Dynamic Investment Behavior Taking Into Account Ageing of the Capital Good.

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Abstract

In standard capital accumulation models all capital goods are equally productive and produce goods of the same quality. However, due to ageing, in reality it holds most of the time that newer capital goods are more productive. Implications of this feature for the firm's investment policies are investigated in an optimal control problem with distributed parameters. It turns out that investing in capital goods of different age is done such that the net present value of marginal investment equals zero. Comparing the returns of investment in capital goods of different age, the higher productivity of younger capital goods has to be weighed against the lower costs of depreciation, discounting and acquisition of older capital goods. In the steady state it holds that, in the most reasonable scenario, the firm should invest at the highest rate in new capital goods, and disinvestment can only be optimal when costs of acquisition are large and machines are old.

1 Introduction

One of the driving forces in a market economy is the growth of firms and industries. In the literature the analysis of firm growth started out in the sixties with Eisner and Strotz (1963). In the framework they considered the firm owns a stock of capital goods that is needed to produce goods, which are sold on the market to obtain revenue. The firm is able to increase capital stock by investing. This profit maximization problem thus involves the choice of investments to expand the stock of capital goods. After this first contribution by Eisner and Strotz (1963), many others have followed (e.g., Lucas (1967), Davidson and

Harris (1981), Barucci (1998)), and they mostly differ in the specifications of revenue and investment cost functions. All these contributions have in common that capital stock is homogeneous. Hence, its features do not change over the years, so that it can be concluded that matters like ageing and technological progress are not taken into account.

The aim of this paper is to analyze a model where capital goods with different ages are distinguished. To do so a vintage capital stock model is developed. We use Haurie, Sethi and Hartl (1984) as basic departure point (see also Appendix 5 of Feichtinger and Hartl (1986)).

In order to show what influence ageing has on the age distribution of the capital stock we consider a situation where there is no technological progress and there is constant returns to scale. Productivity only depends on its age. This means that capital stocks of the same age have the same productivity independent of the year in which they are operating. Thus each capital good of the same age produces a fixed amount.

The vintage capital model has become increasingly popular among economists, especially because it provides an appealing framework for the analysis of investment volatility. However, Barucci and Gozzi (2000) state that, apart from their paper, in the literature the vintage differentiation of the capital goods has not been analyzed in a complete dynamic optimization framework; often capital goods are not durable, they can not be accumulated and therefore the capital accumulation problem either becomes a simple intertemporal budget allocation problem (e.g. Grossman and Helpman (1991)) or capital is completely absent as an explicit input factor (e.g. Chari and Hopenhayn (1991)). Jovanovic (1998) argues that full dynamics are notoriously difficult in such models. Our paper offers a complete dynamic optimization framework, but contrary to Barucci and Gozzi (2000) who concentrate on technological progress, we focus on the effects of ageing on the dynamic investment rates and on the age distribution of capital goods in the steady state. We show that it is in fact optimal for the firm to reach this steady state as soon as possible. The steady state does not exist in Barucci and Gozzi's model due to the technological progress considered there.

By analyzing this model we are able to determine the firm's optimal investment decisions in capital goods of different ages. It turns out that the firm always invests in such a way that the net present value of marginal investment equals zero, so that the discounted extra revenue stream caused by the addition of a capital good exactly balances the marginal investment costs. Investments in younger machines have the advantage that due to ageing they are more productive than older ones, but the disadvantage is that older machines are cheaper and the costs of depreciation and discounting are less. The presence of the latter effects may explain why, according to Chari and Hopenhayn (1991), it is undeniable that new technologies are often adopted on a large scale only after a prolonged period of time (see Mansfield (1968) for empirical evidence). For the steady state it turns out that, provided that the discount rate is sufficiently low, the firm should invest mostly in new capital goods. Disinvestment only occurs if acquisition costs are high and machines are sufficiently old.

The paper is organized as follows. The model is formulated in Section 2.

In Section 3 the optimality conditions are formulated and expressions for the investment rate in capital stocks of different age are derived and economically analyzed. Moreover, Section 3.3 considers the firm in steady state in order to see how the age distribution of capital goods then looks like.

2 The Model

In a recent paper Xepapadeas and De Zeeuw (1999) studied the ideal age composition of the capital stock subject to environmental regulation. Here we consider a related version of the model of Xepapadeas and De Zeeuw (1999): where they concentrate on environmental regulation by specifying pollution output, we leave this out. Instead, we extend their framework by adding discounting and depreciation, so that this paper is a natural extension to the capital accumulation literature mentioned in the first paragraph of the Introduction. As in their paper, here it also holds that the age of the machine is denoted by $\tau \in [0, h]$, so that the maximum age of machines is h.

$v(\tau)$ is the output produced by a machine of age τ, with $v'(\tau) \leq 0$. That is, a newer machine cannot produce less output than an older machine. Since v is independent of time t no technological progress is included.[1]

The stock of capital goods of age τ at time t is denoted by $K(t, \tau)$. Then total output produced in year t is defined as

$$Q(t) = \int_0^h v(\tau) K(t, \tau) d\tau.$$

It is assumed that markets exist for machines of any age from 0 to h. Let $b(\tau)$ be the cost of buying a machine of age τ, with $b'(\tau) \leq 0$ (older machines cannot be more expensive than newer machines) and $b(h) = 0$ (a machine at the maximum age is not worth anything).

Let $I(t, \tau)$ be the number of machines of age τ bought (if $I(t, \tau) > 0$) or sold (if $I(t, \tau) < 0$) in year t. The total cost or revenue to the firm from transactions in the machine market is defined as $b(\tau) I(t, \tau) + \frac{c}{2} [I(t, \tau)]^2$, with the second term reflecting the adjustment costs in buying or selling machines. These costs are, for example, adaptation costs or search costs. The quadratic form of this cost term leads to a simple expression for optimal purchases. It is further imposed that machines of age τ depreciate with rate $\delta(\tau)$, which is the same for every vintage.

[1]This model feature is taken from Xepapadeas and De Zeeuw (1999) (see also Barucci and Gozzi (1998)) who argue that this implies that new machines are more productive because they embody superior technology. However, according to us this argument is wrong. To see this, note that $v(\tau)$ is the same for different t. Now consider two points of time: t_1 and t_2 so that $t_2 > t_1$. Then a machine constructed at time t_2, say m_2, has the same productivity at the same age as a machine constructed at time t_1 (m_1), i.e. m_2 produces at $t_2 + \tau : v(\tau)$, which is also the amount that m_1 produces at $t_1 + \tau$. Hence there is no superior technology embedded in m_2. Therefore, in order to include technological progress, output should be modelled by $v(t, \tau)$ with, at least, $v_t > 0$.

The firm chooses to buy or sell machines of different ages in order to maximize profits, with p the price of output. That is, the firm chooses at each point in time an age distribution of machines to maximize profits. In addition to Xepapadeas and De Zeeuw (1999), our model also includes discounting, where r is the discount rate. The dynamic model of the firm is now given by

$$\max_{I(t,\tau),I_0(t)} \int_0^\infty e^{-rt} \int_0^h \left[pv(\tau)K(t,\tau) - b(\tau)I(t,\tau) - \frac{c}{2}\left[I(t,\tau)\right]^2\right] d\tau dt \quad (1)$$

$$- \int_0^\infty e^{-rt} \left[b_0 I_0(t) + \frac{c_0}{2}\left[I_0(t)\right]^2\right] dt$$

$$\text{subject to } \frac{\partial K(t,\tau)}{\partial t} + \frac{\partial K(t,\tau)}{\partial \tau} = I(t,\tau) - \delta(\tau) K(t,\tau), \quad (2)$$

$$K(t,0) = I_0(t), K(0,\tau) = K_0(\tau) \quad (3)$$

This is an infinite horizon optimal control problem with transition dynamics described by a linear partial differential equation (Carlson, Haurie and Leizarowitz (1991)). The transition equation indicates that the rate of change in the number of machines of a given age, τ, at a given time, t, is determined by two factors. These are the reduction or increase in the number of machines brought about by the sale or acquisition of machines of the given age τ (the first term of the transition equation), and the reduction due to depreciation at rate $\delta(\tau)$. The initial condition on the number of machines implies that the firm starts with given amount $K_0(\tau)$ of machines of age τ. At each time t it is possible to buy new machines. This purchase rate of new machines is denoted by I_0.

3 Analysis of the Model

First, by using the maximum principle analytical expressions are obtained for investment and capital stock in Section 3.1. It is shown that after h years the steady state will be reached. In Section 3.2 the expressions for investment and capital stock are economically analyzed. Section 3.3 focuses entirely at the steady state to see how the optimal age distribution in the steady state looks like.

3.1 Maximum Principle

The current value Hamiltonian H for this problem is given by (see, e.g., Feichtinger and Hartl (1986)):

$$\begin{aligned} H &= pv(\tau)K(t,\tau) - b(\tau)I(t,\tau) - \frac{c}{2}\left[I(t,\tau)\right]^2 \\ &+ \lambda(t,\tau)\left[I(t,\tau) - \delta(\tau) K(t,\tau)\right], \end{aligned} \quad (4)$$

while the boundary Hamiltonian is

$$H_0 = -b_0 I_0(t) - \frac{c_0}{2}\left[I_0(t)\right]^2 + \lambda(t,0) I_0(t).$$

Consequently, the first-order conditions for optimality are

$$\frac{\partial H}{\partial I} = 0, \quad \text{or} \quad cI(t,\tau) = \lambda(t,\tau) - b(\tau), \tag{5}$$

$$\frac{\partial H_0}{\partial I_0} = 0, \text{ or } c_0 I_0(t) = \lambda(t,0) - b_0, \tag{6}$$

$$\frac{\partial \lambda(t,\tau)}{\partial t} + \frac{\partial \lambda(t,\tau)}{\partial \tau} = r\lambda - \frac{\partial H}{\partial K} = (r + \delta(\tau))\,\lambda(t,\tau) - pv(\tau), \tag{7}$$

$$\lambda(t,h) = 0. \tag{8}$$

Solving the partial differential equation (7), while taking into account the boundary condition (8) yields:

$$\lambda(t,\tau) = \int_\tau^h e^{-\int_\tau^s (r+\delta(\rho))d\rho} pv(s)ds. \tag{9}$$

From (5) and (9) the optimal investment rate is obtained:

$$I(t,\tau) = \frac{1}{c}\left[\int_\tau^h e^{-\int_\tau^s (r+\delta(\rho))d\rho} pv(s)ds - b(\tau)\right]. \tag{10}$$

By (6) and (9) it can be concluded that a similar expression holds for the investment in new capital goods:

$$I_0(t) = \frac{1}{c_0}\left[\int_0^h e^{-\int_0^s (r+\delta(\rho))d\rho} pv(s)ds - b_0\right]. \tag{11}$$

An expression for the stock of capital goods can be derived from (2), assuming for the moment that $\tau \leq t$:

$$K(t,\tau) = \left(\int_0^\tau e^{\int_0^\sigma \delta(\rho)d\rho} I\,(t+\sigma-\tau,\sigma)\,d\sigma + A_2\right) e^{-\int_0^\tau \delta(\rho)d\rho}. \tag{12}$$

Note that the initial stock is $A_2 = K\,(t-\tau, 0) = I_0\,(t-\tau)$ (see (3)). Combining the last three expressions, we obtain

$$\begin{aligned} K(t,\tau) \quad = \quad & \left(\int_0^\tau e^{\int_0^\sigma \delta(\rho)d\rho} \frac{1}{c}\left[\int_\sigma^h e^{-\int_\sigma^s (r+\delta(\rho))d\rho} pv(s)ds - b(\sigma)\right] d\sigma\right) e^{-\int_0^\tau \delta(\rho)d\rho} \\ & + \left(\frac{\int_0^h e^{-\int_0^s (r+\delta(\rho))d\rho} pv(s)ds - b_0}{c_0}\right) e^{-\int_0^\tau \delta(\rho)d\rho}. \end{aligned} \tag{13}$$

Note that this formula is only valid for $\tau \leq t$. In case $\tau > t$, i.e. the vintage already exists at the initial time, it is easily obtained via the second boundary condition in (3) that

$$K(t,\tau) = K_0(\tau - t)\, e^{-\int_{\tau-t}^{\tau}\delta(\rho)d\rho} + \int_{\tau-t}^{\tau} e^{-\int_{\sigma}^{\tau}\delta(\rho)d\rho}\frac{1}{c}\left[\int_{\sigma}^{h} e^{-\int_{\sigma}^{s}(r+\delta(\rho))d\rho} pv(s)ds - b(\sigma)\right] d\sigma. \tag{14}$$

An important observation is that (9), (10) and (11) are time invariant. Moreover, $K(t,\tau)$ depends on t only in case $t < \tau$. This means that after h years everything becomes time invariant, that is, the steady state with respect to calendar time is reached.

3.2 Economic Analysis

Let us analyze by what characteristics the investment rate in machines of different years is influenced. The amount of investment is given by

$$I(t,\tau) = \frac{1}{c}\left[\int_{\tau}^{h} e^{-\int_{\tau}^{s}(r+\delta(\rho))d\rho} pv(s)ds - b(\tau)\right]$$

for older machines, and

$$I_0(t) = \frac{1}{c_0}\left[\int_{0}^{h} e^{-\int_{0}^{s}(r+\delta(\rho))d\rho} pv(s)ds - b_0\right] \tag{15}$$

in new machines. It follows that the net present value of marginal investment equals zero: the term with the integral equals the revenue stream (corrected for discounting and depreciation) generated by an extra unit of capital stock of age τ (or 0) bought at time t, and this extra revenue equals total marginal investment costs $b + cI$.

It is clear that no investment will take place in a machine of age h, so that

$$I(t,h) = 0.$$

At a given point of time t the investment rate is influenced by its age as follows:

$$c\frac{\partial I(t,\tau)}{\partial \tau} = (r + \delta(\tau))\left(\int_{\tau}^{h} e^{-\int_{\tau}^{s}(r+\delta(\rho))d\rho} pv(s)ds\right) - pv(\tau) - b'(\tau). \tag{16}$$

Expression (16) shows how investment is affected when the firm compares investing in a machine of age τ with investing in a machine of a marginally older age. According to the RHS of (16), three effects arise. The *first* effect is positive and consists of a discounting and a depreciation effect. The depreciation effect results from the fact that by buying a machine of older age the machine is depreciated less at the moment that its age is s, thus when its productivity equals $v(s)$. The discounting effect is also positive, because the revenue obtained at the moment that the machine is of age s is obtained earlier so that the discounted revenue is higher. The *second* effect is negative and arises from the fact that when buying the machine of a marginally older age than τ, it will not collect the revenue when the machine operates at age τ. The *last* effect is positive which is due to the fact that the acquisition costs of older machines are cheaper.

These effects may help to explain why firms often invest in older technologies even when apparently superior technologies may be available (Chari and Hopenhayn (1991). According to (16) reasons may be that (1) effects of discounting and depreciation are substantial, and (2) an older machine has a lower acquisition price.

Expression (16) also helps to explain the observation that new technologies are often adopted so slowly, as recognized by, e.g., Chari and Hopenhayn (1991). Reasons for such behavior can thus be that effects of discounting and depreciation (especially during the first years that a new capital good operates) are large and/or that the reduction of the acquisition price when the capital good gets older is substantial.

In case $v' \leq 0$ and $\delta' \geq 0$ it can be easily shown that the first effect is always dominated by the second effect, i.e. the discounting and depreciation effects are more than outweighed by the effect that revenues are earned during a shorter time. We illustrate this by taking δ and ν constant, after which expression (16) becomes

$$\begin{aligned}\frac{\partial I(t,\tau)}{\partial \tau} &= \frac{1}{c}\left[\left(pv\left(1-e^{-(h-\tau)(r+\delta)}\right)\right)-pv-b'(\tau)\right]\\ &= \frac{1}{c}\left[-pve^{-(h-\tau)(r+\delta)}-b'(\tau)\right]\end{aligned}$$

Now there are only two contrary effects of age on the investment rate. The advantage of investing in a machine of older age is that investments are cheaper as reflected by the term $-b'(\tau)$. However, the disadvantage is that the planning period during which the firm enjoys revenue from this investment becomes shorter, which is presented by the first term.

Consider now the evolution of the capital stock, where we concentrate on those capital goods for which $\tau < t$, thus at the initial point of time this stock was not present yet. From (12) and $A_2 = I_0(t-\tau)$, it can be obtained that

$$\frac{\partial K(t,\tau)}{\partial \tau} = I(t,\tau) - \delta(\tau)K(t,\tau)\,. \tag{17}$$

Hence, to find out how capital stocks of different age relate to each other at a given point of time, would require substitution of (13) and (10) into (17), and this becomes too messy for drawing clear economic conclusions.

3.3 The Steady State

As remarked at the end of Section 3.1, from time h onwards the firm is in steady state with respect to calendar time. First we consider the optimal age distribution in general, after which we consider a specific example.

3.3.1 The optimal age distribution

From (9) it follows that $\lambda(\tau)$ is given by

$$\lambda(\tau) = \int_\tau^h e^{-\int_\tau^s (r+\delta(\rho))d\rho} pv(s)ds \tag{18}$$

The value of λ as given by (18) reflects the benefits from installing one machine of age τ and keeping it until it becomes of maximum age. From (5) the optimal sales or acquisitions of machines of age τ is given by

$$cI(\tau) = \lambda(\tau) - b(\tau) = \int_\tau^h e^{-\int_\tau^s (r+\delta(\rho))d\rho} pv(s)ds - b(\tau). \tag{19}$$

Note that

$$I(\tau) \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0, \text{ as } \lambda(\tau) \begin{pmatrix} > \\ = \\ < \end{pmatrix} b(\tau),$$

which is intuitively clear because λ denotes the benefits and b denotes the price of new machines.

The stock of machines of age τ is partly determined by sales and acquisitions of machines of that age and partly inherited from sales and acquisitions in the past. The set of stocks of all ages is the optimal age distribution of machines and from (13) it is obtained that

$$\begin{aligned} K(\tau) = {} & \tfrac{1}{c} \int_0^\tau e^{\int_0^\sigma \delta(\rho)d\rho} \left[\int_\sigma^h e^{-\int_\sigma^s (r+\delta(\rho))d\rho} pv(s)ds - b(\sigma) \right] d\sigma e^{-\int_0^\tau \delta(\rho)d\rho} \\ & + \tfrac{1}{c_0} \left[\int_0^h e^{-\int_0^s (r+\delta(\rho))d\rho} pv(s)ds - b_0 \right] e^{-\int_0^\tau \delta(\rho)d\rho}. \end{aligned}$$

3.3.2 Example

In case there is no depreciation $\delta(\tau) = 0$ and no initial investment $I_0 = 0$, as in Xepapadeas-DeZeeuw the solution simplifies to:

$$\lambda(\tau) = \int_\tau^h e^{-r(s-\tau)} pv(s)ds. \tag{20}$$

$$I(\tau) = \frac{\lambda(\tau) - b(\tau)}{c} = \frac{\int_\tau^h e^{-r(s-\tau)} pv(s)ds - b(\tau)}{c}. \tag{21}$$

$$K(\tau) = \int_0^\tau I(\sigma)\, d\sigma = \frac{1}{c} \int_0^\tau \left[\int_\sigma^h e^{-r(s-\sigma)} pv(s)ds - b(\sigma) \right] d\sigma. \tag{22}$$

To see what (21) and (22) look like, consider the following example:

$$v(\tau) = a_0 + a_1 (h - \tau), \tag{23}$$

$$b(\tau) = b(h - \tau), \tag{24}$$

where all parameters are nonnegative and at least a_1 is strictly positive. This implies that acquisition cost b decline linearly with age τ of the machines and output v is linearly decreasing with age τ.

Substitution of these functions into (21) gives

$$\begin{aligned} cI(\tau) &= \int_\tau^h e^{-r(\rho-\tau)} p (a_0 + a_1 (h-\rho))\, d\varrho - b(h-\tau) \\ &= \int_\tau^h p(a_0 + a_1 h) e^{-r\rho} e^{r\tau} d\rho - \int_\tau^h p a_1 \rho e^{-r\rho} e^{r\tau} d\rho - b(h-\tau) \\ &= \left[-\frac{p}{r}(a_0 + a_1 h) e^{-r(\rho-\tau)}\right]_\tau^h + \left[\frac{pa_1}{r} e^{-r(\rho-\tau)} \left(\rho + \frac{1}{r}\right)\right]_\tau^h - b(h-\tau) \\ &= \frac{p}{r} e^{-r(h-\tau)} \left[-a_0 + \frac{a_1}{r}\right] + \frac{p}{r}\left[a_0 - \frac{a_1}{r} + a_1 (h-\tau)\right] - b(h-\tau), \\ &= \left[a_0 - \frac{a_1}{r}\right] \frac{p}{r} \left[1 - e^{-r(h-\tau)}\right] + \left[\frac{pa_1}{r} - b\right](h-\tau). \end{aligned} \tag{25}$$

from which it is obtained that

$$c\frac{\partial I(\tau)}{\partial \tau} = p\left(-a_0 + \frac{a_1}{r}\right) e^{-r(h-\tau)} - \frac{a_1 p}{r} + b, \tag{26}$$

so that

$$c\frac{\partial^2 I(\tau)}{\partial \tau^2} = (-ra_0 + a_1) p e^{-r(h-\tau)}. \tag{27}$$

We can conclude that

Proposition 1 *Under the specifications given by (23) and (24) it holds that*

$$I(h) = 0.$$

Furthermore, for different cases the following results are obtained:

1. Low discount rate: $r < \frac{a_1}{a_0}$*:*

$$\frac{\partial^2 I}{\partial \tau^2} > 0,$$

1.1. Low acquisition cost: $b < pa_0$:

$$\begin{aligned} \frac{\partial I(\tau)}{\partial \tau} &< 0 \ \forall \tau , \\ I(\tau) &> 0 \ \textit{for}\ \tau \in [0, h), \end{aligned}$$

1.2. High acquisition cost: $b \geq pa_0$:

$$\frac{\partial I(\tau)}{\partial \tau} \begin{pmatrix} < \\ = \\ > \end{pmatrix} 0 \text{ iff } \tau \begin{pmatrix} < \\ = \\ > \end{pmatrix} h - \frac{1}{r} \ln \left(\frac{a_1 p - r a_0 p}{a_1 p - r b} \right).$$

And

2. High discount rate: $r > \dfrac{a_1}{a_0}$:

$$\frac{\partial^2 I}{\partial \tau^2} < 0,$$

2.1. High acquisition cost: $b > pa_0$:

$$\begin{aligned} \frac{\partial I(\tau)}{\partial \tau} &> 0 \ \forall \tau \ , \\ I(\tau) &< 0 \ \textit{for} \ \tau \in [0, h), \end{aligned}$$

2.2. Low acquisition cost: $b \leq pa_0$:

$$\frac{\partial I(\tau)}{\partial \tau} \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \text{ iff } \tau \begin{pmatrix} < \\ = \\ > \end{pmatrix} h - \frac{1}{r} \ln \left(\frac{r a_0 p - a_1 p}{r b - a_1 p} \right).$$

We note that the most reasonable cases are probably 1.1 and 1.2. In case 2.1 the solution makes no sense, since I_0 being equal to zero and investments being negative for each age imply that K will become negative too.

Next, let us concentrate on the capital stock rather than investment. To do so, we combine (22) and (25) to obtain:

$$\begin{aligned} cK(\tau) &= \frac{p}{r}\left(-a_0 + \frac{a_1}{r}\right) \int_0^\tau e^{-r(h-z)} dz + \int_0^\tau \left[\frac{p}{r}\left(a_0 - \frac{a_1}{r} + a_1 h\right) - bh\right] dz \\ &\quad + \int_0^\tau \left[-\frac{pa_1}{r} + b\right] z dz \\ &= \frac{p}{r^2}\left(-a_0 + \frac{a_1}{r}\right)\left(e^{-r(h-\tau)} - e^{-rh}\right) + \frac{p}{r}\tau\left[a_0 - \frac{a_1}{r} + a_1 h\right] - bh\tau \\ &\quad + \frac{1}{2}\tau^2\left(\frac{-pa_1}{r} + b\right), \end{aligned}$$

from which it can be derived that (cf. (25))

$$c\frac{\partial K(\tau)}{\partial \tau} = \frac{p}{r}\left(-a_0 + \frac{a_1}{r}\right) e^{-r(h-\tau)} + \frac{p}{r}\left[a_0 - \frac{a_1}{r} + a_1 h\right] - bh + \tau\left(\frac{-pa_1}{r} + b\right) = cI(\tau).$$

Due to the last two equations and Proposition 1 we can conclude that

Proposition 2 *Consider the problem with the specifications presented in (23) and (24). Then it holds that capital stock is age dependent in the following way:*

$$\frac{\partial K}{\partial \tau}\mid_{\tau=h}=0.$$

Furthermore, for different cases the following results are obtained:

1.1. Low discount rate $\left(r<\frac{a_1}{a_0}\right)$ *and low acquisition cost* $(b<pa_0)$:

$$\begin{aligned}\frac{\partial^2 K(\tau)}{\partial \tau^2} &< 0\ \forall\tau\ ,\\ \frac{\partial K}{\partial \tau} &> 0\ \mathit{for}\ \tau\in[0,h)\,,\end{aligned}$$

1.2. Low discount rate $\left(r<\frac{a_1}{a_0}\right)$ *and high acquisition cost* $(b\geq pa_0)$:

$$\frac{\partial^2 K(\tau)}{\partial \tau^2}\left(\begin{array}{c}<\\=\\>\end{array}\right)0\ \mathit{iff}\ \ \tau\left(\begin{array}{c}<\\=\\>\end{array}\right)h-\frac{1}{r}\ln\left(\frac{a_1p-ra_0p}{a_1p-rb}\right),$$

2.1. High discount rate $\left(r>\frac{a_1}{a_0}\right)$ *and high acquisition cost* $(b>pa_0)$:

$$\begin{aligned}\frac{\partial^2 K(\tau)}{\partial \tau^2} &> 0\ \forall\tau\ ,\\ \frac{\partial K}{\partial \tau} &< 0\ \mathit{for}\ \tau\in[0,h)\,,\end{aligned}$$

2.2. High discount rate $\left(r>\frac{a_1}{a_0}\right)$ *and low acquisition cost* $(b\leq pa_0)$:

$$\frac{\partial^2 K(\tau)}{\partial \tau^2}\left(\begin{array}{c}>\\=\\<\end{array}\right)0\ \mathit{iff}\ \ \tau\left(\begin{array}{c}<\\=\\>\end{array}\right)h-\frac{1}{r}\ln\left(\frac{ra_0p-a_1p}{rb-a_1p}\right).$$

Economic Interpretation To understand the age dependent investment level, let us rewrite (26) as follows:

$$c\frac{\partial I(\tau)}{\partial \tau}=b-pa_0e^{-r(h-\tau)}-\int_\tau^h pa_1e^{-r(h-z)}dz. \tag{28}$$

The first term of the r.h.s. of (28) reflects that investing in an older machine is advantageous from the point of view that less investment costs are incurred. The second term indicates that investing in an older machine implies that the lifetime of this machine is shorter which reduces the revenue stream. The third term of the r.h.s. of (28) resembles the fact that production with an older machine leads to a lower revenue flow per time unit.

Explaining Proposition 1 is now an easy job. (28) (cf. (27)) implies that, in case of a low discount rate, $\frac{\partial I(\tau)}{\partial \tau}$ increases with τ (according to the third term of the r.h.s. of (28) the revenue flow reduction takes place during a shorter time interval when τ increases), implying that $\frac{\partial I(\tau)}{\partial \tau}$ reaches its maximum for $\tau = h$. If $pa_0 > b$, $\frac{\partial I(\tau)}{\partial \tau}$ is negative for $\tau = h$, which implies that it will be negative for all possible ages. Since $I(h) = 0$, this in turn implies that the investment rate is positive for all ages of the capital stock, except off course for $\tau = h$. In case acquisition costs are high ($b > pa_0$), it holds that $\frac{\partial I}{\partial \tau} > 0$ for τ sufficiently large, which together with $I(h) = 0$ implies that the firm sells machines (only sufficiently young machines may be bought, because for these machines a large lifetime with positive revenues may counterbalance the high acquisition costs).

The fact that machines are sold in the case of large acquisition costs also holds when the discount rate is large. When acquisition costs are low, the firm again makes use of this by keeping the investment rate positive for all ages (except the maximal age). For high discount rate it further holds that $\frac{\partial I}{\partial \tau}$ decreases with τ. This is due to the fact that future revenues are heavily discounted, so that the effect of the shorter lifetime of the machine (given by the second term on the r.h.s. of (28)) is less.

The results concerning the levels of the capital stocks presented in Proposition 2 follow directly from the investment levels, but additionally it must be taken into account that older machines have a longer investment history. It holds that capital stock increases in a concave way with age if investment is positive but decreasing, capital stock decreases in a concave way with age if investment is negative (machines are sold) and decreasing, while capital stock decreases in a convex way if investment is negative but increasing.

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