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The Control of Age-Structured Populations: A Dynamic Cost-Benefit Approach

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Preliminary Draft

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Abstract. The purpose of this paper is to link two distinct fields, i.e. population dynamics and intertemporal optimization. Our approach imposes a normative structure to population dynamics, and an age structure to dynamic optimization. Recent progress in optimal control theory enables us to carry out intertemporal cost-benefit analysis in an age-specific context. It turns out that age-specific density-dependence, i.e. dependence of transition rates from the age composition, generates non-linearities that may lead to epidemic patterns implying interesting behaviour of optimal policies. A simple model for initiation of illicit drug consumption and violence is presented. It is shown how the optimal mix of instruments changes with time and age. Efficient policies are characterised by an intertemporal trade-off between the social welfare or damage and the costs for the instruments. Furthermore, we sketch several other age-structured compartment models for which cost-benefit analyses might yield valuable insights into intertemporal policy analysis. Among them are models for employment/unemployment, family planning as a well as medium sized HIV incidence/prevalence model.

1. Introduction

Being the most important variable, *age* plays a key role in population dynamics. Clearly, the dimension of *age*, or, more general, of *duration* (i.e. the time since a particular event), is of importance also in demography's neighbourhood disciplines, like sociology, biology, and economics. Examples can be found, among others, in manpower planning, epidemiology, vintage capital theory, population economics, inventory control of perishable items, marketing, and the management of renewable resources.

Thus, virtually all demographic processes and many processes in economics are age-dependent. There are many socio-economic problems where age or duration can be seen as a proxy for other important variables, like income, quality or weight. Hence, age-structured modelling has a broad field of potential applications. The fact that some processes evolve along the dimension of age as well as time makes analysis difficult. Partial differential equations and integral equations provide the adequate set of instruments, which is, however, not contained in the toolkit of most demographers and economists. This might help to explain the fact that in socio-economics age/duration-dependence is a largely neglected field (which is, for instance, not the case in biology). This is in sharp contrast to the number of interesting problems in economics, which rely on age or duration dependence.

The present paper tries to link two distinct traditions, namely population dynamics and intertemporal policy analysis. The following approach imposes a normative structure to formal demography, while an age-structure is brought to dynamic optimization. Early preliminary work in that direction was done by Arthur and McNicoll (1977). Applying integral-equation control techniques these authors tried to create an age-dependent theory of population policy. They combined an aged-structured population model with a vintage capital model. By assuming that both fertility and savings can be controlled, they analysed intertemporal welfare trade-offs and social discounting.

The crucial feature of the Arthur and McNicoll model is that they enlarge the descriptive model of population dynamics by introducing decision variables and a performance function (i.e. fertility, savings and welfare). Generally, we assume that there is an authority (a planner, a decision-maker), who controls an array of interventions. For a given time path of those interventions, and starting from given initial condition, the system will take a certain time course. Some of these time developments are more desirable for the authority, some are less advantageous. Now the basic question that concerns us in the present paper can be stated as follows: how should the decision-maker influence the development to minimise the social loss or to maximize the welfare over time? The mathematical framework of such intertemporal decision-processes is dynamic optimization or, more specifically, optimal control theory.

Recent progress in that field enables us to carry out intertemporal cost-benefit analyses in an age-specific context. Optimal control models of *distributed parameter systems* deal will the optimization of a performance criterion subject to partial differential equations (PDEs) as a dynamics constraint. Taking age as the 'distributed parameter', the state variable depends both on time and on age. The dynamics governed by first-order quasilinear partial differential equation of the McKendrick-von Foerster type[i]. Essential are certain density-dependent effects, modeling the dependence of incidence rates on prevalence numbers in various age groups. The nonlinear interaction of those age-dependent feedbacks with the control variables may generate interesting and complex behaviour.

The goal of a policy is to induce agents to change their behaviour in a desired sense, i.e. to reach a certain target. In many cases those interventions are targeted to individuals who are homogenous with respect to certain characteristics. In the present proposal we focus on age-structured populations of agents. Age-dependent demand for a fashionable good, say, leads to a market segmentation, which is adequately dealt with by an age-specific mix of marketing instruments. As was already mentioned above, the applicability of age-structured models is much broader, since age can be seen as proxy of various economic characteristics like household and family status, income etc.

Sometimes the state variables of dynamic economic models are not only functions of time but also are functions of other dimensions such as duration or spatial coordinates. Such systems are characterized by *partial* differential equations. In natural sciences, distributed parameter systems have been extensively used, e.g., in heat transfer problems. In economics and management science almost all applications of optimal control theory are described by *ordinary* differential equations. There are, however, many socio-economic problems where, in addition to time, other coordinates, mostly age or other characteristics of duration, play a crucial role. Thus, distributed parameter systems have a broad field of potential applications.

The question arises why there is a remarkable gap between several interesting age-structured economic problems and the rather short list of distributed parameter control applications, which exists up to now. First, probably because most economists are not skilled in dealing with *partial* differential equations. Second, and more important, the mathematics behind distributed parameter control is rather sophisticated and *analytical* solutions only exist in exceptionally simple situations. Thus, age-structured optimal control promises a challenging field in *computational dynamic economics*.

One of the intentions of the present paper is to fill the aforementioned gap. In what follows, we will briefly sketch the state of the art. No attempt is made to be exhaustive, but due to the relatively slow progress in the field we hope to include some of the important papers.

There are several books on distributed parameter control (see, e.g., Butkovsky), but most of them are not easy to read and deal with no economic applications. Derzko et al. (1984) provides an introduction to the field; see also Feichtinger and Hartl (1986, Appendix A.5) for a steep introduction.

A primary field of application circulates around *population dynamics*. There a several applications of distributed parameter control in pure demography. Assume that age-specific fertility rates can be controlled[ii] and control is costly. Identify an age-structured as 'desirable' and penalize deviations from that, e.g. by a quadratic function. Starting from a given initial condition, the problem is to find an optimal intertemporal-trade-off between the discounted loss from deviating from the target structure and the costs for the policy instruments. The famous 'momentum of population growth' problem admits a very nice formulation in terms of such a dynamic cost-benefit analysis: find an efficient trade-off between the Scylla of population growth (or decline) and the Charibdis between the 'accordion-like' effect of changing age structures. As far as we know this problem of population dynamics has never been approached within a cost-benefit analysis.

Moreover, demography, population economics, bioeconomics (i.e. the management of renewable resources) contain many problems in distributed parameter control. Gopalsamy (1976) studies age-structured populations with the birth trajectory as boundary control. In an important contribution Brokate (1985) analyzes a similar model for birth control and age-specific harvesting, see also Gurtin and Murphy (1981), Murphy and Smith (1990). An excellent overview on bioeconomic applications up to the late seventies is given in the Ph.D. thesis of Muzicant (1980); see also Derzko et al. (1980, 1984) dealing with the optimal management of a cattle ranching problem. A more recent paper discussing an appropriate maximum principle is Veliov et al. (2000).

Arthur and McNicoll (1977) include the age structure of a population in planning the fertility of a population. By assuming that the fertility level and the savings rate can be influenced, and taking into consideration welfare trade-offs and social discounting, they try to develop a theory of *demo-economic policy*. The optimality conditions imply, among others, that policy should balance the life time value of births and capital against social costs of creating them. It should be noted that this approach up to now delivers only preliminary results. Although it is a rather difficult field, it is important enough to be pursued further.

Other promising applications of the age-dependent processes occur in *manpower modeling*, where the age variable may be interpreted as seniority (or duration in a certain state); see Gaimon and Thompson (1981). Haurie et al. (1984) present a rather general multi-class age-structured framework with interesting applications to social services planning.

Bensoussan et al. (1975) study an *inventory model* where the parameter may be interpreted as quality of a good which decreases with age (in the case of a perishable good like a blood conserve) or increases with age (with quality wine as example); see also Tzafestas (1982) for further applications of distributed optimal control to production/inventory problems.

Robson (1985) analyzes a model for the management of a *housing stock*. A salient feature of housing is its durability. Although the quality of houses declines over time, this decline is slow. Housing stocks, automobiles or other consumer durables often show a 'vintage' structure. We assume that quality and age of a house are inversely related and that quality has a positive impact on value. The maximum principle for partial differential equations provides a powerful tool to get insight into the structure of optimal solutions of vintage models of such kind. Furthermore, capital vintage models might be formulated in a related framework.

The paper is organized as follows. In section 2 we discuss age-specific interventions for density-dependent initiation. In Section 3 an age-structured two-state compartment model for illicit drug dynamics or violence control is presented. Section 4 deals with the solution of the prototype developed in section 3. A suitable extension of Pontryagin's maximum principle will be used to derive insights into the structure of optimal policies. Section 5 sketches a two-class model to control unemployment. In section 6 we list a whole bunch of problems whose analysis can be done in a related framework. In section 7 we draw our conclusions; moreover, some possible extensions are discussed.

2. Age-specific and density-dependent initiation

The object of our study are *multi-class age-structured population models* or, to put it in other words, *age-dependent compartment models*. In some sense they can be seen as deterministic counterparts of semi-Markov population processes in the style of Haurie et al. (1984). Each member of a population may be described by a pair (i,a), where the class i denotes, e.g., parity, socio-economic status, etc., and a gives the age of this individual. Alternatively, we may divide a population according to (i,d), where i is again denotes a compartment, while the duration d measures the elapsed time since an individual entered her current state i. It is a well-known fact that in demography as well as in most of its neighbour disciplines the transition rates q at time t from state i to state j quite often depend on age a/duration d: $q = q_{ij}(t,a)$ or $q_{ij}(t,d)$.

Up to now we have discussed a (purely) descriptive age/duration-specific dynamical system. Let us now assume that there is a single decision-maker who can chose control instruments. In public policy making this might be a central planner, while in manpower planning (see also below) a chief manager will play this role.

To complete our age-structured policy paradigm a performance criterion must be introduced permitting the evaluation of various policies over a given time horizon. In a dynamic cost-benefit analysis the net benefits from the control of transition rates must be compared with the costs of the controls. Pontryagin's maximum principle and various extensions of these necessary optimality conditions allow to determine the optimal intertemporal trade-offs between utilities and costs resulting from control and state paths.

Many interventions in public policy can be characterized as seeking to induce or prevent some discrete change in behaviour, hereafter called an *'initiation'*. One typical example in demography is family planning that seeks to prevent pregnancies. Another are public health interventions that try to prevent initiation into some dangerous behaviour (smoking, drug use, etc.) or to promote a healthful behaviour, e.g. getting vaccinated. A third example is a marketing intervention that tries to convince people to initiate use of some product.

A common debate concerns the relative merits of 'broadcast' vs. age-specific interventions. Broadcast approaches are often less costly per person reached, but when initiation has an age-specific component many of those reached may not be affected by the message.

Suppose one has the capacity to conduct age-specific interventions, how should those interventions be targeted? That is, imagine for the moment that one could focus the intervention on particular ages with an arbitrary degree of precision at no incremental cost, what should one do? On the one hand, it makes sense to concentrate the intervention on people who are at an age at which initiation is common. There is little sense directing pregnancy prevention programs at six-year olds or sixty-year olds. On the other hand, there are generally diminishing returns to increasing intensity of exposure, which would suggest spreading a limited intervention budget over a range of ages. An added complication is interaction between efforts directed at people of different ages at different times. If last year's marketing to 14-year olds convinced many to try a product, there will be fewer uninitiated 15-year olds this year. Such interactions might encourage skewing interventions toward younger ages.

The best balance among these competing factors clearly depends on the details of the objective function, intervention, and response function. Here we focus on the following situation:

1) The (undiscounted) value of generating or preventing an intervention depends on the age of the initiate.
2) There are diminishing returns to increasing the intensity of the intervention for a given age.
3) The intervention either affects behaviour immediately or it is forgotten entirely; there is no delayed or accumulated effect of the intervention over time.
4) The epidemic character of initiation arises by a *feedback-effect* of 'users' to 'non-users'.

Each of these assumptions merits discussion. All could be relaxed, but some set of assumptions is required to get started, and in our opinion these assumptions characterize an important and interesting case.
The first assumption make sense if lifetime expected fertility is dependent of the age of initiation of family planning measures. For example, the expected future utility

generated by getting someone to contracept may depend in any meaningful way on whether the women is 15, 25, or 30.

When trying to persuade someone to do something, or to refrain from doing it, there can be increasing or decreasing returns to increased intensity of exposure. A single message may be simply lost in the noise and have no effect, so sometimes repetition is essential, but not always. Increasing the number of times someone hears that red meat is high in cholesterol from one to five times may induce some change in behaviour. But increasing the number of exposures from five hundred and one to five hundred and five may have no effect. At some point the target individuals have heard the message, and they have either changed their behaviour or they have not. Further repetition is likely to be ignored or, worse, to undermine the credibility of the messenger by becoming annoying or sounding simplistic.

Here we focus on the diminishing returns case because it seems to be the more natural situation, because it is mathematically convenient, and because even when there are initially increasing returns, there is often a point at which decreasing returns take over. In such cases, it is not uncommon for the optimal solution to lie within the region with decreasing returns anyhow.

The third assumption is perhaps most restrictive in the public health context. Some interventions are designed explicitly to build on information given at an earlier age. E.g., 'booster sessions' that reinforce prevention messages delivered a year or two earlier in a more concentrated form. In contrast, whether a restaurant's advertising induces one to visit may depend very little on how many times one saw their advertising three years previously. Again, however, we stress that these four assumptions do not represent the only defensible or interesting case. Rather, we simply need to start with some particular model, and relaxing various subsets of these assumptions would make interesting further work.

3. A two-state prototype model

In this section we will illustrate our ideas by a rather simple age-structured two-state compartment model. It is well suited to deal with several interesting cost-benefit analyses. On the one hand, it is simple enough to allow the derivation of interesting results in an analytic way. On the other hand, it is rich enough to describe the initiation/prevention mechanism or analogous age and density-dependent transition mechanisms. In this sense, the model we are going to formulate may be seen as prototype for a new generation of related models. Or, to put it in other words, it can be interpreted as basic component of many other models describing various intertemporal trade-off problems. The intertemporal policy analysis of most of those models can be done numerically only. In that sense, such dynamic cost-benefit analyses represent a challenging field in computational economics.

A simple age-structured two-state compartment model

To illustrate our model we consider the *initiation for illicit drug consumption*, e.g. marijuana. For simplicity, we distinguish only two different types of individuals, namely non-users and users of the illicit drug.

Figure 1 about here

Denote with n(t,a) the number of non-users in age a at time t. Both variables t and a are continuous, $a \in [o,\omega]$.

The number of users is denoted as x(t,a).

Before we describe the system dynamics we define the control instruments:

w(t,a) is the prevention rate, here identified with the expenditures for prevention; compare, e.g., Behrens et al., 2000, or Almeder et al., 2000, for a discussion of the efficiency and the age-dependence of this invention.

The second decision variable, u(t,a), is the treatment rate in age a at time t.

The objective of the planner is to minimize the discounted sum of the social costs consisting of the social damage due to drug consumption and the costs for steering the system:

$$-J = \min_{u,w} \int_o^\infty e^{-rt} \int_o^\omega \left[\kappa d(a)x(t,a) + c(u(t,a)) + w(t,a)\right] da\, dt. \qquad (1)$$

Here r denotes the time preference rate of the decision-makes. The parameter κ measures the damage in monetary units per consumed gram and per time unit[iii].

The damage function d(a) measures the age-dependent component of the per capita damage. Since consumption is started with light (temporary) use and may be followed by heavy (permanent) use, d(a) is assumed to increase with age. The treatment costs are assumed to be convex. Note that the age interval could be restricted to $[\omega_1, \omega_2]$ with $0 < \omega_1 < \omega_2 < \omega$, where ω_1 denotes the minimal age for initiation (say 10 years), and ω_2 is an age above which the problem of initiation disappears (say 40 years).

The system dynamics is the quasi-linear partial differential equation used by McKendrick (1927) in his epidemiological model; compare Dietz (1997), Keyfitz and Keyfitz (1997) for a nice historic introduction to this neglected population dynamics[iv].

Denoting the partial derivatives by subscripts the governing PDE is given as

$$\frac{\partial x(t,a)}{\partial t}+\frac{\partial x(t,a)}{\partial a}=n(a,t)\bar{\mu}(a)\Phi\left(\int_o^\omega m(a,a')c(t,a')da'\right)\Psi(w(t,a)-$$

$$-\theta\left(\frac{u(t,a)}{x(t,a)}\right)x(t,a)-\delta(a)x(t,a). \qquad (2)$$

Before we simplify the system dynamic (2) we explain the occurring functions:

$\bar{\mu}(a)$ ... autonomous age-specific initiation rate.

Figure 2 about here

Fig. 2 shows the remarkable sharp 'peaking' of the uncontroled initiation rate for marijuana (US date for 1992).

The age-specific density-dependent feedback effect is modeled by the function Φ(.) in (2). Since this mechanisms is crucial for the prevalence-dependence of incidence rates we give a careful description:

(i) The age-specific influence function m(a,a') measures the impact of one individual in age group a' on the initiation rate of a non-users in age group a. This influence depends primarily on the age difference. In particular, for young non-users it is assumed that individuals being at the same age or a little bit older set examples; therefore this influence is very high. On the other hand, persons more than, say twenty years older (e.g. parents) may evoke just an opposite impact (see also Almeder et al., 2000).

Figure 3 about here

(ii) Since the influence rates m(a,a') indicate the per capita influence of one user to another, we have to weigh those rates by the age-structure of users to get an expression for the reputation of the drug. These weights are the relative age-composition of the users, i.e.

$$c(t,a)=x(t,a)\left[\int_o^\omega x(t,a)da\right]^{-1}.$$

(iii) Thus, the reputation of the drug for an a years old individual at time t summarises the global influence as follows:

$$R(t,a)=\int_o^\omega m(a,a')c(t,a')da'. \qquad (3)$$

(iv) Finally, the function Φ(.) maps R(t,a) into an interval [ρ,σ] such that

$$\Phi\begin{pmatrix}-\infty\\0\\\infty\end{pmatrix}=\begin{pmatrix}\rho\\1\\\sigma\end{pmatrix}\quad\text{with}\quad 0<\rho<1<\sigma. \tag{4}$$

The function Φ(.) is a non-negative monotonically increasing function. (4) says that zero reputation has no impact on the initiation, a 'large' negative reputation reduces the initiation by (1-ρ).100%, while a very positive reputation increase the rate by (σ-1).100%. Note that Φ is an S-shaped function with a turning point in (or near) R = 0.

The above consideration enables us to complete the explanation of the right-hand-side of the PDE (2).

The function Ψ(.) describes the impact of the prevention on the initiation:

$$\Psi(w(t,a)) = (1-c)\exp(-\varepsilon w(t,a)) + c, \tag{4}$$

where $(1-c)\in(0,1)$ measures the maximal proportion of reduction of the initiation rate, and ε indicates the efficiency rate of the prevention. Note that here we assume that both parameters, c and ε, do depend neither on time nor on age. Moreover, our ansatz assumes that prevention shows no 'swamping effect', i.e. prevention apply to total population, i.e. both on non-users and users.

On the other hand, treatment exhibits a per capita effect, i.e only offenders are treated. Thus, Θ(u(t,a)/x(t,a)) denotes the impact of the treatment rate to x(t,a) 'applicants'. By assuming Θ as linear, the second term on the r.h.s. of (2) get simply -u(t,a).

Finally, δ(a) measures the mortality rate of users.

A substantial simplification of (2) can be reached by restricting ourselves to a stationary total population {p(a)}[v]. In that case we can use

$$n(t,a) = p(a) - x(t,a),$$

where the stationary population p(a) is given. Then the system dynamics contains only one unknown age-structured variable, x(t,a), instead of two.

4. How to prevent and/or to treat optimally?

In this section we present some insights into the structure of the optimal mix of control instruments. To get them we use an extension of the maximum principle for age-structured populations including certain forms of density-dependence; see Brokate (1985) and Veliov et al. (2000).

The maximum principle is a set of necessary optimality conditions. Generally, the conditions (and their proof) are rather sophisticated due to age/density-dependence.

PDEs and integral equations are by no means simple tools. Although there is no chance to discuss them here, we will sketch below some results obtained by this procedure.

It is on efficient and wise general approach in applied mathematics, to resort to numerical solutions only if analytical methods may not applied. However, in the field of PDEs the power of analytical methods comes rather soon to an end.

However, a nice property for population processes is the (quasi-) linearity of the PDEs. Equations of the McKendrick-type contain only first order derivatives in linear form. Due to this fact, the related stationary problem can be sometimes dealt with (and in special case even solved) by analytic means. The related stationary problem arises in a natural way if we consider the time-age problem for 'very large' values of time, i.e. asymptotically for $t \to \infty$[vi].

All other cases, in particular the transitory behaviour of the optimal prevention and treatment paths must be calculated *numerically*. There exist several packages to solve such problems.

Let us now discuss some of the results we obtained for the drug initiation model described in section 3 (see also Almeder et al., 2000).

Let us briefly mention some preliminary observations we derived for a distributed parameter model to control drug initiation in the US (see Almeder et al., 2000). The optimality conditions may yield *qualitative* insight into the *structure* of the *optimal mix of control instruments* depending on age and time. Preliminary calculations show, e.g., the following results: For the Marijuana consumption in the US the uncontrolled age-specific initiation rate $\overline{\mu}(a)$ shows a sharp rise between 11 and 16 years and a steep decline afterwards. It turns out that the optimal prevention path $w^*(t,a)$ follows the age-specific pattern for any period t. The recommendation for drug policy is that (primary) prevention which controls initiation should be high in age groups with high risk and low elsewhere. Note, however, that such optimal prevention policies may be hardly implemented due to institutional restrictions (preventive education must mostly take place before the age of 14 years). A more adequate analysis should include delayed systems.

The result that the optimal prevention rate should be high in those age-groups where the problem is severe, i.e. where the initiation is high, may not surprise. The next one, however, is less trivial. Our calculations show that the effectiveness of prevention and treatment depend critically on the stage in the epidemic in which they are employed (cp. Behrens et al. 2000). Preliminary considerations suggest that prevention is more appropriate in young years, while treatment is more effective later. Thus, it turns out that the optimal mix of interventions varies not only over time but also with age.

We already mentioned the fact that the optimality conditions of the general age-structured density-dependent model are rather sophisticated. Nevertheless all these conditions admit nice, but generally intricated economic interpretations.

In special cases, however, the 'spirit' of this powerful tool in economic analysis may be revealed more easily. For an illustration we omit density-dependence (which overburdens the formulas), i.e. we set $\Phi(.) = 1$, and assume $\theta(.)$ as linear.

The key for all maximum principles is the adjoint variable $\lambda(t,a)$ measuring the shadow price of one additional user[vii]. Since drug-use (and violence) creates damage, $\lambda(t,a)$ is clearly negative in our case. Being an easy special case of Brokate (1985) or Veliov et al. (2000) the maximum principle delivers the following adjoint equation:

$$\frac{\partial\lambda(t,a)}{\partial t}+\frac{\partial\lambda(t,a)}{\partial a}=\kappa d(a)+\lambda(t,a)\left[r+\delta+\overline{\mu}(a)\Psi(w(t,a))\right]. \tag{5}$$

This relation has a striking economic interpretation. To simplify it, assume that (5) is in equilibrium, i.e. its left-hand-side vanishes. Then, (5) can be written as

$$\kappa d(a)+\lambda(t,a)\overline{\mu}(a)\Psi(w(t,a))=-\lambda(t,a)(r+\delta). \tag{6}$$

(6) says the following: along an optimal path the planner is indifferent between the following two options: to accept one additional user in the system or to remove her. In the first case (l.h.s. of (6)) an instantaneous marginal loss of $\kappa d(a)$ monetary units is charged, but there is one non-user less reducing the risk to become a later user. On the other hand, if a user is removed from the system, the present value of the process increases by $-\lambda(t,a)$, and the r.h.s. of (6) denotes the interest rate of this capital. Note that in addition to r the depreciation rate δ (due to mortality) is taken into account. The non-stationary relation (5) admits a slightly more general, but similar economic interpretation.

To conclude our review of the results, it can be shown that the optimal prevention rate w decreases both with the number of users x as well as with the shadow price λ. Moreover, the optimal treatment rate generally increases with x, but decreases with λ, a behaviour which makes economic sense. Generally, the optimal mix of instruments is such that it influences the state variable in the same direction (for a proof of this general property in 'ordinary' optimal control theory compare Feichtinger (1984).

Up to now we interpreted our model as initiation model for illicit drug use. Another interpretation would be in the context of violence that shows typical age-pattern. First the initiation rate $\overline{\mu}(a)$ has qualitatively a similar sharp as for marijuana consumption (although its increase is less sharp). The age-specific pattern of the per-capita damage in the case of violence has probably also a similar shape, i.e. it increases with age.

5. Minimizing the social loss of unemployment

In this section we present a cost-benefit analysis for unemployment. Since the model is formally similar to those analyzed in section 3, we can be rather brief. Our preliminary considerations should illustrate the possibility of another interesting interpretation of the prototype proposed in section 3. Thus, let us identify non-users as employed, and the users as unemployed individuals.

The state variable x(t,a) is the number of unemployed aged a years at time t.

n(t,a) = p(a) - x(t,a) denotes the number of employed, and p(a) is the population at age a assumed to be stationary.

The authority tries to prevent that employed get unemployed. A good example for such measures would be the support of 'on the job training'. Such investment could be pay out more for young people than for old workers. On the other hand, treatment against unemployment may be seen as educational measures for new jobs.

The autonomous incidence rate for unemployment, $\overline{\mu}(a)$, is highly age-dependent. It seems realistic to assume an U-shaped age-pattern, since the risk to get unemployed is usually rather high in young ages as well as in high age classes, but smaller in between.

Both the objective functional as well as the system dynamics are given by (1) and (2), respectively. Changing the interpretations is either straightforward or has been already mentioned above.

See the work of Shimer (1999) for same sort of feedback mechanism (we owe this reference to T. Lindh and B. Malmberg). The social damage per unemployed individual is clearly age-dependent. A plausible form for d(a) is an inverse U-sharp, i.e. $d'(a) > 0$ for low a, but $d'(a) < 0$ for high age groups.

6. Optimal Control of further age-structured models

The usual models to study the dynamics of human populations are age-structured, descriptive and linear. Beside of age the population can be classified with respect to other characteristics like parity, region, educational status etc.

Population dynamics is usually *descriptive*. However, policy questions deal largely with *normative* issues. In a remarkable paper, Arthur and McNicoll (1977) attempted to link two distinct traditions, namely formal intertemporal policy analysis and mathematical demography. In our multi-class models, sometimes also denoted as compartment models, the flows between the various classes of a given classification are the control variables. Given that some sub-populations, e.g. the drug consumers or the unemployed individuals generate a certain social loss per time unit.

Let us briefly explain a fluid mechanic analogon as follows: consider a network of boxes, inputs, outputs and connections between some of the boxes. The flows between the boxes can be steered, i.e. increased or decreased, but this control is costly. The level (volume) of the fluid in some of the boxes is negatively evaluated. We are looking for a program to control the flows in an optimal way. The dimension of age or/and duration as well as the density-dependence of the flow rates makes analysis not an easy task.

Nonlinearities play an increasing rate not only in general population dynamics, but also in demography. Let us only refer to three important issues: two sex models, marriage market models, and Easterlin-type models.

Table 1 summarizes a bundle of applications of our prototype in various fields.

While the problems assembled in table 1 deal with population of individuals, the cases shown in table 2 contain vintage capital models in various forms, i.e. as financial, physical or human capital stock. Since feedback effects in the style of table 1 are not easily introduced we omit that column for the moment.

Table 1: Cost-benefit analyses for selected examples of age-structured compartment models

Field	Control(s)	Feedback effects	References
epidemiology	vaccination	yes	Anderson & May (1991) Noymer (2000)
HIV	prevention, treatment	yes	Sanderson (2000)
illicit drug initiation	prevention, treatment	yes	Behrens et al. (2000) Almeder et al. (2000)
manpower planning	promotion rates, input rates	some	Gaimon & Thompson (1981)
education	enrolment rates for various steps of education	negligible	IIASA, Lutz & Goujon (2000)
unemployment	on-the-job training	some	Shimer (1999)
marketing	advertising, pricing product quality	yes	Caulkins et al. (2000)
population dynamics	fertility	some	Gopalsamy (1976)
bioeconomics (management of renewable resources)	harvesting	no	Muzicant (1980) Brokate (1985) Murphy & Smith (1990)

Table 2: Age-structured vintage capital models

Field	Control(s)	References
vintage capital	investment	Kort et al. (2000)
demo-economics	fertility, savings rate	Arthur & McNicoll (1977)
environmental planning	investment, prevention, maintenance	Xepapadeas & de Zeeuw (1999)

7. Conclusions and extension

The fact that many processes in biology, demography and economics involve simultaneously time and age makes its analysis difficult. Conventional tools of policy analysis do not apply.

The first aim of this paper is a methodological one. Our intention is to show how optimal control theory may be applied to intertemporal policy questions. For a certain class of age-dependent problems an approach is proposed to determine the intertemporal trade-off between policy instruments and social loss. It turns out that the special 'flavour' of this approach, i.e. the qualitative characterization of optimal paths, remains valid for feedbacks on incidence rates of the age composition.

The second aim of our contribution is to identify a whole bunch of relevant policy questions which fit into our proposed umbrella. Admittedly, for virtually all models we sketched in section 5 and 6 an empirical validation is still missing. Even the model for marijuana consumption introduced in section 2 is only partially validated. It would be an interesting, task to estimate the influence functions m(a,a') measuring the impact of an a' years old individual to the propensity for drug initiation of an individual aged a years (see Fig. 1 for a rough qualitative picture of this function).

Our intention is to contribute to shaping a skeleton of population policy. The flesh of empirical work is largely left for future research. In a certain sense the purpose of this paper may be seen as that of Arthur and McNicoll (1977) who tried to create some building stones of a theory of population policy. The optimal control approach in public policy-making allows us to derive results on the qualitative structure of policies without a full and exact knowledge of the underlying functions and parameters.

We are aware on the fact that we pointed to many more open problems than solved ones. The road ahead to a public policy based on solid fundaments is still long. An adequate full analysis of vintage capital models is still missing. Recent preliminary work reveals the power of the maximum principle approach in related fields; compare, e.g., Xepapadeas and de Zeeuw (1999), and Kort et al. (2000).

Another direction of research are *spatial aspects* leading to PDEs of second order. The work of Puu (1999) and Brito (1999) may be seen as promising start. Mathematical niceties, like travelling waves, and policy relevance could be trademarks of spatial models.

References:

Almeder, C., G. Feichtinger and G. Tragler (2000), Age-specific multi-stage drug initiation models - innovation towards heterogeneity. Forschungsbericht **246** des Instituts für Ökonometrie, OR und Systemtheorie, TU Wien.

Anderson, R.M. and R.M. May (1991), *Infectious Diseases of Humans. Dynamics and Control*. Oxford University Press, Oxford.

Arthur, W.B. and G. McNicoll (1977), Optimal time paths with age-dependence: a theory of population policy. *Review of Economic Studies* **54**, 111-123.

Behrens, D., J.P. Caulkins, G. Tragler and G. Feichtinger (2000), Optimal control of drug epidemics: prevent and treat - but not at the same time? *Management Science* **46**, 333-347.

Bensoussan, A., G., Nissen and C.S. Tapiero (1975), Optimum inventory and product quality control with deterministic and stochastic deterioration - an application of distributed parameters control systems. *IEEE Trans. Autom. Contr.* AAC-20, 407-412.

Brito, P. (1999), Spatial heterogeneity and growth. Working Paper, Dept. of Economics, Technical University, Lisbon.

Brokate, M. (1985), Pontryagin's principle for control problems in age-dependent population dynamics. *Journal of Mathematical Biology* **23**, 75-101.

Butkovsky, A.G. (1969), *Distributed Control Systems*. American Elsevier Publ., New York.

Carlson, D.A., A.B. Haurie and A. Leizarowitz (1991), *Infinite Horizon Optimal Control. Deterministic and Stochastic Systems*. Springer, Berlin.

Caulkins, J. Feichtinger, G. and G. Tragler (2000), The 'jeans model' market segmentation by age. Preliminary draft. TU Vienna.

Derzko, N.A., S.P. Sethi and G.L. Thompson (1980), Distributed parameter systems approach to the optimal cattle ranching problem. *Optimal Control Applications and Methods* **1**, 3-10.

Derzko, N.A., S.P. Sethi and G.L. Thompson (1984), Necessary and sufficient conditions for optimal control of quasilinear partial differential systems. *Journal of Optimization Theory and Applications* **43**, 89-101.

Dietz, K. (1997), Introduction to McKendrick (1926), Applications of mathematics to medical problems. In: *Breakthroughs in Statistics*, S. Kotz and N.L. Johnson (eds.), Springer, Berlin 17-26.

Dorfman, R. (1969), An economic interpretation of optimal control theory. *American Economic Review* **59**, 817-831.

Everingham, S.S. and C.P. Rydell (1994), *Modeling the Demand for Cocaine*. RAND, Santa Monica, CA.

Feichtinger, G. (1984), On the synergistic influence of two control variables on the state of nonlinear optimal control models. *Journal of Operational Research Society* **35**, 907-914.

Feichtinger, G. and Hartl, R.F. (1986), *Optimale Kontrolle ökonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften*. de Gruyter, Berlin.

Feichtinger, G., R.F. Hartl and P. Kort (2000), Dynamic investment behavior taking into account heterogeneity of the capital stock and technological progress. Working Paper, TU Vienna.

Gaimon, C. and G.L. Thompson (1981), A distributed parameter cohort personnel planning model. Working paper, No. 43-80-81, Carnegie-Mellon Univ. Pittsburgh.

Gopalsamy, K. (1976), Optimal control of age-dependent populations. *Math. Biosci.* **32**, 155-163.

Gurtin, M.E. and L.F. Murphy (1981), On the optimal harvesting of persistent age-structured populations. *Journal of Mathematical Biology* **13**, 131-148.

Haurie, A., S. Sethi and R. Hartl (1984), Optimal control of an age-structured population model with applications to social services planning. *Large Scale Systems* **6**, 133-158.

Keiding, N. (2000), Mortality measurement in the 1870s: diagrams, stereograms, and the basic differential equation. Working Paper, Dept. of Biostatistics, University Copenhagen.

Keyfitz, B.L. and N. Keyfitz, (1997), The McKendrick partial differential equation and its uses in epidemiology and population study. *Math. Comput. Modelling* **26**, 1-9.

Lindh, T. and B. Malmberg (1999), Age distribution and the current account. A changing relation? Working Paper, Dept. of Economics, Uppsala University.

Lutz, W. and A. Goujon (2000), The World's Changing Human Capital Stock: Population Forecasts by Educational Attainment. Forthcoming in *Population and Development Review*.

McKendrick, A.G. (1926), Applications of mathematics to medical problems. *Proc. Edinburgh Math. Soc*. **44**, 98-130.

Murphy, L.F. and S.J. Smith (1990), Optimal harvesting of an age-structured population. *Journal of Mathematical Biology* **29**, 77-90.

Muzicant, J. (1980), Systeme mit verteilten Parametern in der Bioökonomie: Ein Maximumprinzip zur Kontrolle altersstruktuierter Modelle. Diss., Technische Universität Wien.

Noymer, A. (2000), Demographic-epidemiologic models of measles transmission in developing countries: the case of muyinga sector, Burundi. Working Paper, Dept. of Sociology/Dept. of Demography, UC-Berkeley.

Puu, T. (1999), Travelling waves in a modified Hotelling population model. Working Paper, Dept. Economics, Umea University.

Robson, A.J. (1985), Optimal control of systems governed by partial differential equations: economic applications. In: G. Feichtinger (Ed.) *Optimal Control Theory and Economic Analysis 2*. North-Holland, Amsterdam, 105-118.

Sanderson, W. (2000), Diamonds and deaths. Part 1. Paper presented at the Workshop 'Macroeconomic Demography' held at the MPI for Demographic Research, Rostock.

Shimer, R. (1999), The impact of young workers on the aggregate labor market. Working Paper 7306, National Bureau of Economic Research, Cambridge, MA.

Tragler, G., J.P. Caulkins and G. Feichtinger (1997), The impact of enforcement and treatment on illicit drug consumption. Forthcoming in *Operations Research*.

Tzafestas, S.G. (1982), Optimal and modal control of production-inventory systems. In: S.G. Tzafestas (Ed.) *Optimization and Control of Dynamic Operational Research Models*. North-Holland, Amsterdam, 1-71.

Xepapadeas, A. and A. de Zeeuw (1999), Environmental policy and competitiveness: the Porter hypothesis and the composition of capital. *Journal of Environmental Economics and Management* **37**, 165-182.

Veliov, V., G. Feichtinger and G. Tragler (2000), A maximum principle for age-structured density-dependent population processes. Forschungsbericht **245** des Instituts für Ökonometrie, OR und Systemtheorie, TU Wien.

List of Captions:

Fig. 1: Schematic representation of a simple age-specific model (thicker arrows indicate higher initiation rates)

Fig. 2: Influence rates for illicit drug initiation, m(a,a') measuring the impact of one individual in age group a' on the initiation rate of a non-users in age group a (stylized picture). (The right graph shows the age-specific basic initiation rate, which was used for the calculations and the left figure shows a typical result for an age-specific prevention program without a memory effect. The results suggest directing the prevention program to those age groups, which have the highest basic initiation rates.)

Fig. 3: Age-specific initiation rates for marijuana consumption in the US (data for 1992).

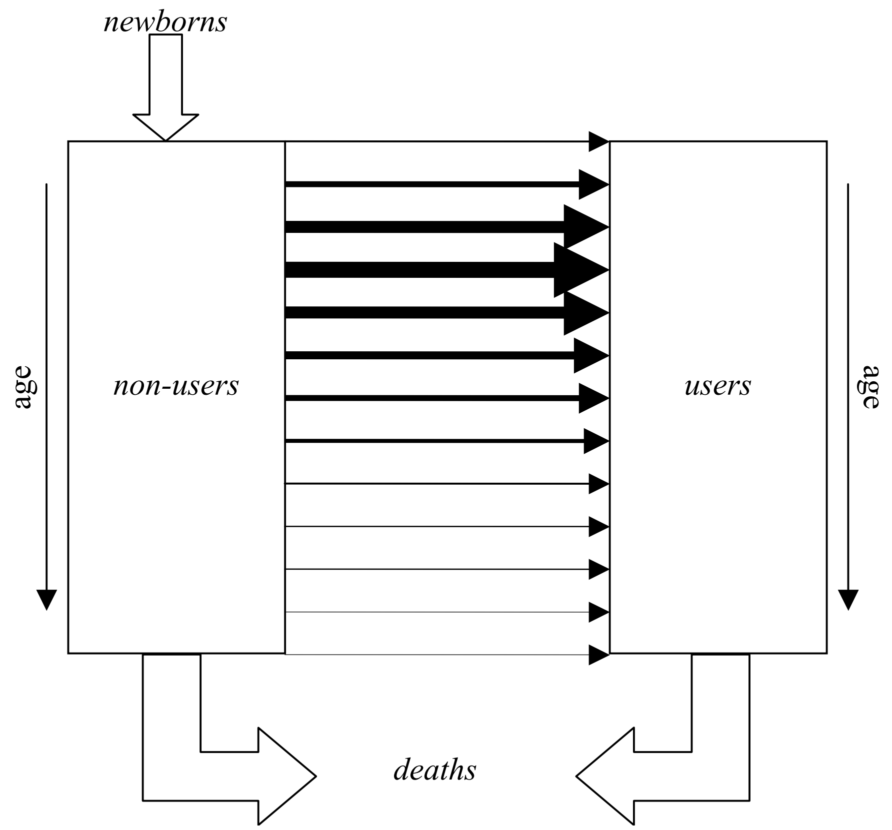

Figure 1

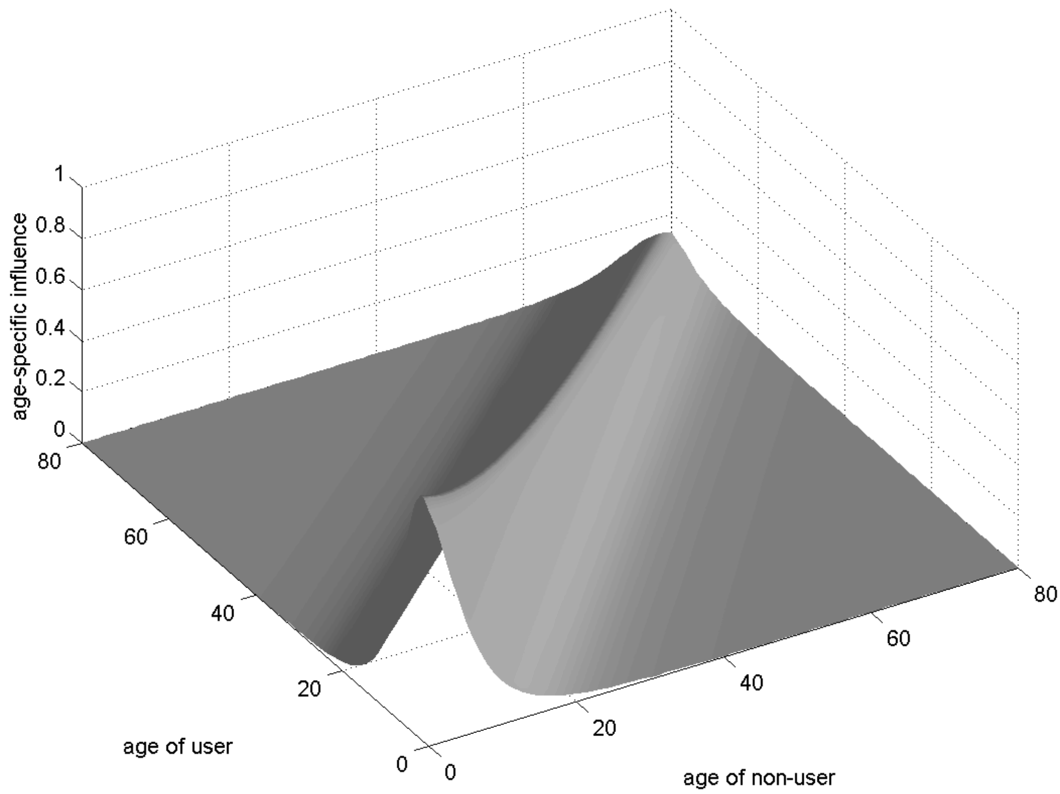

Figure 2

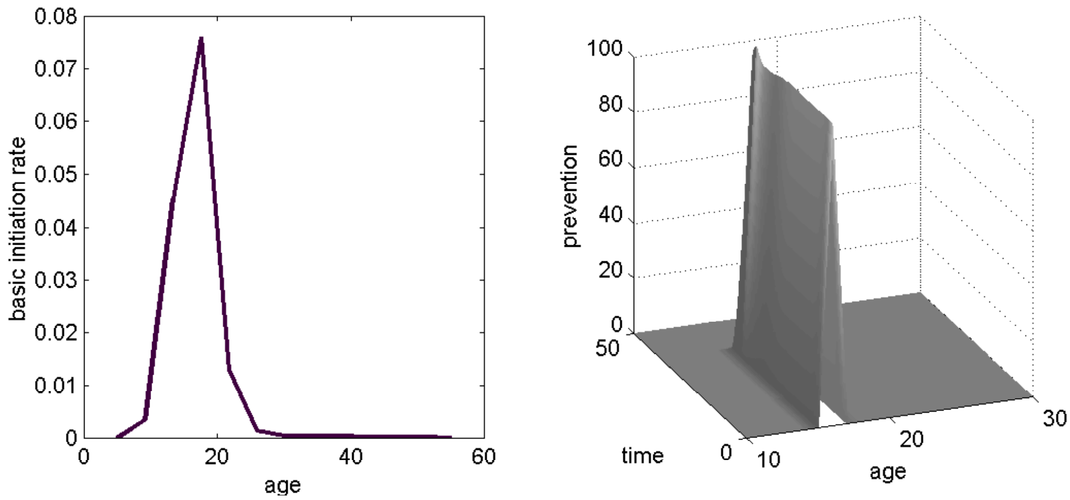

Figure 3

[i] In a recent interesting contribution, Keiding (2000) showed how this PDE model is related to another, but well-known approach in population dynamics, i.e. to the Lexis diagram.

[ii] In addiction to that (or alternatively), age-dependent migration rates might be another possibility to control the system.

[iii] For the US cocaine epidemic Everingham and Rydell (1994) have estimated the unit damage κ as 67,6 US for the year 1992; see also Tragler et al. (1997).

[iv] In epidemiology, however, age-related problems play an important role; see the standard reference Anderson and May (1991).

[v] Our aim to study the initiation as well as the mix of prevention and treatment justifies this stationary assumption. Note also, that in an earlier draft of this paper we focused on the non-users as the relevant state variable.

[vi] In the literature this problem is also known as OSSP (optimal steady state problem); compare Carlson et al. (1991).

[vii] More precisely, $\lambda(t,a)$ denotes the change of the optimal value of the process if there appears at time t one a years old additional user. Note that this a an internal price for a rational (optimizing) planner, and no market price. The interested reader is referred to the early work by Dorfman (1969) who gave a brilliant introduction to the economic interpretation of optimal control models.