begin{aligned} x(t) &= K_1(1 - e^{\delta(t_0-t)}) \\ &+ K_2\left(e^{(r_1+\delta)(t-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-t)}\right) \\ &+ K_3\left(e^{(r_2+\delta)(t-T)} - e^{(r_2+\delta)(t_0-T)+\delta(t_0-t)}\right) \\ &+ x_0 e^{\delta(t_0-t)}\,. \end{aligned} \tag{45}$$

For some $\hat{t} \in [t_0, T]$ we get

$$\begin{aligned} x(\hat{t}) &= K_1(1 - e^{\delta(t_0-\hat{t})}) \\ &+ K_2\left(e^{(r_1+\delta)(\hat{t}-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-\hat{t})}\right) \\ &+ K_3\left(e^{(r_2+\delta)(\hat{t}-T)} - e^{(r_2+\delta)(t_0-T)+\delta(t_0-\hat{t})}\right) \\ &+ x_0 e^{\delta(t_0-\hat{t})}\,. \end{aligned} \tag{46}$$

Now, we consider $\hat{t}$ to be the new initial time t_0 and apply $x(\hat{t})$ regarding to (46) as the initial state x_0. If we substitute these expressions into (45) and call the new trajectory $\tilde{x}(t)$, then we have to show that $\tilde{x}(t) = x(t)$.

$$\begin{aligned} \tilde{x}(t) &= K_1\left[(1 - e^{\delta(\hat{t}-t)}) + (1 - e^{\delta(t_0-\hat{t})})e^{\delta(\hat{t}-t)}\right] \\ &+ K_2\left[e^{(r_1+\delta)(t-T)} - e^{(r_1+\delta)(\hat{t}-T)+\delta(\hat{t}-t)}\right. \\ &+ \left.\left(e^{(r_1+\delta)(\hat{t}-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-\hat{t})}\right) e^{\delta(\hat{t}-t)}\right] \\ &+ K_3\left[e^{(r_2+\delta)(t-T)} - e^{(r_2+\delta)(\hat{t}-T)+\delta(\hat{t}-t)}\right. \\ &+ \left.\left(e^{(r_2+\delta)(\hat{t}-T)} - e^{(r_2+\delta)(t_0-T)+\delta(t_0-\hat{t})}\right) e^{\delta(\hat{t}-t)}\right] \end{aligned}$$

$$
\begin{aligned}
&+ \quad x_0 e^{\delta(t_0-\hat{t})} e^{\delta(\hat{t}-t)} \\
&= \quad K_1 \left[1 - e^{\delta(\hat{t}-t)} + e^{\delta(\hat{t}-t)} - e^{\delta(t_0-\hat{t}+\hat{t}-t)}\right] \\
&+ \quad K_2 \left[e^{(r_1+\delta)(t-T)} - e^{(r_1+\delta)(t_0-T)+\delta(t_0-\hat{t}+\hat{t}-t)}\right] \\
&+ \quad K_3 \left[e^{(r_2+\delta)(t-T)} - e^{(r_2+\delta)(t_0-T)+\delta(t_0-\hat{t}+\hat{t}-t)}\right] \\
&+ \quad x_0 e^{\delta(t_0-\hat{t}+\hat{t}-t)} \\
&= \quad x(t) \qquad \square
\end{aligned}
$$

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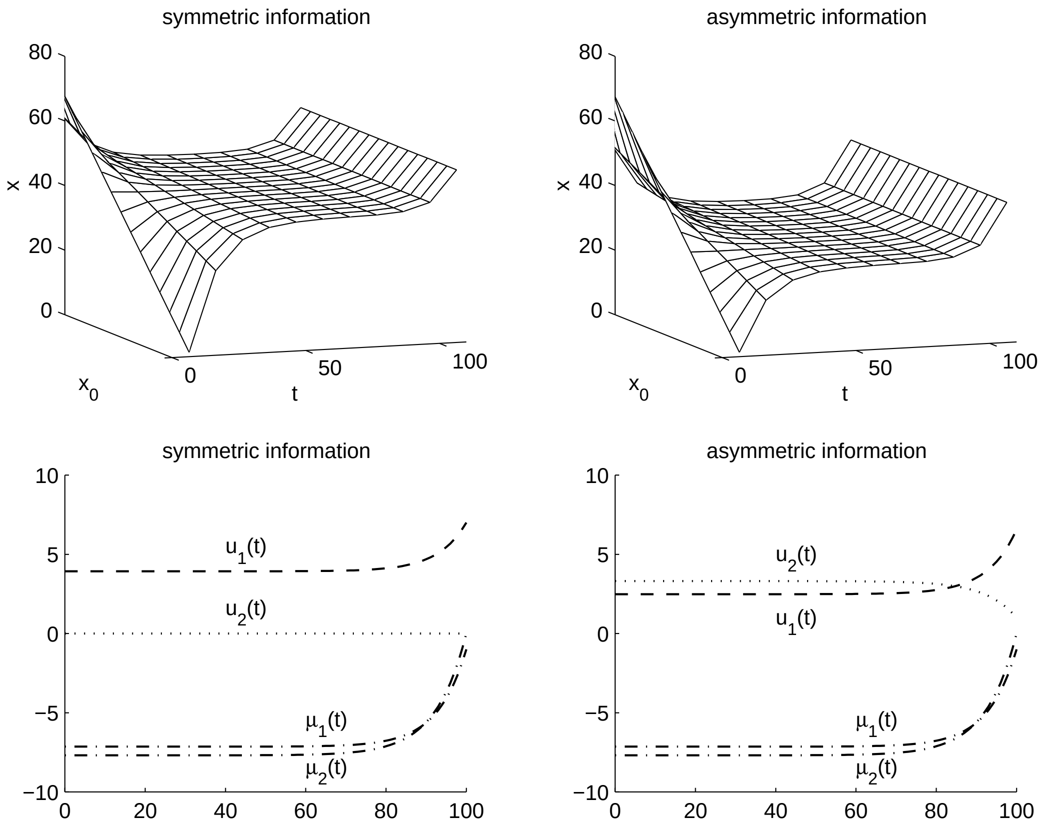

Figure 1: trajectories of models 2 and 3 for $x_0 \in [0, 80]$

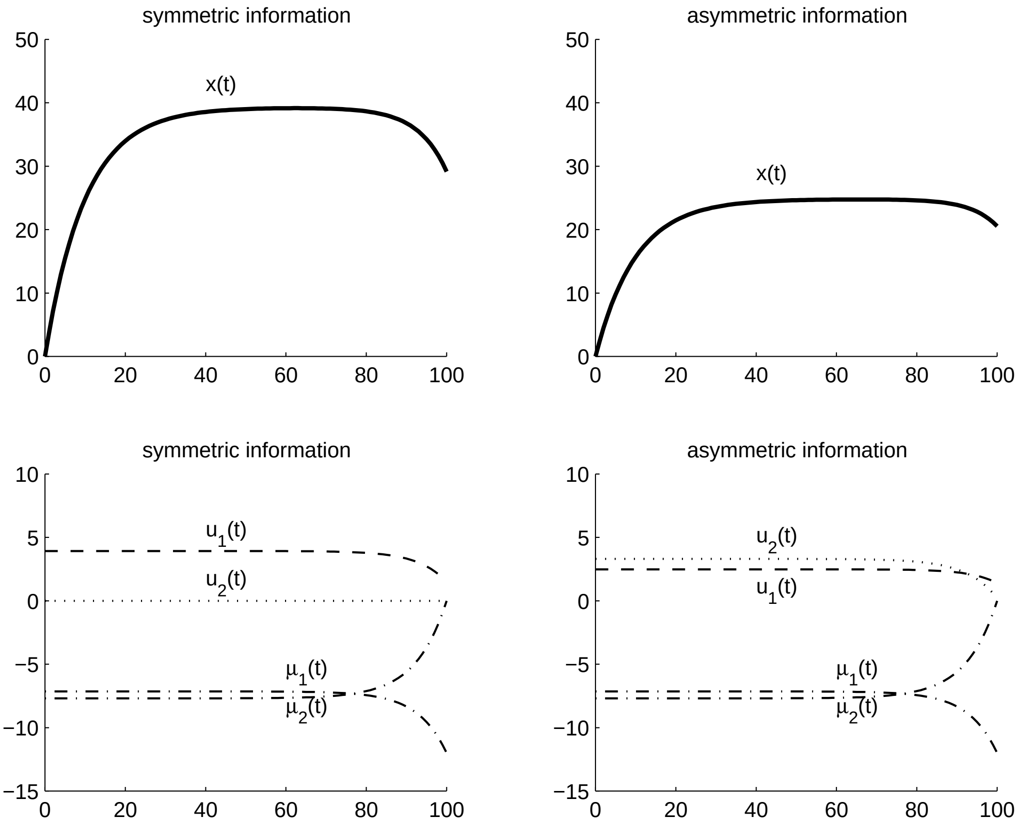

Figure 2: trajectories of models 2 and 3 for $\eta = 12$

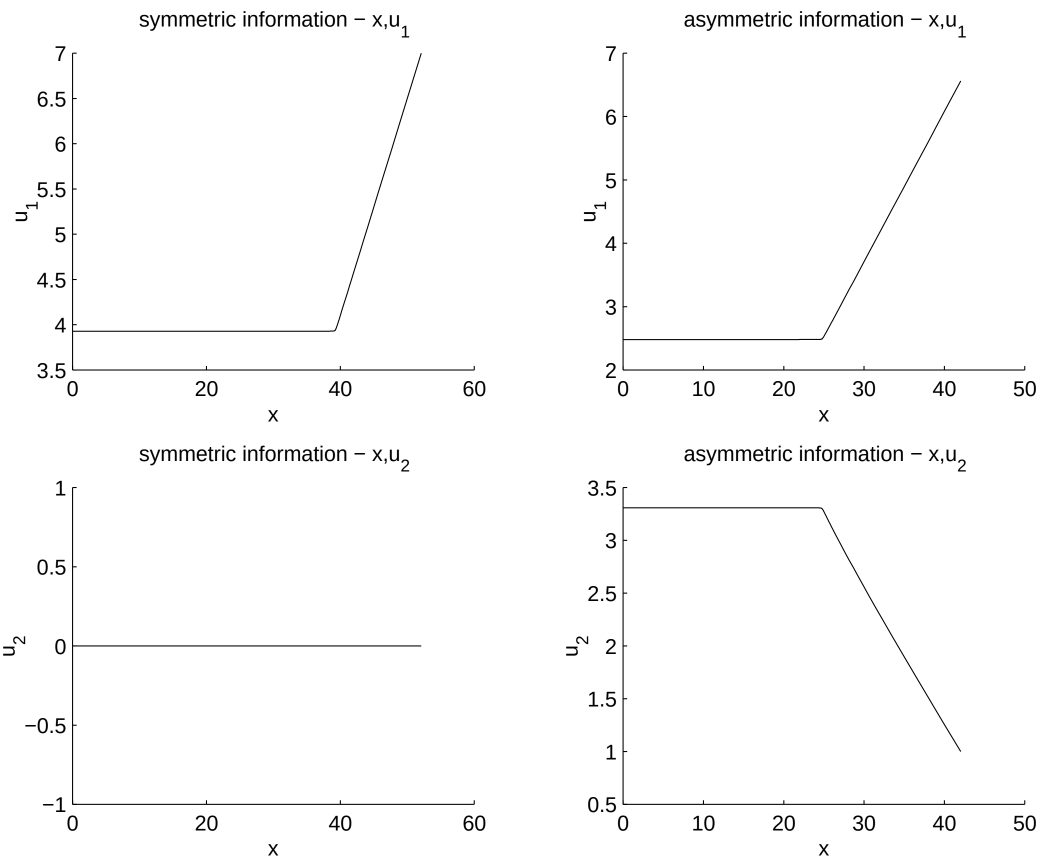

Figure 3: phase portraits of models 2 and 3

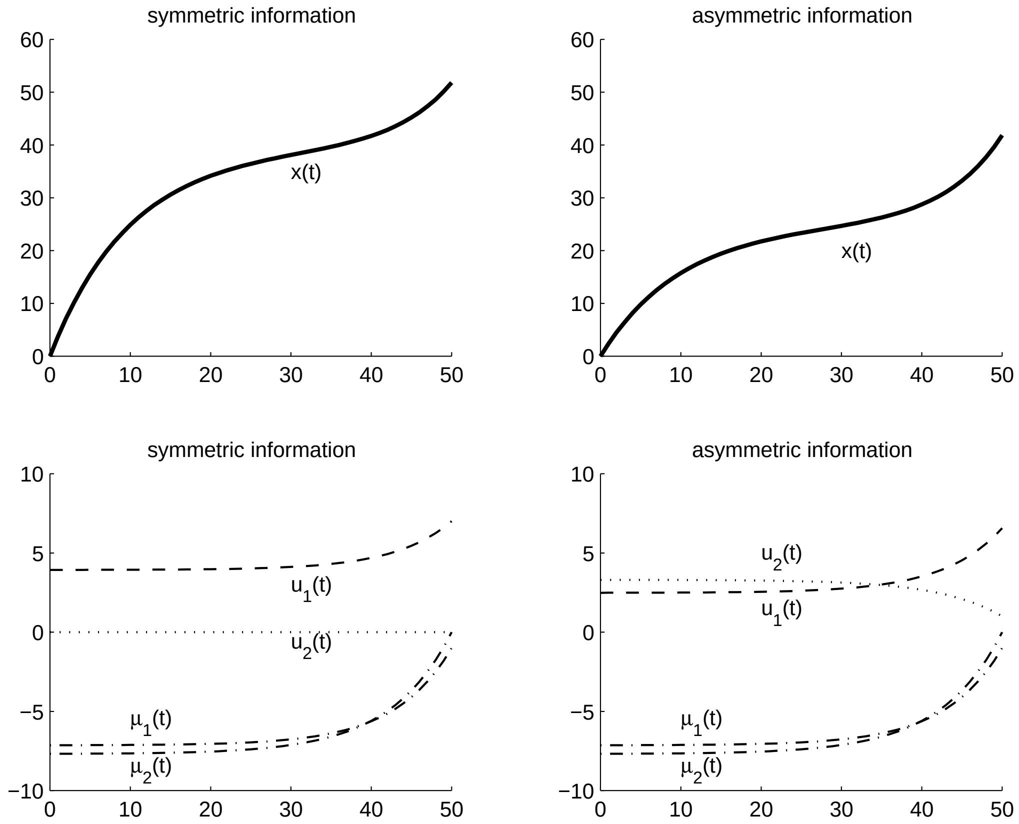

Figure 4: trajectories of models 2 and 3 for $T = 50$

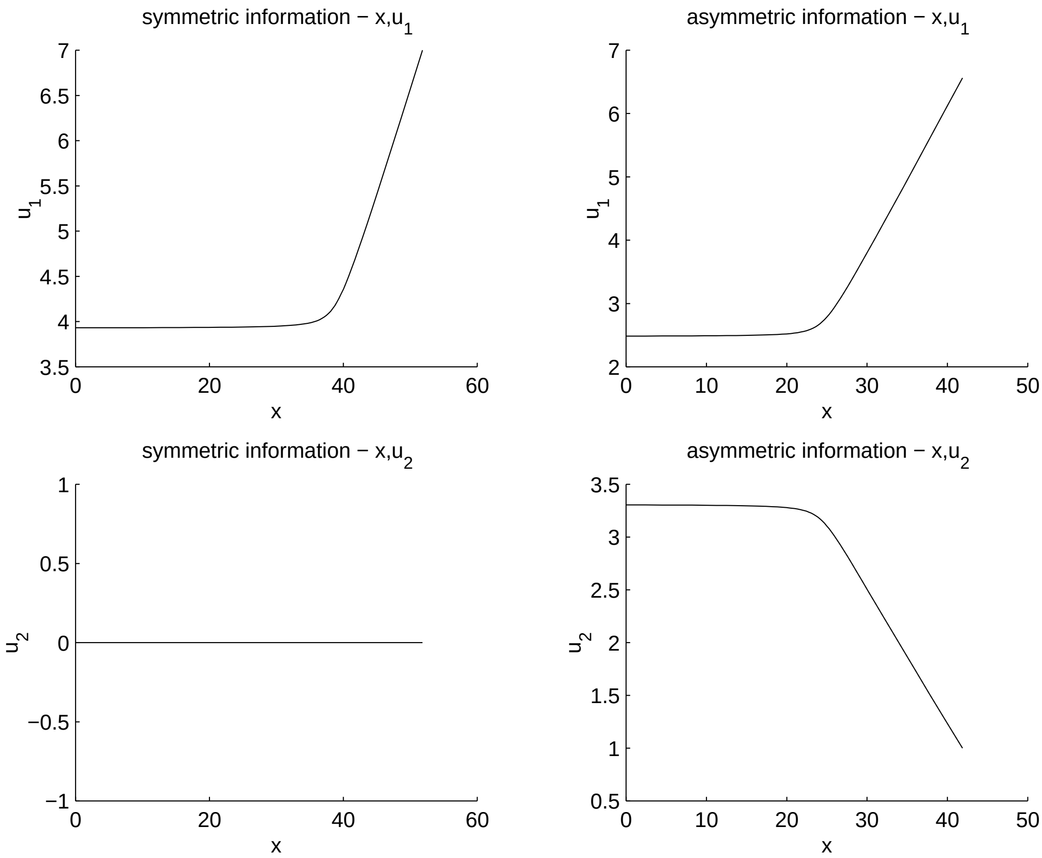

Figure 5: phase portraits of models 2 and 3 for $T = 50$

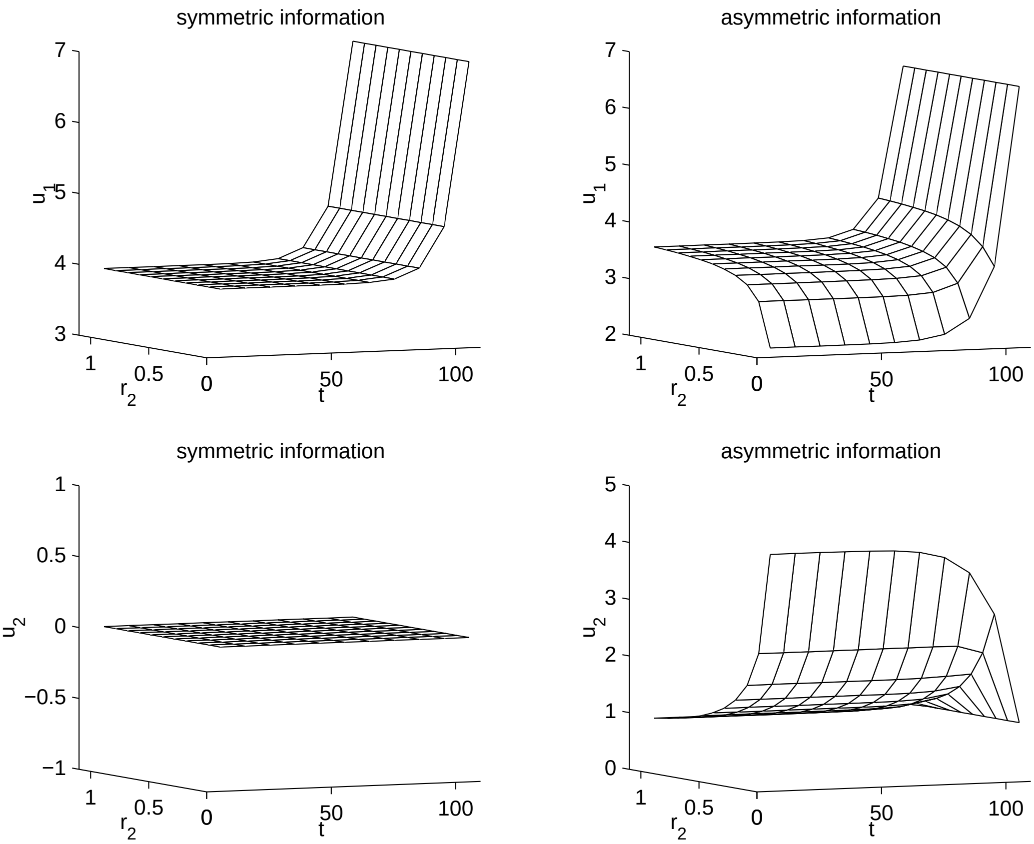

Figure 6: trajectories of models 2 and 3 for $r_2 \in [0, 1]$

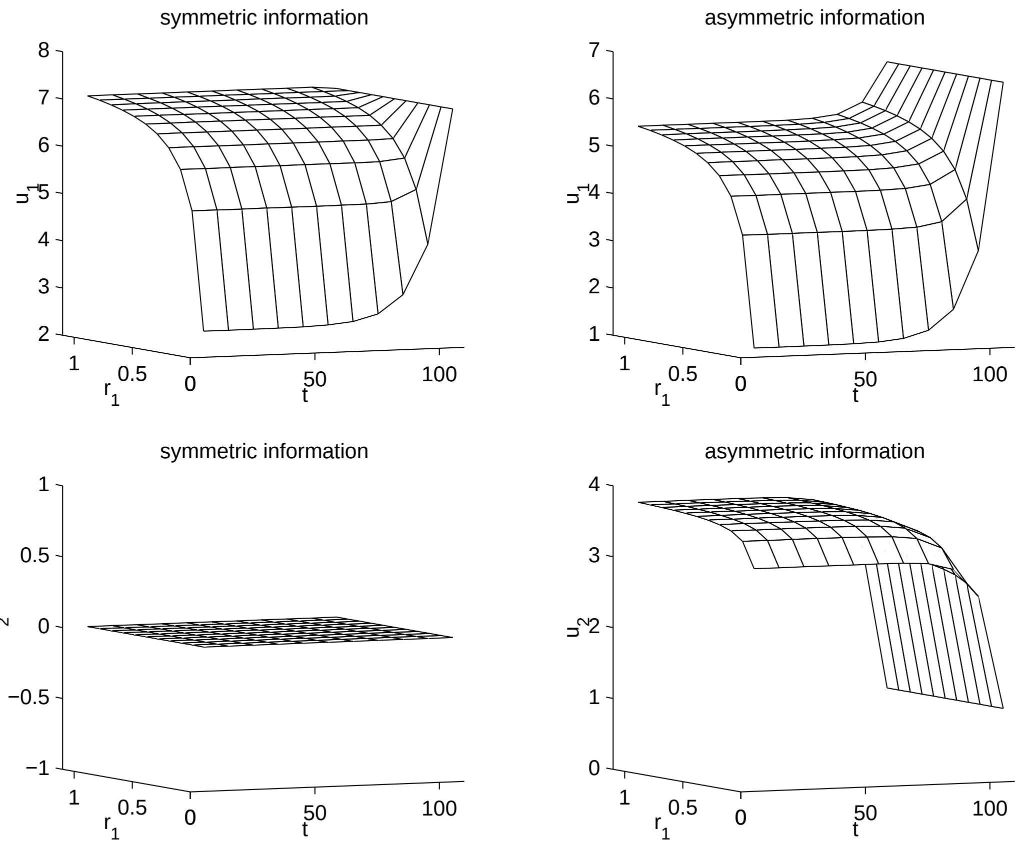

Figure 7: trajectories of models 2 and 3 for $r_1 \in [0, 1]$