# BIFURCATIONS TO PERIODIC SOLUTIONS IN A PRODUCTION/INVENTORY MODEL

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ABSTRACT. Total production costs show sometimes an S-shaped form. There are several ways in which a plant with given capacity can be adapted to a specific demand rate, one of them being adaptation of intensity per work hour. In this paper we present an application of the Hamilton-Hopf bifurcation to an inventory/production intensity splitting model with a nonconvex cost function. Our analysis provides a new proof that persistent oscillations may be optimal for arbitrary small discount rates. For zero discounting a "Hamilton Hopf bifurcation" occurs, leading to a family of periodic solutions bifurcating from a steady state. If the discount rate becomes positive, almost all periodic solutions vanish; only an unique branch of periodic solutions is obtained.

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## 1. Introduction

Usually, machines or production plants are constructed for a certain production intensity. In a neighbourhood of that intensity there is an efficient way of production, whereas any deviation below or above those levels leads to decreasing efficiency of production, i.e. to increasing unit production cost. Under those conditions the unit production cost show a $U$-shaped form. Writing these costs as function of the production intensity $v$, i.e. $\varphi(v)$, the (total) production cost, $\Phi(v)$, may be written as

$$\Phi(v) = v\,\varphi(v).$$

These costs show typically a concave-convex form.

We consider a production facility operating at production speed $v$. The production cost $\Phi(v)$ behaves as indicated in Fig. 1: There are two ranges of efficient production speeds, one at low level ("do nothing") and one at normal level ("operate regularly"). In between these levels $\Phi(v)$ is non-convex, indicating suboptimal resource utilization.

Let us assume that the produced good sells at a constant demand rate $d$ and the production beyond $d$ is stored in an inventory. If the production rate $v$ is below $d$, the demand is satisfied from the storage. If the inventory is exhausted and there is still an excess demand, it is backlogged, i.e. shortages are allowed. Since only small oscillations about the equilibrium state $v_0 = d$ are investigated, we need not explicitely restrict the production speed to non-negative values. These restrictions would have to be imposed, if the numerically calculated large amplitude motion shows negative production speeds.

When the production intensity can be adapted instantaneously and without cost, a policy of switching intensity between a high and a low level turns out to be optimal ("intensity splitting"). If the adjustment of production intensity is charged with costs and inventories are allowed, then even a constant demand can lead to persistent oscillations, i.e. to gradually intensity splitting of production rates.

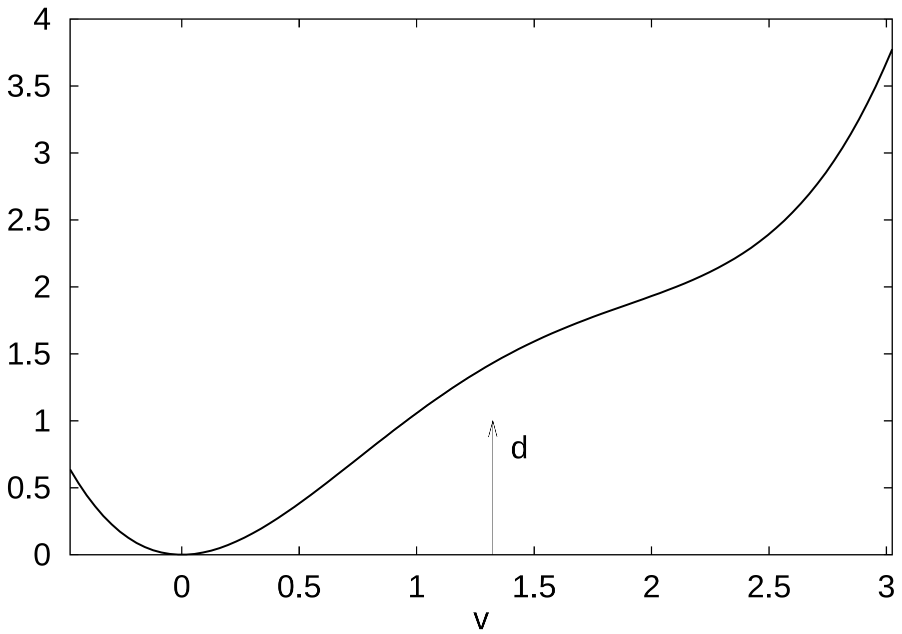


FIGURE 1. Production cost $\Phi(v)$ described by a fourth order polynomial with concave segment. At the efficient production levels $\Phi(v)$ is convex.

A well-known result of production planning says that non-convexities of the production cost function may lead to *intensity splitting.* This production pattern means that the speed of the production, i.e. its intensity, varies within a certain interval. Intensity splitting due to concave production costs occur both in statistic as well as in dynamic production planning. For a detailed discussion of the latter case compare Feichtinger and Sorger (1986) and Feichtinger et al. (1988).

Feichtinger and Sorger (1986) analyzed a simple production-inventory model for one good by Pontryagin's maximum principle; see also Feichtinger et al. (1988). Using the Hopf bifurcation theorem, these authors showed the existence of a cyclical solution of the necessary optimality conditions. In particular, it was proved that a sufficiently strong curvature of the concave part of the production costs is responsible for the occurrence of a stable limit cycle provided that the constant demand is "near" to the point of maximal curvature of the concave past of the production cost function. These results have been illustrated by a numerical example with various parameter values showing their influence on the shape of the optimal production program.

If we assume quadratic costs for the store and speed adjustments for the production device, we want to minimize the total costs

$$W = \int_0^\infty e^{-\varrho t} F(x, v, u) dt = \int_0^\infty e^{-\varrho t} \left[ \frac{h}{2} x^2 + \Phi(v) + \frac{c}{2} u^2 \right] dt, \tag{1}$$

subject to

$$\dot{x} = v - d, \tag{2}$$

$$\dot{v} = u. \tag{3}$$

Here $x$ denotes the stock of inventory and $u$ is the rate of change for the production speed; $(\cdot)^{\cdot}$ denotes differentiation with respect to time. For simplicity the inventory and shortage costs as well as the production adjustment costs are assumed to be quadratic, i.e. $h x^2/2$ and $c u^2/2$, respectively. The discount rate $\varrho$ is assumed small and non-negative. The variables $x$ and $v$ are considered as state variables, while $u$ acts as control variable.

For values of $d$ outside the nonconvex domain of $\Phi(v)$ the optimal strategies will converge to a steady state solution. If $d$ lies in the concave region of $\Phi(v)$, and if $h$ and $c$ are sufficiently small, we expect cyclic switching behaviour.

In Appendix B we show that for $\varrho > 0$ a Hopf bifurcation creates a branch of periodic solutions. For realistically small values of $\varrho$ the Hopf bifurcation theorem is valid in a very small parameter range, because close to the Hopf bifurcation boundary a nonsemi-simple pair of eigenvalues occurs, leading to terms of order $1/\varrho$ in the Center Manifold and Normal Form calculations. When we investigated the (ordinary) Hopf bifurcation in 1984, the Hamiltonian Hopf bifurcation was already well known in (celestial) mechanics, see e.g. Van der Meer [1]; but it took the authors several years to understand the terminology and become familiar with the presented methods to analyze the bifurcation.

The present investigation treats the discount rate $\varrho$ as small perturbation term of an autonomous Hamiltonian system. In the unperturbed case $\varrho = 0$ a so-called *Hamiltonian Hopf bifurcation* occurs: Two pairs of eigenvalues coincide along the imaginary axis. Shortly after the collision the system possesses 2 families of periodic solutions. If $\varrho$ is increased from zero, almost all periodic solutions are destroyed; only a single nontrivial periodic orbit survives. We will show that this solution is a proper candidate for the optimal strategy and fits well with the periodic solution calculated with the (ordinary) Hopf bifurcation.

Two things should be stressed in this context. First, we are dealing with a standard inventory-production model with no exceptional or pathological features. A substantial concavity in the production cost suffices to generate an optimal limit cycle even for an almost perfect farsighted production manager. Second, the proof of this result is new. The result that complex behaviour and optimality is compatible for arbitrary small discount rates is known since the work of Montrucchio (1989). In a continuous-time framework, he showed that arbitrary low myopia does not exclude the possibility of chaotic dynamics.[1]

In section 2 we state the equations of motion in dimensionless variables and consider the linearized equations at the steady state solution. In Section 3 and Appendix B we shortly state and prove the results for the (ordinary) Hopf bifurcation case.

Section 4 concentrates on the Hamiltonian Hopf bifurcation for $\varrho = 0$. The system is transformed into Normal Form, which provides a simple equation for the periodic solutions. In section 5 we show the existence of a one-parameter family of periodic orbits around the equilibrium for the conservative case.

In section 6 we re-introduce the discount rate $\varrho$ and study its effect on the periodic orbits.

[1]Montrucchio (1989) used a time transformation; the orbits of any dynamical system defined by a sufficiently smooth differential equation can be the optimal paths of a strictly concave model with an arbitrary low discount rate. In discrete-time models, however, the ‘trick’ with the time transformation does not work. Only very recently a rigorous proof of the possibility of chaos for low time preference rates was given by Nishimura and Yano (2000) whose proof is analytically complex. Nishimura et al. (2000) provided a proof using a simpler framework.

## 2. Necessary Optimality Conditions and Linearization of the Canonical System

Following Feichtinger and Hartl [7] we introduce the current-value Hamiltonian

$$H(x,v,u,\lambda,\mu) = -\left[\frac{h}{2}x^2 + \Phi(v) + \frac{c}{2}u^2\right] + \lambda(v-d) + \mu u, \tag{4}$$

where $\lambda$ and $\mu$ are the adjoint variables.

According to the maximum principle the optimal value $u^*$ of the control variable $u$ is now obtained by maximizing $H$ over $u$:

$$\frac{\partial H(x,v,u,\lambda,\mu)}{\partial u} = 0 \Longrightarrow u^* = \frac{\mu}{c}. \tag{5}$$

Eliminating $u$ in (3) by (5), we obtain the Hamiltonian system of equations in $(x,v,\lambda,\mu)$-space:

$$\dot{x} = \partial H/\partial\lambda = v - d, \tag{6}$$

$$\dot{v} = \partial H/\partial\mu = \mu/c, \tag{7}$$

$$\dot{\lambda} = \varrho\lambda - \partial H/\partial x = hx + \varrho\lambda, \tag{8}$$

$$\dot{\mu} = \varrho\mu - \partial H/\partial v = \Phi'(v) - \lambda + \varrho\mu. \tag{9}$$

Since we are looking for optimal strategies for infinite planning horizon, we first locate the stationary solution. If this steady state is a saddle point, – the linearized system has two stable and two unstable eigenvalues – then the candidates for optimal trajectories will lie on the stable manifold and converge to the stationary solution. If the real parts of all eigenvalues are positive, the steady state cannot be attained from neigbouring initial values and we have to search for different solutions, especially periodic orbits.

The steady state solution is found easily by equating the right hand sides in (6)...(9) to zero:

$$v_0 = d, \qquad \mu_0 = 0, \qquad \lambda_0 = \Phi'(d), \qquad x_0 = -\varrho\lambda_0/h. \tag{10}$$

The behaviour close to the stationary state is governed by the Jacobian

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/c \\ h & 0 & \varrho & 0 \\ 0 & \Phi''(d) & -1 & \varrho \end{pmatrix} \tag{11}$$

with characteristic polynomial

$$P(\sigma) = \sigma^2(\varrho - \sigma)^2 + \frac{\Phi''(d)\sigma(\varrho - \sigma) + h}{c}. \tag{12}$$

Equations (11) and (10) show that the slope of $\Phi$ at $v = d$ has no effect on the dynamics, but changes only the steady state $\lambda_0$ and $x_0$.

Now let

$$\gamma_{1,2} = \frac{-\Phi''(d) \pm \sqrt{\Phi''(d)^2 - 4hc}}{2c}$$

be the solutions of the quadratic equation $c\gamma^2 + \Phi''(d)\gamma + h = 0$, then the eigenvalues $\sigma$ of $\mathbf{A}$ are the solutions of $\sigma(\varrho - \sigma) = \gamma$

$$\sigma_{1,2,3,4} = \frac{\varrho}{2} \pm \sqrt{\frac{\varrho^2}{4} - \gamma_{1,2}}. \tag{13}$$

Setting $a_2 = \Phi''(d)$ and assuming $h > 0$, $c > 0$, and $0 \leq \varrho \ll 1$, we find the following possibilities for the arrangement of the eigenvalues

(1) $a_2 > 2\sqrt{hc}$: All eigenvalues are real. Since $\gamma_i < 0$, two eigenvalues are negative and the other two ones are positive. This situation applies in the strongly convex domain of $\Phi(v)$.

(2) $a_2 = 2\sqrt{hc}$: Two pairs of real eigenvalues meet on the real axis at $\sigma_{1,2} = (\varrho - \sqrt{\varrho^2 + 2a_2/c})/2$.

(3) $\varrho^2 c - \sqrt{\varrho^4 c^2 + 4hc} < a_2 < 2\sqrt{hc}$: All eigenvalues have nonzero imaginary parts; two eigenvalues lie in the left half plane.

(4) $a_2 = \varrho^2 c - \sqrt{\varrho^4 c^2 + 4hc} < 0$: A pair of previously stable eigenvalues crosses the imaginary axis leading to a Hopf bifurcation for $\varrho > 0$.

(5) $-2\sqrt{hc} < a_2 < \varrho^2 c - \sqrt{\varrho^4 c^2 + 4hc}$: All four complex valued eigenvalues lie in the right half plane. Note that the size of this domain is

proportional to $\varrho^2 \ll 1$; the critical eigenvalues move very quickly to the complex line $\operatorname{Re}\sigma = \varrho/2$.

(6) $a_2 = -2\sqrt{hc}$: Two pairs of complex eigenvalues meet at the double eigenvalues

$$\sigma_{1,2} = \frac{\varrho}{2} \pm \sqrt{\frac{\varrho^2}{4} + \frac{a_2}{2c}}.$$

(7) $a_2 < -2\sqrt{hc}$: All four eigenvalues lie on the complex line $\operatorname{Re}\sigma = \varrho/2$. This situation corresponds to a strongly concave shape of $\Phi(v)$ at $v = d$.

For a purely Hamiltonian system with $\varrho = 0$ the cases 4 and 6 coincide, two pairs of eigenvalues coincide at the imaginary axis. This event is called "Hamiltonian Hopf bifurcation" or "$1 : -1$-resonance". The corresponding bifurcation was investigated by Van der Meer [1] and Meyer and Hall [2].

From now on we will assume that $a_2 = \Phi''(d) < 0$, $\Phi^{(iv)}(d) > 0$, $h > 0$, and $c > 0$.

By properly shifting and rescaling the state variables, the time and the objective function we get rid of several constants and obtain simpler formulas: We may assume

$$d = 0, \quad a_2 = -1, \quad \Phi^{(iv)}(d) = 6, \quad h = 1.$$

The details of the rescaling calculations are given in Appendix A. The scaled equations of motion up to third order become

$$\begin{aligned} x' &= v, \\ v' &= \mu/c, \\ \lambda' &= x + \varrho\lambda, \\ \mu' &= \Phi'(v) - \lambda + \varrho\mu, \end{aligned} \tag{14}$$

with $\Phi'(v) = -v + a_3v^2 + v^3$ and $a_3 = \Phi'''(d)/2$.

Since by the rescaling the parameters $h$ and $a_2$ are now fixed values, ($h = 1$, $a_2 = -1$), the (rescaled) cost coefficient $c$ will serve as bifurcation parameter. The Hamiltonian Hopf bifurcation occurs at $c = 1/4$.

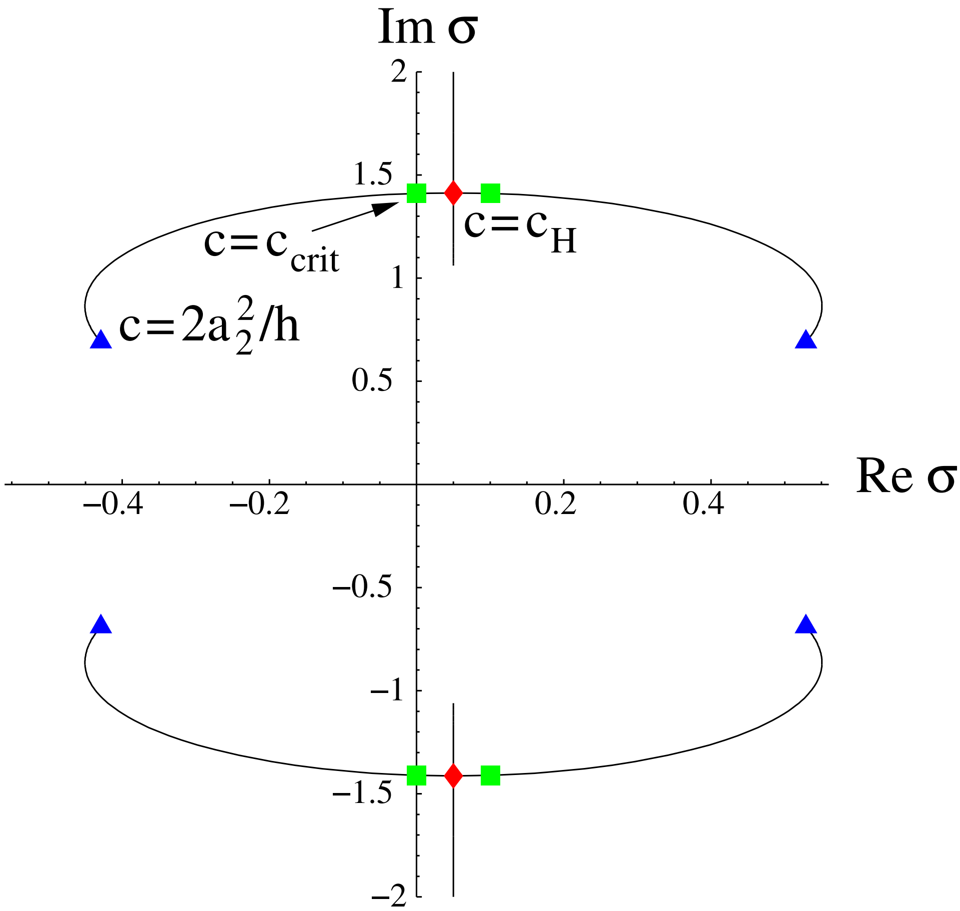


FIGURE 2. Motion of the eigenvalues in the complex plane for positive discount rate $\varrho$ and $a_2 = -1$, $h = 1$. For large values of $c$ – indicated by triangles – two eigenvalues are located in the left half plane and two eigenvalues in the right half plane. At $c_{\text{crit}} = a_2^2/(4 - 2\varrho^2)h$ (boxes) a pair of eigenvalues crosses the imaginary axis; at $c_H = a_2^2/4h$ (diamonds) the eigenvalue pairs meet along the line $\operatorname{Re}\sigma = \varrho/2$; in the conservative case $\varrho = 0$ the double eigenvalues occur at the imaginary axis, giving rise to a Hamiltonian Hopf bifurcation.

## 3. HOPF BIFURCATION FOR $\varrho > 0$

For positive discount rates $\varrho > 0$ we can show the following lemma.

**Lemma 1.** *For $\varrho > 0$ a family of periodic solutions bifurcates from the steady state at the parameter value*

$$c_{crit} = \frac{1}{4 - 2\varrho^2} \approx \frac{1}{4} + \frac{\varrho^2}{8}. \tag{15}$$

*If*

$$a_3^2 < \frac{27}{16} + O(\varrho), \tag{16}$$

*the bifurcating solution exists for $c < c_{crit}$.*

The proof of this lemma is given in Appendix B.

Using the original units, (16) is satisfied if

$$(\Phi'''(d))^2 < \frac{9}{8}|\Phi''(d)|\Phi^{(iv)}(d). \tag{17}$$

Inequality (17) says that the third derivative of the production cost function evaluated at the given constant demand $d$ is sufficiently small. This indicates that a hyperbolic limit cycle exists if the given constant demand $d$ is 'near' to the point of maximal concavity of the production cost function. Hence, (17) is satisfied, e.g., for $\Phi'''(d) = 0$.

If the inequality (17) is violated, the periodic solution bifurcates subcritically, that is it exists for $c > c_{\text{crit}}$ and cannot be attained, because it has one neutral and three unstable Floquet multipliers.

## 4. The Hamiltonian Hopf bifurcation

Now we investigate the Hamiltonian Hopf bifurcation with $\varrho = 0$. At $c_{\text{crit}} = 1/4$ two pairs of eigenvalues meet at $\pm i\sqrt{2}$. For $c < c_{\text{crit}}$ the eigenvalues move along the imaginary axis, while for $c > c_{\text{crit}}$ all eigenvalues have nonvanishing real part.

Since immediately after the bifurcation point the purely imaginary eigenvalues are non-resonant, Liapunov's Center theorem Siegel and Moser, [4]) applies: Two families of periodic solutions bifurcate from the stationary solution. But without knowledge about the nonlinear terms we cannot know whether these solutions are connected and whether they are stable. These questions can be answered by using Normal Form theory for the Hamiltonian Hopf bifurcation.

We start by transforming the linearized system at the critical parameter value to its real Normal Form by a symplectic change of coordinates: The matrix

$$\mathbf{E} = \begin{pmatrix} 0 & -\sqrt{2}/4 & -1 & 0 \\ 1/2 & 0 & 0 & \sqrt{2} \\ 3/4 & 0 & 0 & -\sqrt{2}/2 \\ 0 & -3\sqrt{2}/8 & 1/2 & 0 \end{pmatrix}$$

satisfies

$$\mathbf{E}^T \cdot \mathbf{J} \cdot \mathbf{E} = \mathbf{J}, \tag{18}$$

where $\mathbf{E}^T$ denotes the transpose of $\mathbf{E}$ and

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

With the symplectic change of coordinates ([2, 5])

$$(x, v, \lambda, \mu)^T = \mathbf{E} \cdot (q_1, q_2, p_1, p_2)^T \tag{19}$$

we obtain the transformed Hamiltonian function

$$K(q_1, q_2, p_1, p_2) = \sqrt{2}(p_2 q_1 - p_1 q_2) + (q_1^2 + q_2^2)/2 + a_3 v^3/3 + v^4/4, \tag{20}$$

with $v = q_1/2 + \sqrt{2} p_2$ by (19). Let

$$K_2(q_1, q_2, p_1, p_2) = \sqrt{2}(p_2 q_1 - p_1 q_2) + (q_1^2 + q_2^2)/2 \tag{21}$$

denote the quadratic part of $K$ and $K_j$ the terms of order $j$.

The Jacobian of the linearized system becomes

$$\mathbf{A}_c = \begin{pmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ -1 & 0 & 0 & -\omega \\ 0 & -1 & \omega & 0 \end{pmatrix} \qquad \text{with } \omega = \sqrt{2}. \tag{22}$$

4.1. **Normal Form for the unfolding of the linearized system.** Since we want to investigate the behaviour of the system in a small vicinity of the bifurcation point, we need to know the variation of $K_2$ and $\mathbf{A}$ for small deviations of $c$ from $c_{\text{crit}}$. Neglecting higher order terms in $\Delta c = c - c_{\text{crit}}$ the perturbed Hamiltonian becomes

(23)

$$\begin{aligned} K_2(q,p;c) =& \sqrt{2}(p_2q_1 - p_1q_2) + (q_1^2+q_2^2)/2 - \frac{\mu^2}{2c_{\text{crit}}^2}(c - c_{\text{crit}}) \\ =& \sqrt{2}(p_2q_1 - p_1q_2) + (q_1^2+q_2^2)/2 - \frac{(4p_1 - 3\sqrt{2}q_2)^2}{8}(c - c_{\text{crit}}). \end{aligned}$$

The Jacobian of the linearized system becomes

$$\mathbf{A}(c) = \mathbf{A}_c + \begin{pmatrix} 0 & 3\sqrt{2} & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -9/2 & 3\sqrt{2} & 0 \end{pmatrix}(c - c_{\text{crit}}). \tag{24}$$

**Lemma 2.** *By a symplectic near identity change of coordinates the quadratic Hamiltonian* (23) *can be transformed to*

$$\tilde{K}_2 = \frac{q_1^2+q_2^2}{2} + \nu\frac{p_1^2+p_2^2}{2} + (\omega+\alpha)(q_1p_2 - q_2p_1), \tag{25}$$

*with the unfolding parameters* $\nu = -2(c - c_{crit})$ *and* $\alpha = -3\sqrt{2}(c - c_{crit})/2$.

*The Jacobian of the perturbed system becomes*

$$\mathbf{A} = \begin{pmatrix} 0 & -(\omega+\alpha) & \nu & 0 \\ (\omega+\alpha) & 0 & 0 & \nu \\ 1 & 0 & 0 & -(\omega+\alpha) \\ 0 & 1 & (\omega+\alpha) & 0 \end{pmatrix}. \tag{26}$$

The proof of this lemma is given in Appendix C

While $\alpha$ represents only a small detuning of the frequency, the parameter $\nu$ determines the behaviour of the linearized system near the bifurcation point: For $\nu < 0$ all 4 eigenvalues move away from the imaginary axis, while for $\nu > 0$ the eigenvalues move along the imaginary axis. Therefore $\nu$ will act as bifurcation parameter.

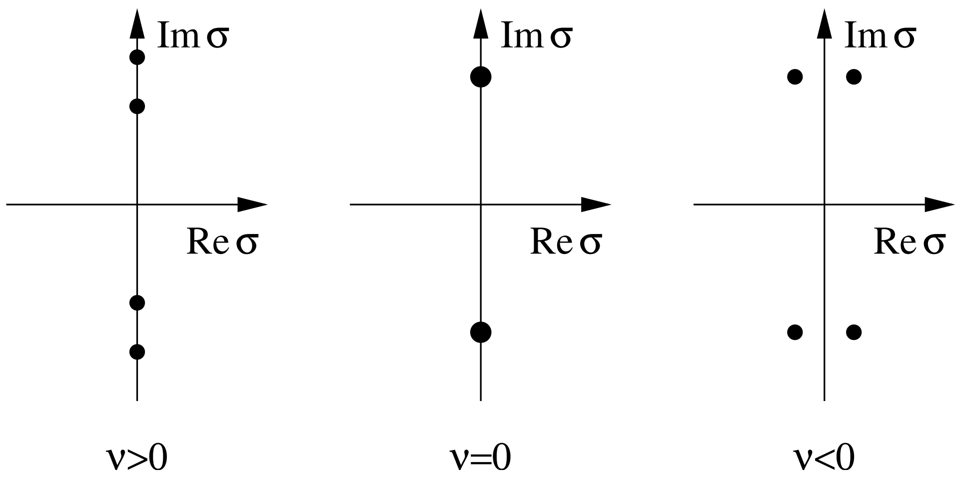


FIGURE 3. Location of eigenvalues of the linearized system close to the Hamiltonian Hopf bifurcation depending on the unfolding parameter $\nu = -2(c - c_{\text{crit}})$.

4.2. **Transformation of the nonlinear terms to Normal Form.** Now we look for a nonlinear, canonical change of coordinates to simplify the higher order terms in $K(q,p)$. According to Meyer and Hall [2] we look for functions

$$\tilde{K}(q,p) = K_2(q,p) + \tilde{K}_3(q,p) + \tilde{K}_4(q,p),$$
$$W(q,p) = W_3(q,p) + W_4(q,p),$$

where $\tilde{K}_j(q,p)$ and $W_j(q,p)$ contain monomials of order $j$, and satisfy

$$\tilde{K}_3 = K_3 + \{K_2, W_3\}, \tag{27}$$

$$\tilde{K}_4 = K_4 + \{K_2, W_4\} + \Big(\{K_3, W_3\} + \{\tilde{K}_3, W_3\}\Big)/2. \tag{28}$$

where

$$\{K, W\} = \sum_j \frac{\partial K}{\partial q_j} \cdot \frac{\partial W}{\partial p_j} - \frac{\partial K}{\partial p_j} \cdot \frac{\partial W}{\partial q_j} \tag{29}$$

denotes the Poisson bracket ([5]) of $K$ and $W$.

The function $\tilde{K}(q,p)$ is the fourth order expansion of the Normal Form Hamiltonian in the new coordinate system and $W(q,p)$ can be regarded as Hamilton function, whose flow determines the symplectic, near-identity coordinate transform.

It turns out that all cubic terms in $K$ can be eliminated ($\tilde{K}_3 = 0$) and $\tilde{K}_4$ contains only 3 terms. More important than the reduction of the number of coefficients is the fact that the transformed system can be chosen to be symmetric with respect to a continuous one-parameter group of transformations. This symmetry will make the system integrable.

Instead of working with the real variables $(q, p)$ it is convenient to use the canonical complex coordinates

$$z_1 = (q_1 + iq_2)/\sqrt{2}, \qquad z_2 = (p_1 - ip_2)/\sqrt{2}, \tag{30}$$

yielding the quadratic Hamiltonian

$$G_2(z_1, \overline{z}_1, z_2, \overline{z}_2) = i\omega(z_1 z_2 - \overline{z}_1\overline{z}_2) + z_1\overline{z}_1 \tag{31}$$

and linearized equations

$$z_1' = \partial G_2/\partial z_2 = i\omega z_1, \tag{32}$$

$$z_2' = -\partial G_2/\partial z_1 = -\overline{z}_1 - i\omega z_2. \tag{33}$$

The outcome of the Normal Form calculation is summarized in the

**Theorem 1.** *By a symplectic near-identity change of coordinates and a scaling the Hamiltonian* (20) *can be transformed to*

$$G = \omega S + \varepsilon(X + \nu Y + \alpha_1 Y^2) + O(\varepsilon)^2, \tag{34}$$

*where*

$$X = \frac{q_1^2 + q_2^2}{2} = z_1\overline{z}_1, \tag{35a}$$

$$Y = \frac{p_1^2 + p_2^2}{2} = z_2\overline{z}_2, \tag{35b}$$

$$S = q_1 p_2 - q_2 p_1 = 2\,\mathrm{Im}\, z_1 z_2, \tag{35c}$$

$$Z = q_1 p_1 + q_2 p_2 = 2\,\mathrm{Re}\, z_1 z_2 \tag{35d}$$

*are the Hilbert generators ([6]) for the action of the circle group* $\mathbf{S}^1$

$$z_1 \to \exp(i\vartheta)\, z_1, \quad z_2 \to \exp(-i\vartheta)\, z_2, \quad \textit{for } \vartheta \in \mathbf{S}^1. \tag{36}$$

*The coefficient $\alpha_1$ of the quartic term is given by*

$$\alpha_1 = -\frac{3}{2} + \frac{16}{9} a_3^2. \tag{37}$$

The proof of this theorem is given in [1, 2] and the lengthy details of the calculation are given in Appendix D.

## 5. Discussion of the bifurcation equations

The Hamiltonian (34) determines the behaviour of the dynamical system close to the stationary solution. In the literature about the Hamiltonian Hopf bifurcation (at least) two different approaches are used to calculate the periodic solutions: While the calculations in Meyer and Hall [2] are algebraically easier, the geometric view in Van der Meer [1] provides a better imagination of the dynamics close to the bifurcation point. Since the understanding of the conservative situation is also essential for the nonconservative problem ($\varrho > 0$), we present both approaches.

5.1. **Singularity theory approach by Van der Meer.** In [1] the discussion of the bifurcation equations is carried out using singularity theory. The new Hamiltonian (34) is expressed by the Hilbert Generators (35), which are invariant under the rotation (36). Since the generator $S$ is an integral of motion and the functions $S$, $X$, $Y$, $Z$ satisfy the relation

$$4\,X\,Y = S^2 + Z^2, \tag{38}$$

the Hamilton function can be regarded as function of 2 real variables with the constant parameter $S$. For fixed $S \neq 0$ the relation (38) defines a hyperbolic surface $M_S$ in the space $(X, Y, Z)$ and for $S = 0$ it defines a cone. Since $S$ is a first integral, each surface $M_S$ is an invariant manifold for the flow: Trajectories starting on a particular $M_S$ remain on that $M_S$ for all times. The level curves of $H$ on the surface $M_S$ correspond to trajectories. Since the "coordinates" $X$, $Y$, $S$ and $Z$ are related to orbits in $(p, q)$-space, stationary points correspond to steady solutions (trivial solution for $Y = 0$) or periodic orbits. Closed trajectories represent quasiperiodic motions of the system.

The dynamics can be rewritten using the Hilbert generators $X$, $Y$, $Z$ and $S$ by calculating the Poisson bracket ([5]) between these coordinates

$$\begin{array}{c|cccc} \{\cdot,\cdot\} & X & Y & Z & S \\ \hline X & 0 & Z & 2X & 0 \\ Y & -Z & 0 & -2Y & 0 \\ Z & -2X & 2Y & 0 & 0 \\ S & 0 & 0 & 0 & 0 \end{array} \tag{39}$$

and using formula (6.9) in [21]

$$\dot{X} = \{X, G\} = \{X, Y\}\,\partial G/\partial Y = Z\varepsilon(\nu + 2\alpha_1 Y), \tag{40a}$$

$$\dot{Y} = \{Y, G\} = \{Y, X\}\,\partial G/\partial X = -Z\varepsilon, \tag{40b}$$

$$\begin{aligned} \dot{Z} = \{Z, G\} &= \{Z, X\}\,\partial G/\partial X + \{Z, Y\}\,\partial G/\partial Y \\ &= \varepsilon\left(-2X + 2Y(\nu + 2\alpha_1 Y)\right), \end{aligned} \tag{40c}$$

$$\dot{S} = \{S, G\} = 0. \tag{40d}$$

Using the Poisson bracket relations (39) it can be shown that

$$\{S^2 + Z^2 - 4XY, f\} = 0 \quad \text{for } f \in \{X, Y, Z, S\},$$

so that $F = S^2 + Z^2 - 4XY$ is a Casimir function and the dynamical system leaves the level sets $M_S$ of $F = 0$ invariant.

5.1.1. *Stationary solutions in the reduced problem.* Since the stationary solutions in the reduced problem relate to periodic and steady state solutions of the full problems, we have a closer look at these stationary solutions: Although the Normal Form expression (34) for the Hamilton function looks quite simple, following the investigation in [1], we prefer to use the Hilbert generators $Y$ and $Z$ as coordinates to work with, because the projection of the surfaces $S^2 + Z^2 = 4XY$ to the $(X, Y)$-plane is singular at $Z = 0$, where the stationary solutions are located. Therefore we express the variable $X$ by $(S^2 + Z^2)/4Y$

and determine local extrema of the function

$$G = \omega S + \varepsilon\left(\frac{S^2 + Z^2}{4Y} + \nu Y + \alpha_1 Y^2\right) \tag{41}$$

for fixed values of $S$ and $Y \geq 0$. The complete investigation of this problem is carried out in [1].

**Hyperbolic case $\alpha_1 < 0$:** The behaviour of the system close to the bifurcation point depends on the sign of the coefficient $\alpha_1$ in (41). For $\alpha_1 < 0$ we obtain a family of hyperbolic periodic solutions, which are candidates for the long term behaviour of our optimal control problem (1).

Since $G$ is an even function in $Z$, its extreme values occur along the line $Z = 0$. For $\nu < 0$, which corresponds to $c > c_{\text{crit}}$, the function $S^2/4Y+\nu Y+\alpha_1 Y^2$ is monotonically decreasing in $Y$; the only stationary solution of the system is the trivial solution $S = 0$, $Y = 0$, $Z = 0$.

For $\nu > 0$ and $S = 0$ the function $G$ has a saddle point at $Z = 0, Y = -\nu/2\alpha_1$ and a local minimum at the origin. For small values of $|S|$ these stationary points persist; at $S^2 = 4\nu^3/27\alpha_1^2$ the steady states collide and for larger values of $|S|$ no stationary solutions exist.

Figures 5 shows two surfaces for $\alpha_1 < 0$ and different values of $S$. In the left picture two different steady states are visible: a center close to the origin encircled by closed trajectories and a saddle point, corresponding to an unstable periodic orbit. The right picture shows the situation for a larger value of $S$: Both steady states have disappeared in a collision and all trajectories are unbounded.

**Elliptic case $\alpha_1 > 0$:** If the coefficient $\alpha_1 > 0$, the reduced Hamiltonian $G$ in (41) is strictly convex in $Y$ along the line $Z = 0$. Therefore it has exactly one global minimum for all values of the bifurcation parameter $\nu$. When all four eigenvalues of the Jacobian lie on the imaginary axis, that is for $\nu > 0$, the reduced system possesses a family of equilibrium points for both signs of $S$. Since these equilibria correspond to minima of the Hamiltonian $G$, they are stable.

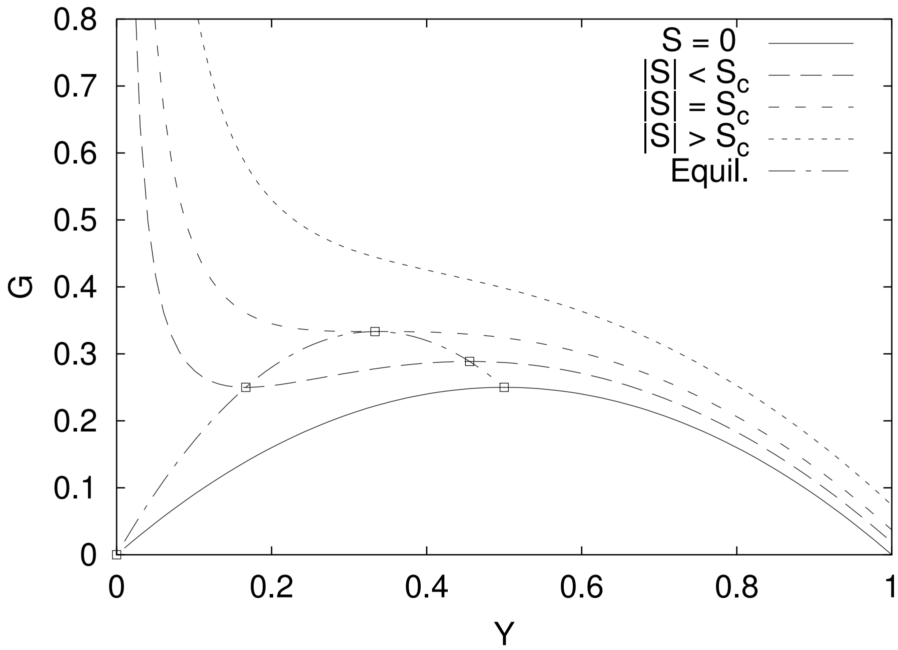


FIGURE 4. Reduced Hamilton function (41) with $Z = 0$ for $\nu > 0$, $\alpha_1 < 0$, and different values of the constant $S$.

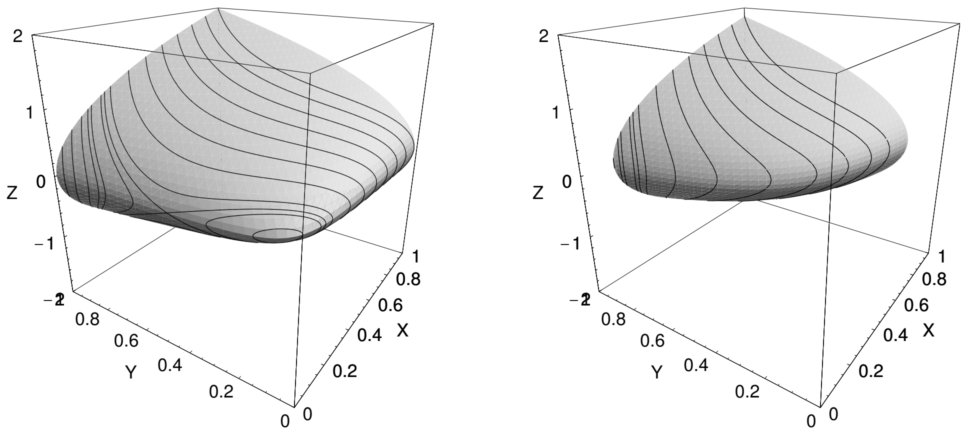


FIGURE 5. Trajectories of the reduced system on the manifold $Z^2 + S^2 = 4XY$ for $\alpha_1 < 0$, $\nu > 0$ and different values of the first integral $S$.

Also for $\nu < 0$ the reduced system has a family of stable equilibria (Fig. 6), but this family is separated from the trivial solution, which is a hyperbolic equilibrium (Fig. 3).

While the nontrivial equilibria exist only for $\nu > 0$ in the hyperbolic case $\alpha_1 < 0$, stable equilibria exist for all $\nu \neq 0$ in the elliptic case $\alpha_1 > 0$.

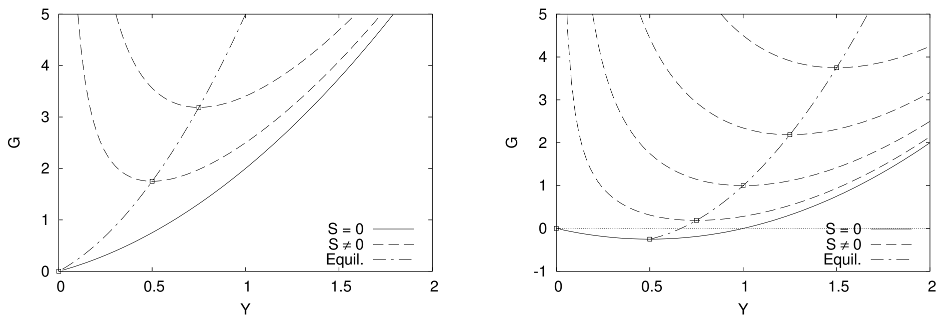


FIGURE 6. Reduced potential $G$ for $Z = 0$, $\alpha_1 > 0$ and $\nu > 0$ (left) and $\nu < 0$ (right) and different values of the first integral $S$. The dash-dotted curve connects the equilibrium solutions. For $\nu < 0$ the equilibria curve is separated from the origin.

The trivial solution $(X, Y, Z, S) = \mathbf{0}$ is a saddle point for $\nu < 0$ and a global minimum of $G$ for $\nu > 0$. Furthermore there exists a minimum of $G$ on all surfaces $M_S$ for $S \neq 0$ and all values of $\nu$.

5.2. **Dynamic approach by Meyer and Hall.** Let us now follow the discussion in [2]. The vector field induced by the Hamiltonian $H$ in (34) up to $O(\varepsilon)$ is given by

$$z_1' = i\omega z_1 + \varepsilon\,(\nu\overline{z}_2 + 2\alpha_1|z_2|^2\overline{z}_2), \tag{42}$$

$$\overline{z}_2' = -\varepsilon z_1 + i\omega\overline{z}_2. \tag{43}$$

The corresponding equations for $\overline{z}_1$ and $z_2$ are obtained by taking the complex conjugates of (42) and (43), respectively.

Since by Normal Form reduction the system (42) and (43) is equivariant with respect to the one-parameter group $\mathbf{S}^1$ of rotations

$$z_1 \to e^{i\tau}z_1, \qquad \overline{z}_2 \to e^{i\tau}\overline{z}_2, \tag{44}$$

we expect the bifurcating periodic solutions to be rotational motions of the form $z_1 = w_1 e^{i(\omega+\varepsilon\vartheta)t}$ and $z_2 = w_2 e^{-i(\omega+\varepsilon\vartheta)t}$ for some fixed amplitudes $w_j$ and detuning $\vartheta$. Inserting this ansatz into (43) gives

$$w_1 = -i\vartheta\overline{w}_2. \tag{45}$$

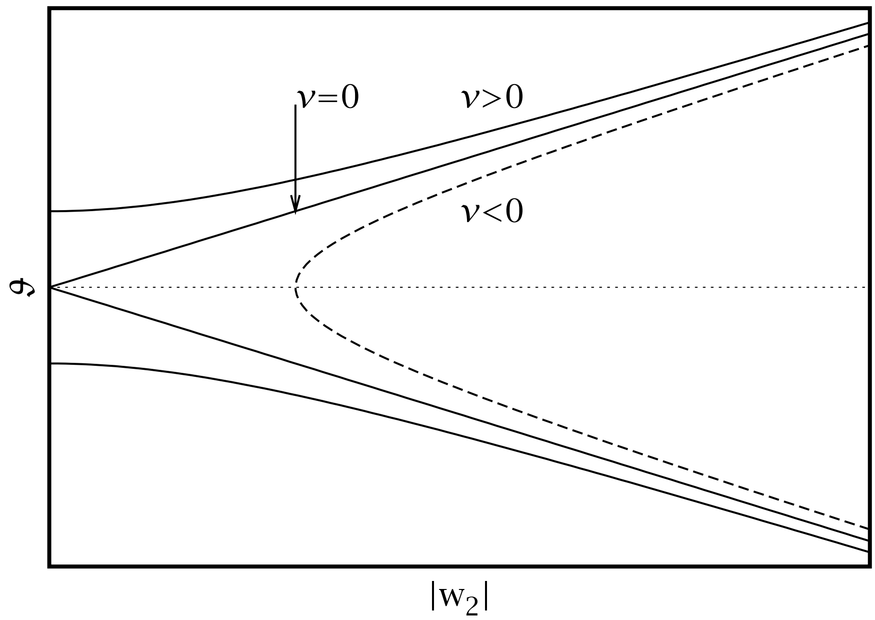


FIGURE 7. Graph of (46) when $\alpha_1 > 0$.

Inserting (45) in (42) leads first to

$$w_1' = -i\vartheta\,(i(\omega + \varepsilon\vartheta))\overline{w}_2 = -i\omega(i\vartheta\,\overline{w}_2) + \varepsilon\,(\nu + 2\alpha_1|w_2|^2)\overline{w}_2$$

and by division by $\varepsilon\overline{w}_2$ to

$$\vartheta^2 = \nu + 2\alpha_1|w_2|^2. \tag{46}$$

Depending on the sign of $\alpha_1$, there are 2 different cases: Case A when $\alpha_1$ is positive, and Case B when $\alpha_1$ is negative.

**Case A:** $\alpha_1 > 0$. For fixed values of the distinguished parameter $\nu$, the graph of (46) is a hyperbola, but only the part where $|w_2| \geq 0$ is of interest. Each point on the graph corresponds to a periodic solution of the system. For $\nu > 0$ two unbounded families of periodic orbits exist. Also for $\nu < 0$ a family of periodic orbits exists, but it is not connected with the trivial state.

**Case B:** $\alpha_1 < 0$. For fixed $\nu > 0$ the graph of (46) is an ellipse. The two branches of periodic solutions predicted by Lyapunov's Center theorem are globally connected. As $\nu$ tends to zero, the family of solutions shrinks to a point and disappears. For $\nu < 0$ there are no periodic solutions.

Along the lines $\vartheta^2 = 4|\alpha_1|\,|w_2|^2$ the stability of the periodic solutions changes: For $\vartheta^2 > 4|\alpha_1|\,|w_2|^2$ the solutions are elliptic, while for $\vartheta^2 < 4|\alpha_1||w_2|^2$ they are hyperbolic.

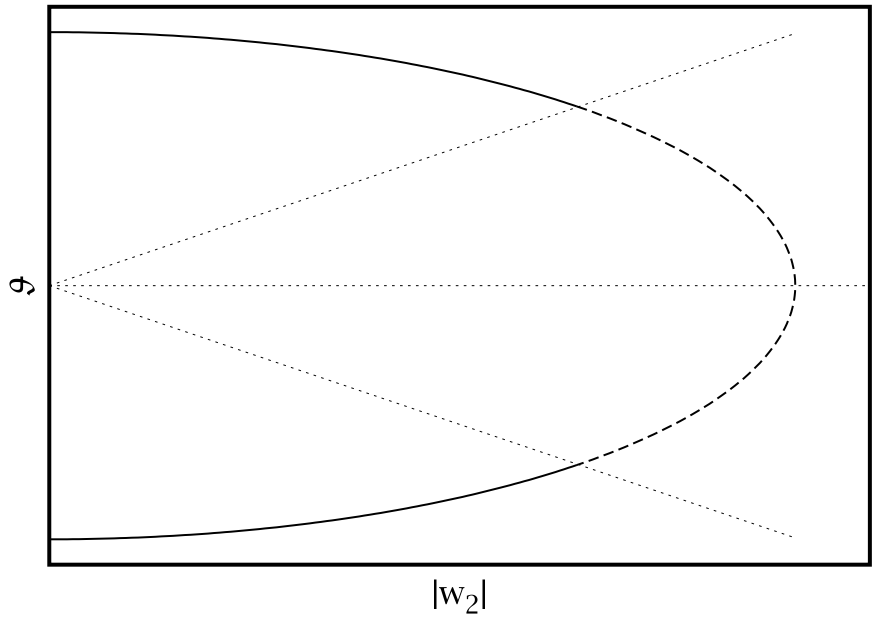


FIGURE 8. Graph of (46) when $\alpha_1 < 0$ and $\nu > 0$. Between the dotted lines the period solutions are hyperbolic; outside of this sector the solutions are elliptic.

In [2] it is shown that the terms of order $O(\varepsilon^2)$ do not change the bifurcation diagrams significantly.

## 6. NON-CONSERVATIVE PERTURBATION OF THE HAMILTONIAN HOPF BIFURCATION

Up to now we treated the purely conservative system to obtain information about the Hamiltonian structure of our model. Next we assume a small positive discount rate $\varrho > 0$. In section 2 it was shown that the eigenvalues of the linearized system at the steady state shift to the right by $\varrho/2$. In theorem 2 we apply Normal Form theory of matrices ([8]) to calculate the influence of the discount rate on the linear part of the bifurcation equations (46). In theorem 3 we study the periodic solutions under the perturbation by the discount rate $\varrho$.

6.1. **Normal Form for the non-conservative perturbed linear system.** Since the introduction of the discount rate destroys the Hamiltonian structure of the system, the Normal Form of the perturbed matrix cannot be derived by the method used in subsection 4.1. Instead we have to use the method for general matrices presented in [8].

**Theorem 2.** *For small discount rates* $0 < \varrho \ll 1$ *the Jacobian of the perturbed linear system can be rewritten in the form*

$$\mathbf{A} = \begin{pmatrix} \varrho/2 & -(\omega+\alpha) & \nu & 0 \\ (\omega+\alpha) & \varrho/2 & 0 & \nu \\ -1 & 0 & \varrho/2 & -(\omega+\alpha) \\ 0 & -1 & (\omega+\alpha) & \varrho/2 \end{pmatrix} \tag{47}$$

*by a small linear change of coordinates.*

The proof of this theorem is given in Appendix E.

6.2. **Discussion of the perturbed bifurcation equations.** In this subsection we study the periodic solutions of the (nonconservatively) perturbed system.

**Theorem 3.** *For small positive discount rates* $\varrho$ *a branch of periodic orbits bifurcates from the trivial solution.*

*Proof.* Introducing the Normal Form terms $\varrho\mathbf{I}_4/2$ for non-zero discount rate into the bifurcation equation (46) and rescaling $\varrho \to \varepsilon\varrho$ we get the perturbed equations

$$z_1' = (\varepsilon\varrho/2 + i\omega)z_1 + \varepsilon(\nu + 2\alpha_1|z_2|^2)\bar{z}_2, \tag{48a}$$

$$\bar{z}_2' = -z_1 + (\varepsilon\varrho/2 + i\omega)\bar{z}_2. \tag{48b}$$

Since the system is still equivariant with respect to rotations (36), we proceed as in the purely Hamiltonian case: Using the ansatz $z_1 = e^{i(\omega+\varepsilon\vartheta)t}w_1$ and $\bar{z}_2 = e^{i(\omega+\varepsilon\vartheta)t}\bar{w}_2$, eliminating $w_1$ from (48b) and cancelling the common factor

$\varepsilon\overline{w}_2$ yields the bifurcation equation

$$-(i\vartheta - \varrho/2)^2 = (\nu + 2\alpha_1|w_2|^2). \tag{49}$$

Since the left hand side in (49) contains imaginary entries, we solve (49) separately for the imaginary and real parts:

Setting the imaginary part of (49) to zero, yields the condition

$$\vartheta = 0.$$

Therefore only periodic orbits with $\vartheta = 0$ survive the perturbation by the nonconservative perturbation.

Now we can solve for the real part and obtain the equation

$$-\varrho^2/4 = (\nu + 2\alpha_1|w_2|^2), \tag{50}$$

admitting a one-parameter family of periodic solutions:

$$\nu = -2\alpha_1|w_2|^2 - \varrho^2/4. \tag{51}$$

The solution branches off from the trivial state at $\nu = -\varrho^2/4$. If $\alpha_1 < 0$ it exists for $\nu > -\varrho^2/4$, that is for $c < c_{\text{crit}}$. By (37) this case applies if the cubic coefficient $a_3$ in $\Phi(v)$ is sufficiently small. If $\alpha_1 > 0$ the periodic solution bifurcates subcritically and is unstable. □

Comparing that result with the outcome of the ordinary Hopf bifurcation, we find that we discovered the same solutions and the same conditions (16) and (37) for the criticallity of the bifurcation.

For a geometrical visualization of the dynamics for positive discount rate $\varrho$, we consider again the equations for the Hilbert generators $X$, $Y$, $Z$ and $S$. The perturbed equations in Normal Form become

$$\dot{X} = \varepsilon\varrho X + \{X, G\} = \varepsilon\left(\varrho X + Z(\nu + 2\alpha_1 Y)\right), \tag{52a}$$

$$\dot{Y} = \varepsilon\varrho Y + \{Y, G\} = \varepsilon\left(\varrho Y - Z\right), \tag{52b}$$

$$\dot{Z} = \varepsilon\varrho Z + \{Z, G\} = \varepsilon\left(\varrho Z - 2X + 2Y(\nu + 2\alpha_1 Y)\right), \tag{52c}$$

$$\dot{S} = \varepsilon\varrho S + \{S, G\} = \varepsilon\varrho S. \tag{52d}$$

By (52d) the function $S$ grows exponentially for nontrivial initial values. Therefore the surfaces $M_S$ for $S \neq 0$ are not invariant surfaces for $\varrho \neq 0$; the trajectories cross these surfaces. But the surface

$$M_0 = \{(X, Y, Z) \,|\, Z^2 = 4XY\}$$

remains invariant, because

$$\frac{d(Z^2 - 4XY)}{dt} = \frac{\varepsilon\varrho}{2}(Z^2 - 4XY).$$

The bounded solutions of the system are therefore constrained to the cone $M_0$. As can be seen in Fig. 9, the introduction of the discount rate changes the dynamics on this surface considerably: While there is a family of closed orbits in the conservative problem, all these periodic orbits are broken by the perturbation.

The discount rate $\varrho$ acts like a negative damping parameter in mechanical systems: The family of closed orbits around the trivial solution is replaced by spirals. Since the saddle point is robust under small perturbations, the perturbed system has a saddle point nearby. One branch of the stable manifold (heavy curve in Fig. 9) spirals towards the origin for $t \to -\infty$. The trajectories emanating on the stable manifold converge to the saddle point and remain finite, while all other initial values lead to unbounded solutions.

## 7. Conclusions

In this paper a standard inventory/production model is analyzed by using the Hamilton-Hopf bifurcation. It is true that the analysis is sometimes

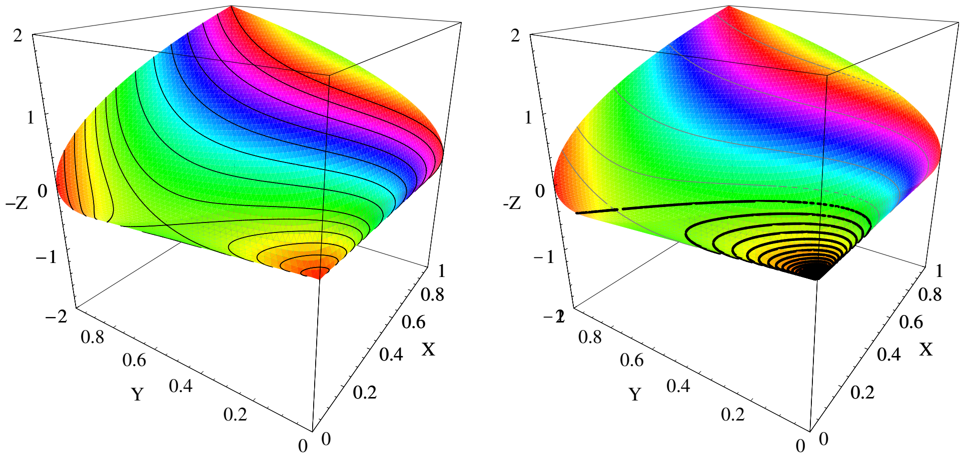


FIGURE 9. Phaseflow of the reduced system on the surface $S = 0$ for $\varrho = 0$ (left) and $\varrho > 0$ (right). The orientation of the vertical axis is reversed to display the major part of the stable manifold (heavy curve) on the upper sheet.

tedious, but some of the results obtained are interesting and economically important. Let us briefly summarize the main results.

We found a new proof that stable limit cycles can be optimal for arbitrary small discount rates, i.e. for farsighted decision-makers. It would be an interesting task to study whether our analysis might be extended to cover even more complex optimal paths.

While the ordinary Hopf bifurcation theorem works only for 'large' values of the discount rate and for very small variations of the distinguished parameter $c$, the approach by the Hamiltonian Hopf bifurcation is also valid for infinitesimally small values of the discount rate and modest variations of the parameter $c$.

Furthermore the treatment of the system as small perturbation of the conservative Hamilton-Hopf system provides geometric insights into the system, which are not available with the usual Hopf bifurcation.

## APPENDIX A. NONDIMENSIONALIZATION OF THE SYSTEM

In this section we shift and rescale the state variables $x$ and $v$, the control variable $u$, the time $t$ and the cost function $F$ to get rid of several parameters and obtain simple equations.

Let $a_2 = -\Phi''(d)$, $a_4 = \Phi^{(iv)}(d)/6$ and set

$$v = d + \alpha_v \tilde{v}, \quad x = x_0 + \alpha_x \tilde{x}, \quad t = \alpha_t \tilde{t}$$
$$u = \alpha_u \tilde{u}, \qquad \tilde{F} = \alpha_t \alpha_F F, \tag{53}$$

where the factor $\alpha_t$ in $\tilde{F}$ takes care of the time scaling. Since we study the system in the concave region of the cost function $\Phi(v)$, the coefficient $a_2$ is positive.

Inserting into $\tilde{F}$ we get the expression for the new cost function

$$\tilde{F} = \alpha_t \alpha_F \left[ \frac{h}{2} \alpha_x^2 \tilde{x}^2 + \Phi(d + \tilde{v}) + \frac{c}{2} \alpha_u \tilde{u}^2 \right].$$

Now we choose the scaling parameters $\alpha_v \dots \alpha_u$, such that the differential equations for $\tilde{x}$ and $\tilde{v}$ become

$$\tilde{x}' = \tilde{v}, \tag{54a}$$

$$\tilde{v}' = \tilde{u}, \tag{54b}$$

where $(\cdot)'$ denotes differentiation with respect to the rescaled time $\tilde{t}$. We require also that the coefficient of $\tilde{x}^2$ in $\tilde{F}$ becomes $1/2$ and $\tilde{F}''(0) = -1$ and $\tilde{F}^{(iv)}(0) = 6$. This leads to the set of equations

$$\alpha_x = \alpha_t \alpha_v, \tag{55a}$$

$$\alpha_v = \alpha_t \alpha_u, \tag{55b}$$

$$\alpha_t \alpha_F \alpha_x^2 h = 1, \tag{55c}$$

$$\alpha_t \alpha_F \alpha_v^2 a_2 = 1, \tag{55d}$$

$$\alpha_t \alpha_F \alpha_v^4 a_4 = 1. \tag{55e}$$

Positive solutions for the coefficients are given by

$$\alpha_v = \sqrt{a_2/a_4}, \tag{56a}$$

$$\alpha_x = a_2/\sqrt{a_4\, h}, \tag{56b}$$

$$\alpha_t = \sqrt{a_2/h}, \tag{56c}$$

$$\alpha_F = a_4\sqrt{h/a_2^5}, \tag{56d}$$

$$\alpha_u = \sqrt{h/a_4}. \tag{56e}$$

Inserting the scaled variables in the cost function $F$ yields the modified control problem

$$\max_{\tilde{u}} - \int_0^\infty e^{-\tilde{\varrho}\tilde{t}} \left[ \frac{(x_0+\tilde{x})^2}{2} + \tilde{\Phi}(\tilde{v}) + \frac{\tilde{c}\tilde{u}^2}{2} \right] d\tilde{t} \tag{57}$$

with (54). The adaption cost parameter $\tilde{c}$ is given by

$$\tilde{c} = h\, c/a_2^2, \tag{58}$$

and the rescaled discount rate is $\tilde{\varrho} = \alpha_t \varrho$.

The cubic term in $\tilde{\Phi}(\tilde{v})$ becomes $\Phi'''(d)\tilde{v}^3/(6\sqrt{a_2 a_4})$.

Next we get rid of the constant and linear terms in the cost function $\tilde{F}$: Since the constant terms $x_0^2/2$ and $\tilde{\Phi}(0)$ do not enter the dynamics at all, they can be simply neglected. The linear terms $x_0\tilde{x}$ and $\tilde{\Phi}'(0)\, v$ can be removed by adding $(\lambda_0 - \tilde{\Phi}'(0))v$ to $\tilde{F}$ and shifting the co-state variable $\lambda$: $\lambda = \lambda_0 + \tilde{\lambda}$ with $\lambda_0 = \tilde{\Phi}'(0)$.

## Appendix B. Proof of Lemma 1

To prove the existence of the periodic orbits for the (ordinary) Hopf bifurcation, we show that the sufficient conditions of Theorem 2 in [16] are satisfied:

(1) Since we assume that the cost function $\Phi(v)$ is in $C^4$, the vector field (14) is in $C^3$.

(2) The trivial solution $(x, v, \lambda, \mu) = \mathbf{0}$ satisfies (14) for all $c > 0$.

(3) At $c = c_{\text{crit}}$ the Jacobian $\mathbf{A}$ at the trivial solution has the eigenvalues

$$\sigma_{1,2} = \pm i\omega \qquad \text{and } \lambda_{3,4} = \varrho \pm i\omega, \tag{59}$$

with $\omega = \sqrt{2 - \varrho^2}$.

The eigenvalue $\lambda_1$ crosses the imaginary axis with non-zero speed

$$\frac{d\,\mathrm{Re}\,\sigma_1}{dc}\bigg|_{c=c_{\mathrm{crit}}} = \mathrm{Re}\,\frac{-2i\omega^3\,(1 - i\varrho\omega - \omega^2)}{\varrho\,(\varrho - 2i\omega)} \approx \left(\frac{-2}{\varrho} + \frac{9\varrho}{4}\right).$$

By Theorem 2 in [16] these conditions ensure that a family of periodic orbits bifurcates from the trivial solution at the critical parameter value $c = c_{\mathrm{crit}}$. To determine, whether the bifurcation is supercritical, we have to calculate the cubic coefficient $\alpha_3$ in the bifurcation equation

$$\dot{z} = \sigma_1 z + \alpha_3 |z|^2 z. \tag{60}$$

The calculation of this coefficient is outlined in [3,16] and involves three steps:

(1) By a linear change of coordinates the linear system is transformed to Jordan Normal Form and decouples into two scalar complex equations.

The matrix of eigenvectors is given by

$$\mathbf{E} = \begin{pmatrix} 2 & 2 & -\dfrac{(\varrho - i\omega)\omega^2}{\varrho + i\omega} & -\dfrac{(\varrho + i\omega)\omega^2}{\varrho - i\omega} \\ 2i\omega & -2i\omega & -\dfrac{2\omega^2}{\varrho + i\omega} & -\dfrac{2\omega^2}{\varrho - i\omega} \\ -\varrho - i\omega & -\varrho + i\omega & \dfrac{(\omega + i\varrho)\omega}{\varrho + i\omega} & \dfrac{(\omega - i\varrho)\omega}{\varrho - i\omega} \\ -1 & -1 & -1 & -1 \end{pmatrix}, \tag{61}$$

where the $j$-th column contains the eigenvector corresponding to $\sigma_j$.

Let $x = (x, v, \lambda, \mu)^T$ and substitute

$$x = \mathbf{E}z, \quad \text{with } z = (z_1, \overline{z}_1, z_2, \overline{z}_2)^T,$$

where $\overline{z}_j$ denotes the complex conjugate of $z_j$.

In the new coordinates $z$ the differential equation reads

$$z_1' = i\omega z_1 + g_1(z), \tag{62a}$$

$$z_2' = (\varrho + i\omega) z_2 + g_3(z), \tag{62b}$$

where $g_j(z)$ denotes the $j$-th component of

$$g = \mathbf{E}^{-1} \cdot f(\mathbf{E}x),$$

and $f$ contains the nonlinear terms in the original system (14). Since the only nonlinear terms in (14) are contributed by $\tilde{\Phi}'(v)$, we just have to compute the quantities

$$v = (\mathbf{E}z)_2 = 2i\omega z_1 - 2i\omega\overline{z}_1 - \frac{2\omega^2}{\varrho - i\omega}z_2 - \frac{2\omega^2}{\varrho + i\omega}\overline{z}_2,$$

and

$$\gamma = \mathbf{E}^{-1}e_4 = \begin{pmatrix} \dfrac{\omega}{(\omega^2-2)\omega - i(\omega^2+2)\varrho} \\ \dfrac{\omega}{(\omega^2-2)\omega + i(\omega^2+2)\varrho} \\ \dfrac{-(\varrho + i\omega)}{(2+\omega^2)\varrho + i(\omega^2-2)\omega} \\ \dfrac{-(\varrho - i\omega)}{(2+\omega^2)\varrho - i(\omega^2-2)\omega} \end{pmatrix},$$

from which we find the nonlinear terms

(63) $$g_j(z) = \gamma_j \cdot (a_3 v^2 + v^3).$$

(2) Using Center Manifold reduction we get rid of the unstable variable $z_2$. In order to obtain the proper cubic term in the bifurcation equation, we need the expansion of the Centre Manifold to second order. Setting

(64) $$z_2 = h_{11}z_1^2 + h_{12}z_1\overline{z}_1 + h_{22}\overline{z}_1^2$$

and inserting in (62), we obtain to second order in $z_1$

(65a) $$z_2' = 2i\omega h_{11}z_1^2 - 2i\omega h_{22}\overline{z}_1^2$$

$$= (\varrho + i\omega)(h_{11}z_1^2 + h_{12}z_1\overline{z}_1 + h_{22}\overline{z}_1^2)$$

(65b) $$+ g_{311}z_1^2 + g_{312}z_1\overline{z}_1 + g_{322}\overline{z}_1^2$$

where $g_{3kj}$ denote the coefficients of $z_1^2$, $z_1\overline{z}_1$, and $\overline{z}_1^2$ in $g_3$.

Comparing coefficients in (65) we obtain

$$h_{11} = g_{311}/(i\omega - \varrho),$$

$$h_{12} = -g_{312}/(i\omega + \varrho),$$

$$h_{22} = -g_{322}/(3i\omega + \varrho).$$

Inserting (64) into the mixed quadratic terms in (62a)

$$g_{113}z_1z_2 + g_{114}z_1\overline{z}_2 + g_{123}\overline{z}_1z_2 + g_{124}\overline{z}_1\overline{z}_2$$

yields the correction term for the coefficient of $z_1^2\overline{z}_1$

$$\alpha_{3,\mathrm{cm}} = g_{113}h_{12} + g_{114}\overline{h}_{22} + g_{123}h_{11} + g_{124}\overline{h}_{22}. \tag{66}$$

After some calculations we obtain the expression

$$\begin{aligned}\alpha_{3,\mathrm{cm}} = \gamma_1 \Bigg( &\frac{64i\gamma_3\omega^5}{(\varrho + i\omega)^2} + \frac{64i\bar{\gamma}_3\omega^5}{(\varrho - i\omega)^2} \\ &+ \frac{32i\gamma_3\omega^5}{\varrho^2 + \omega^2} + \frac{32i\bar{\gamma}_3\omega^5}{\varrho^2 - 4i\varrho\omega - 3\omega^2}\Bigg) a_3^2.\end{aligned} \tag{67}$$

(3) After reducing the system to a scalar complex differential equation we apply Normal Form Theory to get rid of all quadratic and most cubic terms. With the nonlinear near-identity change of coordinates

$$z_1 = w + k_{11}w^2 + k_{12}w\overline{w} + k_{22}\overline{w}^2 \tag{68}$$

we obtain the equation

$$\dot{z}_1 = \dot{w} + 2k_{11}w\dot{w} + k_{12}(\dot{w}\overline{w} + w\dot{\overline{w}}) + 2k_{22}\overline{w}\dot{\overline{w}} \tag{69a}$$

$$\begin{aligned}&= i\omega(w + k_{11}w^2 + k_{12}w\overline{w} + k_{22}\overline{w}^2) \\ &\quad + g_{111}w^2 + g_{112}w\overline{w} + g_{122}\overline{w}^2 + O(|w|^3).\end{aligned} \tag{69b}$$

Since all second order terms in the equation for $w$ will be eliminated, we can replace $\dot{w}$ and $\dot{\overline{w}}$ in (69a) by $i\omega w$ and $-i\omega w$, respectively. Comparing coefficients we obtain

$$k_{11} = g_{111}/(i\omega), \tag{70a}$$

$$k_{12} = -g_{112}/(i\omega), \tag{70b}$$

$$k_{22} = -g_{122}/(3i\omega). \tag{70c}$$

Due to the change of coordinates the coefficient of $w^2\overline{w}$ is modified by

$$\begin{aligned}\alpha_{3,\mathrm{nf}} &= 2g_{111}k_{12} + g_{112}(\overline{k}_{12} + k_{11}) + 2g_{122}\overline{k}_{22} \\ &= \frac{64i\omega^3}{3}\gamma_1\,(2\gamma_1 - 7\bar{\gamma}_1) \\ &= \frac{-64i\omega^5(4 - 5i\omega\varrho + 2\omega^2)}{3\varrho^2(2 - i\varrho\omega + \omega^2)^2(2 + i\varrho\omega + \omega^2)}.\end{aligned} \tag{71}$$

After eliminating all quadratic terms from the bifurcation equation, it is possible to remove all cubic terms, except $g_{1112}w^2\overline{w}$, by a further change of coordinates

$$w = z + k_{111}z^3 + k_{112}z^2\overline{z} + k_{122}z\overline{z}^2 + k_{222}\overline{z}^3.$$

Since this transformation doesn't alter the coefficient of $z^2\overline{z}$, we need not calculate the coefficients $k_{ijl}$.

The cubic term $\alpha_3$ in (60) is given by the sum of three terms

$$\alpha_3 = \alpha_{\mathrm{cm}}a_3^2 + \alpha_{\mathrm{nf}}a_3^2 + g_{1,21}, \tag{72}$$

where

$$g_{1,21} = -\frac{24i\omega^4}{\varrho\,(\omega\varrho + i(2 + \omega^2))} \tag{73}$$

is the (original) coefficient of $z_1^2\overline{z}_1$ in $g_1(z)$ in (62).

Since the periodic solution should exist for $c \le c_{\mathrm{crit}}$, we require

$$\mathrm{Re}(\alpha_3) < 0,$$

which holds at least for $a_3 = 0$, because

$$\mathrm{Re}(g_{1,21}) = -\frac{24\omega^4\,(2 + \omega^2)}{\varrho\,(\varrho^2\omega^2 + (2 + \omega^2)^2)} < 0.$$

This situation occurs, if the demand $d$ assumes the mean value of the optimal operating conditions.

The condition $\mathrm{Re}(\alpha_3) < 0$ is satisfied as long as

$$\begin{aligned}\mathrm{Re}(\alpha_{\mathrm{cm}} + \alpha_{\mathrm{nf}}) \\ &= \frac{16(\varrho^2-2)^3}{\varrho(3\varrho^2-8)} - \frac{16\varrho(\varrho^2-2)^3}{(8-3\varrho^2)^2} \\ &\quad + \frac{8(\varrho^2-2)^3}{\varrho(3\varrho^2-8)} + \frac{8(\varrho^2-2)^3(16-13\varrho^2+2\varrho^4)}{\varrho(3\varrho^2-8)^2(9-4\varrho^2)} \\ &\quad - \frac{64\omega^6\,(2+\omega^2)\,(4+(4+\varrho^2)\,\omega^2+\omega^4)}{\varrho|\varrho\omega+(2+\omega^2)|^6} \\ &\approx \frac{128}{9\varrho} + O(\varrho) \\ &< |\,\mathrm{Re}(g_{1,21})|/a_3^2.\end{aligned} \tag{74}$$

In the limit $\varrho \to 0$ this inequality reduces to

$$a_3^2 < \frac{27}{16}. \tag{75}$$

Now it follows from theorem II in [16], that the small periodic solutions exist for $\sigma_1 > 0$ (i.e. $c < c_{\mathrm{crit}}$), if the coefficient $\alpha_3 < 0$.

## Appendix C. Proof of Lemma 2

Following [1] we let $W_2(q,p)$ denote a quadratic Hamiltonian with unknown coefficients and consider the equation

$$\Delta\tilde{K}_2 = \Delta K_2 + \{W_2, K_2^0\}, \tag{76}$$

where $K_2^0$ denotes the unperturbed Hamiltonian, $\Delta K_2$ contains the perturbations and $\Delta\tilde{K}_2$ contains the simplified perturbations. We try to make $\Delta\tilde{K}_2$ as simple as possible by a proper choice of $W_2$.

While (76) is solved easily using complex coordinates, we want to give the calculations and results in real coordinates.

With respect to the monomial basis

$$(q_1^2, q_1q_2, q_2^2, q_1p_1, q_1p_2, q_2p_1, q_2p_2, p_1^2, p_1p_2, p_2^2)$$

for $W_2$, the map $W_2 \to \mathbf{ad}_{K_2^0} W_2 = \{K_2^0, W_2\}$ is represented by the matrix

$$\mathbf{A}_{K_2^0} = \begin{pmatrix} 0 & -\omega & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\,\omega & 0 & -2\,\omega & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega & -\omega & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & -\omega & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & -\omega & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \omega & \omega & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\,\omega & 0 & -2\,\omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \end{pmatrix} \tag{77}$$

with corank 2. It can be seen easily that the lines 5 and 6 are identical and lines 8 and 10 differ only by sign.

Since the matrix $\mathbf{A}_{K_2^0}$ in (77) is singular, it is impossible to remove all entries from $\Delta K_2$. For example, by chosing a proper coefficient for $p_1 p_2$ in $W_2$ we could eliminate either the $p_1^2$- or $p_2^2$-term in $\Delta K_2$, but not both of them, unless they sum up to 0. A convenient choice for a complementary subspace to the image of $\mathbf{A}_{K_2^0}$ is given by the kernel of the transposed matrix $\mathbf{A}_{K_2^0}^T$

$$\mathbf{ker}\mathbf{A}_{K_2^0}^T = \mathbf{span}(e_8 + e_{10}, e_5 - e_6),$$

corresponding to the functions

$$\begin{aligned} Y &= (p_1^2 + p_2^2)/2, \\ S &= q_1 p_2 - q_2 p_1. \end{aligned} \tag{78}$$

The Normal Form of the second order Hamiltonian close to the bifurcation point is given by

$$\tilde{K}_2 = X + \nu Y + (\omega + \alpha) S, \tag{79}$$

where in the general case $\nu$ and $\alpha$ are given by

$$\begin{aligned} \nu &= \left(\frac{\partial^2 \Delta K_2}{\partial p_1^2} + \frac{\partial^2 \Delta K_2}{\partial p_2^2}\right)/2, \\ \alpha &= \left(\frac{\partial^2 \Delta K_2}{\partial q_1 \partial p_2} - \frac{\partial^2 \Delta K_2}{\partial q_2 \partial p_1}\right)/2. \end{aligned} \tag{80}$$

In our example these quantities are given by

$$\begin{aligned} \nu &= -2(c - c_{\text{crit}}), \\ \alpha &= -3\sqrt{2}(c - c_{\text{crit}})/2. \end{aligned} \tag{81}$$

The linear, symplectic coordinate transformation $(q, p) \mapsto (\tilde{q}, \tilde{p})$ is obtained as the general solution of the Hamiltonian differential equation

$$\frac{dq}{d\epsilon} = \frac{\partial W_2}{\partial p}, \qquad \frac{dp}{d\epsilon} = -\frac{\partial W_2}{\partial q},$$
$$q(0) = \tilde{q}, \qquad p(0) = \tilde{p},$$

evaluated at $\epsilon = 1$.

## Appendix D. Normal Form transformation of the cubic and quartic terms

In this section we apply the Normal Form transformation, as it is outlined in [2] to the Hamiltonian (20). In the first step we show that it is possible to remove all cubic terms. Since the equation for the quartic terms is singular, some quartic terms cannot be removed. The Normal Form is then obtained by projecting the original quartic terms and the terms, which are introduced by the previous step, onto the complement of the image of the linear equation. Finally the terms introduced by the elimination of the cubic terms, which show up in the final Normal Form, are calculated explicitely.

### D.1. **Elimination of cubic terms.**

**Lemma 3.** *By a symplectic change of coordinates all cubic terms in the Hamilton function* (20) *can be eliminated.*

*Proof.* Following [2] we solve the normal form equation (27) with complex variables

$$\tilde{G}_3 = G_3 + \{G_2, W_3\}. \tag{82}$$

We write $W_3$ and $G_3$ in multi-index notation

$$W_3(z_1, \overline{z}_1, z_2, \overline{z}_2) = \sum_{|m|=3} w_m z^m := \sum_{|m|=3} w_m z_1^{m_1} \overline{z}_1^{m_2} z_2^{m_3} \overline{z}_2^{m_4}, \tag{83}$$

and $G_3(z_1, \overline{z}_1, z_2, \overline{z}_2) = \sum_m g_m z^m$, with $m = (m_1, m_2, m_3, m_4)$ and $|m| = \sum_{j=1}^4 m_j = 3$ and $m_j \geq 0$. Since the Poisson bracket acts linearly on each argument, we can calculate $\{G_2, W_3\}$ individually for each monomial in $W_3$:

$$\begin{aligned} \{G_2, z_1^{m_1} \overline{z}_1^{m_2} z_2^{m_3} \overline{z}_2^{m_4}\} &= -m_1\, z_1^{(m_1-1)} \overline{z}_1^{m_2} z_2^{m_3} \overline{z}_2^{m_4} i\omega z_1 \\ &\quad + m_2\, z_1^{m_1} \overline{z}_1^{(m_2-1)} z_2^{m_3} \overline{z}_2^{m_4} i\omega \overline{z}_1 \\ &\quad + m_3\, z_1^{m_1} \overline{z}_1^{m_2} z_2^{(m_3-1)} \overline{z}_2^{m_4} (i\omega z_2 + \overline{z}_1) \\ &\quad - m_4\, z_1^{m_1} \overline{z}_1^{m_2} z_2^{m_3} \overline{z}_2^{m_4-1} (i\omega \overline{z}_2 - z_1) \\ &= -i\omega(m_1 - m_2 - m_3 + m_4) z^m \\ &\quad + m_3 z^{(m_1, m_2+1, m_3-1, m_4)} \\ &\quad + m_4 z^{(m_1+1, m_2, m_3, m_4-1)}. \end{aligned} \tag{84}$$

So each monomial $z^m$ is mapped to a multiple $-(m_1 - m_2 - m_3 + m_4)\, i\omega$ of itself plus at most 2 further terms with the same value of $\zeta(m) = m_1 - m_2 - m_3 + m_4$. This observation suggests the following ordering of the monomials: The monomials $z^m$ are divided into sets

$$S_k := \{z^m : \zeta(m) = k\}$$

with the same value of $\zeta(m)$; within each set $S_k$ they are ordered by the weight $m_3 + m_4$. With this arrangement the linear map

$$\mathbf{ad}_{G_2} W_3 = \{G_2, W_3\}$$

becomes block-diagonal: The sets $S_k$ are invariant under the mapping $\mathbf{ad}_{G_2}$. The diagonal entries in the matrix representation are given by $-i\omega k$. Due to the ordering by the weight $m_3 + m_4$ the matrix blocks are upper-triangular.

$\zeta(m) = -3$: $S_{-3} = \{\bar{z}_1^3, \bar{z}_1^2 z_2, \bar{z}_1 z_2^2, z_2^3\}$

$$\mathbf{ad}_{G_2}\big|_{S_{-3}} = 3i\omega\mathbf{I}_4 + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\zeta(m) = -1$: $S_{-1} = \{z_1\bar{z}_1^2, z_1\bar{z}_1 z_2, \bar{z}_1^2\bar{z}_2, z_1 z_2^2, \bar{z}_1 z_2\bar{z}_2, z_2^2\bar{z}_2\}$,

$$\mathbf{ad}_{G_2}\big|_{S_{-1}} = i\omega\mathbf{I}_6 + \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\zeta(m) = 1$: $S_1 = \{z_1^2\bar{z}_1, z_1^2 z_2, z_1\bar{z}_1\bar{z}_2, z_1 z_2\bar{z}_2, \bar{z}_1\bar{z}_2^2, z_2\bar{z}_2^2\}$,

$$\mathbf{ad}_{G_2}\big|_{S_1} = -i\omega\mathbf{I}_6 + \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\zeta(m) = 3$: $S_3 = \{z_1^3, z_1^2\bar{z}_2, z_1\bar{z}_2^2, \bar{z}_2^3\}$

$$\mathbf{ad}_{G_2}\big|_{S_3} = -3i\omega\mathbf{I}_4 + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since all submatrices are regular, the equation

$$0 = G_3 + \{G_2, W_3\} \tag{85}$$

can be solved for all $G_3$. Therefore we can eliminate all cubic terms and have $\tilde{G}_3 = 0$. □

D.2. **Normal Form for quartic terms.** The leading nonlinear terms in the bifurcation equation are generically given by the fourth order terms in the Hamiltonian. The following lemma shows that these quartic terms can be transformed to a very simple form.

**Lemma 4.** *By a symplectic change of coordinates the quartic terms in the Hamiltonian* (20) *can be transformed to*

$$G_4 = \alpha_1 Y^2 + \alpha_2 Y S + \alpha_3 S^2.$$

*Proof.* Since we are only interested in the terms in the Normal Form, we need not calculate all coefficients in $W_4$ and it is sufficient to know a proper complementary subspace to the image $\mathbf{im}(\mathbf{ad}_{G_2})$ in the space of quartic polynomials. Since (84) holds at all orders, we concentrate on the subspace with $\zeta(m) = 0$, which is spanned by the monomials

$$S_0 = \{z_1^2\overline{z}_1^2, z_1^2\overline{z}_1 z_2, z_1\overline{z}_1^2\overline{z}_2, z_1^2 z_2^2, z_1\overline{z}_1 z_2\overline{z}_2, \overline{z}_1^2\overline{z}_2^2, z_1 z_2^2\overline{z}_2, \overline{z}_1 z_2\overline{z}_2^2, z_2^2\overline{z}_2^2\}.$$

The matrix $\mathbf{ad}_{G_2}|_{S_0}$ becomes

$$\mathbf{A}_0 = \mathbf{ad}_{G_2}\big|_{S_0} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{86}$$

A complementary subspace to $\mathbf{im}(\mathbf{A}_0)$ is spanned by

$$v_1 = (0,0,0,0,0,0,0,0,1),$$

$$v_2 = (0,0,0,0,0,0,-1,1,0),$$

$$v_3 = (0,0,0,1,-2,1,0,0,0),$$

corresponding to the polynomials

$$P_1 = z_2^2\overline{z}_2^2 = |z_2|^4 = Y^2,$$

$$P_2 = z_2\overline{z}_2(\overline{z}_1\overline{z}_2 - z_1z_2) = -2i|z_2|^2 \operatorname{Im} z_1z_2 = iY S,$$

$$P_3 = z_1^2z_2^2 - 2z_1\overline{z}_1z_2\overline{z}_2 + \overline{z}_1^2\overline{z}_2^2 = -4(\operatorname{Im} z_1z_2)^2 = -S^2.$$

All remaining monomials can be eliminated by a proper choice of the coefficients in $W_4$. □

Following ([2]) we perform a further rescaling with a small parameter $\varepsilon$

(87a) $$z_1 \to \varepsilon^2 z_1, \qquad z_2 \to \varepsilon z_2,$$

(87b) $$\nu \to \varepsilon^2 \nu, \qquad \alpha \to \varepsilon^2 \alpha,$$

(87c) $$H \to \varepsilon^{-3} H.$$

and obtain the following

**Corollary 1.** *By a symplectic change of coordinates the fourth order Hamiltonian can be transformed to*

(88) $$G = \omega S + \varepsilon(X + \nu Y + \alpha_1 Y^2) + O(\varepsilon^2).$$

*Proof.* Combining Lemma 2, Lemma 3 and Lemma 4 yields the Hamiltonian

$$G = (\omega + \alpha)S + X + \nu Y + \alpha_1 Y^2 + \alpha_2 Y S + \alpha_3 S^2.$$

The scaling (87) gives

$$X \to \varepsilon^4 X, \qquad S \to \varepsilon^3 S, \qquad Y \to \varepsilon^2 Y.$$

Collecting equal powers of $\varepsilon$ we obtain (88). □

*Remark* 1. Actually the rescaling (87) is only "symplectic with multiplier $\varepsilon^3$ ([2]). The division of $G$ by this multiplier preserves the Hamiltonian structure.

The term $\varepsilon\alpha_1 Y$ is the only nonlinear term in $G$.

**Lemma 5.** *The coefficient $\alpha_1$ is given by*

(89) $$\alpha_1 = G_{4,3344} + \alpha_{1,nf} = -\frac{3}{2} + \frac{16}{9}a_3^2.$$

*Proof.* Similarly to the coefficient $\alpha_3$ in the ordinary Hopf bifurcation $\alpha_1$ consists of 2 parts:

(90) $$G_{4,3344} = -3/2$$

is the coefficient of $|z_2|^4$ in $G_4 = -v^4/4$ with

(91) $$v = q_1/2 + \sqrt{2}p_2 = \frac{z_1 + \overline{z}_1}{2\sqrt{2}} + i(z_2 - \overline{z}_2)$$

and

(92) $$\alpha_{1,\mathrm{nf}} = 16a_3^2/9$$

is the coefficient of $|z_2|^4$ in the correction term $\{G_3, W_3\}$ in (28). The calculation of $\alpha_{1,\mathrm{nf}}$ is given in the next subsection. □

D.3. **Calculation of the intermediate terms in the Normal Form reduction.** In order to calculate the correction term $\alpha_{1,\mathrm{nf}}$ in (92) we have to solve the equation (27) – where $K_2$ is replaced by $G_2$ – for $W_3$ and insert the solution into the Poisson bracket $\{G_3, W_3\}$ to find the coefficient of $|z_2|^4$.

The cubic nonlinearities are contributed by the term $a_3v^3/3$, which has to be expressed in the coordinates $(z_1, \overline{z}_1, z_2, \overline{z}_2)$: With (91) we obtain the multi-index representation

$$v^3 = \sum_m g_m z^m$$

sorted by the weight $\zeta \cdot m$:

$$
\begin{aligned}
S_{-3}: \quad & g_{0300} = \sqrt{2}/32, && g_{0210} = 3i/8, && g_{0120} = -3\sqrt{2}/4, \\
& g_{0030} = -i \\
S_{-1}: \quad & g_{1200} = 3\sqrt{2}/32, && g_{1110} = 3i/4, && g_{0201} = -3i/8, \\
& g_{1020} = -3\sqrt{2}/4, && g_{0111} = 3\sqrt{2}/2, && g_{0021} = 3i, \\
S_{1}: \quad & g_{2100} = 3\sqrt{2}/32, && g_{2010} = 3i/8, && g_{1101} = -3i/4, \\
& g_{1011} = 3\sqrt{2}/2, && g_{0102} = -3\sqrt{2}/4, && g_{0012} = -3i, \\
S_{3}: \quad & g_{3000} = \sqrt{2}/32, && g_{2001} = -3i/8, && g_{1002} = -3\sqrt{2}/4, \\
& g_{0003} = i.
\end{aligned} \tag{93}
$$

Solving the equations $\mathbf{ad}_{G_2} w + g = 0$ in the sets $S_{-3} \ldots S_3$ gives

$$
\begin{aligned}
S_{-3}: \quad & w_{0300} = -i/864, && w_{0210} = -5\sqrt{2}/144, \\
& w_{0120} = -i/12, && w_{0030} = \sqrt{2}/6 \\
S_{-1}: \quad & w_{1200} = 117i/32, && w_{1110} = 21\sqrt{2}/8, \\
& w_{0201} = 15\sqrt{2}/16, && w_{1020} = -9i/4, \\
& w_{0111} = -3i/2, && w_{0021} = -3\sqrt{2}/2, \\
S_{1}: \quad & w_{2100} = -117i/32, && w_{2010} = 15\sqrt{2}/16, \\
& w_{1101} = 21\sqrt{2}/8, && w_{1011} = 3i/2, \\
& w_{0102} = 9i/4, && w_{0012} = -3\sqrt{2}/2, \\
S_{3}: \quad & w_{3000} = i/864, && w_{2001} = -5\sqrt{2}/144, \\
& w_{1002} = i/12, && w_{0003} = \sqrt{2}/6.
\end{aligned} \tag{94}
$$

The coefficient of $|z_2|^4$ in $\{G_3, W_3\}$ is

$$
\begin{aligned}
c_{0022} =& 3(g_{0120}w_{0003} - g_{0003}w_{0120}) - 3(g_{0030}w_{1002} - g_{1002}w_{0030}) \\
& + 2(g_{0111}w_{0012} - g_{0012}w_{0111}) - 2(g_{0021}w_{1011} - g_{1011}w_{0021}) \\
& + g_{1020}w_{0012} - g_{0012}w_{1020} + g_{0102}w_{0021} - g_{0021}w_{0102} \\
=& 16.
\end{aligned} \tag{95}
$$

Considering the coefficient $a_3/3$ of $v^3$ yields (92).

## Appendix E. Proof of Theorem 2

*Proof.* The perturbations proportional to $\nu$ and $\alpha$ have already been treated in section 4.1. Introducing $\varrho$ at the Hamiltonian Hopf bifurcation point the Jacobian of the linearized system (22) is perturbed by the matrix

$$\mathbf{A}_\varrho = \mathbf{E}_c^{-1}\mathbf{diag}(0,0,1,1)\mathbf{E}_c$$

(96)
$$= \begin{pmatrix} 3/4 & 0 & -\sqrt{2}i/2 & 0 \\ 0 & 3/4 & 0 & \sqrt{2}i/2 \\ 3\sqrt{2}i/16 & 0 & 1/4 & 0 \\ 0 & -3\sqrt{2}i/16 & 0 & 1/4 \end{pmatrix},$$

where $\mathbf{E}_c$ is the symplectic transformation matrix to our complex coordinates

$$\mathbf{E}_c = \mathbf{E}\cdot\frac{\sqrt{2}}{2}\cdot\begin{pmatrix} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{pmatrix}.$$

Introducing the near-identity change of coordinates

(97)
$$z = w + \varrho\mathbf{H}w,$$

with an unknown matrix $\mathbf{H}$, the linear equations are given by

(98)
$$(1+\varrho\mathbf{H})w' = \mathbf{A}_c(w+\varrho\mathbf{H}w) + \varrho\mathbf{A}_\varrho w + O(\varrho^2).$$

Neglecting terms of order $O(\varrho^2)$ and writing the equation for $w$ in the symbolic form

$$w' = (\mathbf{A}_c + \varrho\tilde{\mathbf{A}}_\varrho)w,$$

we obtain the Normal Form equation for $\tilde{\mathbf{A}}_\varrho$ and $\mathbf{H}$

(99)
$$\tilde{\mathbf{A}}_\varrho = \mathbf{A}_\varrho + \mathbf{A}_c\mathbf{H} - \mathbf{H}\mathbf{A}_c = \mathbf{A}_\varrho + [\mathbf{A}_c, \mathbf{H}],$$

by comparing the coefficients of $\varrho$. Equation (99) is a singular, linear equation for the transformation matrix $\mathbf{H}$. The Normal Form is given by a properly chosen complement of the image of the map

$$\mathbf{ad}_{\mathbf{A}_c}(H) = [\mathbf{A}_c, \mathbf{H}].$$

Rearranging the variables $z$ in the order

$$P(z_1, \overline{z}_1, z_2, \overline{z}_2) = z_1, \overline{z}_2, \overline{z}_1, z_2 \tag{100}$$

and ordering $\mathbf{H}$ by columns, we obtain the following block diagonal matrix representation for $\mathbf{ad}_{\mathbf{A}_c}$:

$$\mathbf{ad}_{\mathbf{A}_c} = \begin{pmatrix} \mathbf{A}_c - i\omega\mathbf{I}_4 & \mathbf{I}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_c - i\omega\mathbf{I}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_c + i\omega\mathbf{I}_4 & \mathbf{I}_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_c + i\omega\mathbf{I}_4 \end{pmatrix} \tag{101}$$

As the second block in (101) takes care of the complex conjugate values, it is sufficient to investigate the first block and to append the conjugates afterwards. The first block in (101) reads

$$\mathbf{ad}^1_{\mathbf{A}_c} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2i\omega & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2i\omega & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2i\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2i\omega \end{pmatrix}. \tag{102}$$

Since the fifth row is zero and the sixth row is just the negative of the first row, a complement to $\mathbf{im}(\mathbf{ad}^1_{\mathbf{A}_c})$ is spanned by the kernel of its transpose:

$$\mathbf{ker}(\mathbf{ad}^1_{\mathbf{A}_c})^T = \mathbf{span}(e_5, e_1 + e_6).$$

Now we project the perturbation matrix $\delta_\varrho\mathbf{A}$ onto $\mathbf{ker}(\mathbf{ad}^1_{\mathbf{A}_c})^T$. Since by the reordering (100) the fifth line in (102) corresponds to the entry $(\mathbf{A}_\varrho)_{14} = 0$ and the first and sixth line to $(\mathbf{A}_\varrho)_{11} = 3/4$ and $(\mathbf{A}_\varrho)_{44} = 1/4$, respectively, we obtain the Normal Form entries for the perturbation by the discount rate $\varrho$

$$\tilde{\mathbf{A}}_\varrho = \mathbf{I}_4/2, \tag{103}$$

restating our observation (13) that the spectrum moves by $\varrho/2$ to the right. □

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