A DNS-Curve in a Two State Capital Accumulation Model: a Numerical Analysis

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Abstract

Since the end of the seventies several contributions appeared in which one-state-variable-optimal-control-models where treated, whose outcome exhibits a so-called DNS (Dechert-Nishimura-Skiba)-point. These models have in common that their solution reveals the existence of (at least) two stable steady states (saddle points). At the DNS-point it holds that the decision maker is indifferent between converging to either one of these steady states. Intuition suggests that extending this feature to the class of optimal control models with two state variables will lead to the occurrence of a DNS-curve, which is defined as being the set of points in the state space at which converging to either of two stable steady states can both be optimal. Determination of such a DNS-curve appeared to be extremely difficult, which is reflected by the fact that until now no paper

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appeared in which such a DNS-curve was designed. In this paper we determine a DNS-curve by analyzing numerically the optimal trajectories of a two dimensional capital accumulation model.

Key words: DNS-curve; capital accumulation; optimal control; numerical methods; multiple steady states; invariant manifolds; discontinuous feedback rule.

JEL classification: C61, C62, D92

1 Introduction

A DNS (Dechert-Nishimura-Skiba)-point can be defined as a treshold at which there are two optimal paths. Each optimal path approaches a different steady state. The decision maker is indifferent concerning the choice of these two paths. Skiba (1978) analyzes a continuous time growth model with a non-convex technology where this phenomenon occurred for the first time in the literature.

Davidson et al. (1981) and Dechert (1983) analyze a firm capital accumulation model where the revenue function contains a convex segment. For this model it is possible that two saddle points occur. It then depends on the initial capital stock value to which saddle point the firm will converge to in the long run. Both Davidson and Harris (1981) and Dechert (1983) show that under a particular scenario a treshold value (or DNS-point) of the initial capital stock exists, above which it is optimal to converge to the larger saddle point and below which convergence to the smaller saddle point is preferable.

Dechert and Nishimura (1983) analyze a discrete time Ramsey model in which the production function is convex-concave. They show that, provided that the interest rate has an intermediate value, the optimal path converges to a steady state only if the initial capital stock is above a critical value, otherwise it converges to zero. All these papers have in common that the fact that the optimal paths are history dependent is caused by (local) convexities. An unstable steady state is crucial for determining the treshold separating a desirable and less desirable long run outcome. In a recent paper by Wirl and Feichtinger (1999) it was found that (local) convexities are by no means necessary for the occurrence of such tresholds. They provide two mechanisms, i.e. growth and control state interactions, that can lead to history dependence in a strictly concave framework.

All the above mentioned contributions rely on the one-dimensional structure of the dynamic optimization problem. Extending the analysis to two dimensions would suggest that the DNS-point would change into a DNS-curve on which the decision maker is indifferent between converging to different steady states. In the literature contributions that deal with this topic are scarce. Brock et al. (1983) prove the existence of a DNS-curve, but the exact location of it is not determined. Doing the latter is the aim of our paper. To our knowledge the present paper is the first one in which the exact location of a DNS-curve is determined.

Designing this DNS-curve is done in the context of a classical problem in continuous time investment theory: the optimal accumulation of capital by a firm maximizing its present value over an infinite horizon. This problem has been extensively analyzed in the literature assuming a constant or decreasing returns to scale technology and adjustment costs of investment (see Eisner, 1963; Lucas, 1967; Gould, 1968). Here the specific feature of our model is that besides the "normal" adjustment costs associated with investment, also the changes in the investment rate are made costly. Furthermore, like in Davidson et al. (1981) and Dechert (1983), the revenue function contains one convex segment. This means that for capital stock values occurring in that segment the firm's production function exhibits increasing returns to scale. We impose that increasing returns to scale hold for intermediate values of the capital stock, so that the revenue function is concave for small and large capital stock values. Opposite to Barucci (1998), who studies the classical framework with the exception that the revenue function is convex throughout.

The contents of the paper is as follows. The next section specifies the model. In Section 3 we analyze the model both mathematically and economically. Section 4 concludes the paper.

2 Model Formulation

Consider a firm that needs capital goods to produce goods which are sold on the output market. The more capital goods the firm owns the more goods can be sold and thus the more revenue is obtained. Of course, in case the firm has some market power the output price decreases with the number of goods that are sold, which implies that decreasing returns to scale will be present especially if the capital stock is sufficiently large. On the other hand scale economies can cause increasing returns to scale. To analyze the effect of this on optimal firm behavior it is imposed that there exists an interval of capital stock values for which there are increasing returns to scale.

Denoting revenue by R and capital stock by K[1], it is imposed that $R(K)$ is a positive, twice continuously differentiable, increasing function with one convex segment for intermediate values of the capital stock (e.g. Davidson et al., 1981, Figure 2b). This convex segment arises due to the fact that for these values of the capital stock the firm's production technology exhibits increasing returns to scale.

The firm can increase its capital stock by investing, where the investment rate is denoted by I. Besides the purchase costs, the cost of investments $c(I)$ also consists of adjustment costs which are assumed to be convex. This makes that $c(0) = 0, c' > 0, c" > 0$.

As usual, capital stock increases with investments and decreases with depreciation. Assuming a constant depreciation rate $\delta > 0$, the following state equation for capital stock arises:

$$\dot{K} = I - \delta K. \tag{1}$$

The concept of investment adjustment costs is refined here by penalizing changes in the investment rate (see also Jorgensen et al., 1993). Such costs can arise in case an organization is used to a certain rate of investment, so that it has to reorganize at the moment that changes in this investment rate occur. Representing the change of investment by v, these costs equal $g(v)$. For the sake of illustration, let g be quadratic,

$$g(v) = \frac{\alpha}{2} v^2,$$

[1] Since the model is dynamic, all variables are functions of time, i.e. $K = K(t)$. For notational convenience, in what follows this dependence of variables on time is not explicitly denoted.

where

$$\dot{I} = v. \tag{2}$$

To include (2) in the optimization problem, investment I will be modelled as a state variable.

The firm's objective is to maximize the discounted cash flow stream over an infinite planning horizon. Collecting the revenue function and both types of adjustment costs described above, and assuming a constant discount rate ρ, we arrive at the following expression for the criterion function:

$$\max_{v} \int_{0}^{\infty} e^{-\rho t} \left[R(K) - c(I) - \frac{\alpha}{2} v^2 \right] dt. \tag{3}$$

The optimal control model now consists of the expressions (1)-(3). It has two state variables, K and I, and one control variable, v. Summarizing, the following model is obtained:

$$\begin{aligned} \max_{v} \quad & \int_{0}^{\infty} e^{-\rho t} \left[R(K) - c(I) - \frac{\alpha}{2} v^2 \right] dt, \\ \text{s.t.} \quad & \dot{K} = I - \delta K, \\ & \dot{I} = v. \end{aligned}$$

If $\alpha = 0$, v can be deleted and the problem becomes an optimal control model with state variable K and control variable I. Then we are back in the classical capital accumulation models. For $R(K)$ being linear or concave, and $c(I)$ convex the basic framework arises which is analyzed extensively (e.g. Lucas, 1967; Gould, 1968)). In this model Dechert (1983) proves the existence of a DNS-point with $R(K)$ being a convex-concave function, while Davidson et al. (1981) studies the implications of convex segments in $R(K)$ and concave segments in $c(I)$. Barucci (1998) considers convex quadratic functions for both $R(K)$ and $c(I)$.

3 Analysis of the Model

This section consists of five subsections. First, in Subsection 3.1 the optimality conditions are listed, after which in Subsection 3.2 the stability behavior around the steady states is studied. In Subsection 3.3 the stable manifolds are analyzed, while in Subsection 3.4 the DNS-curve is determined. Finally, an economic intuition of the mathematical results is provided in Subsection 3.5.

3.1 Optimality Conditions

To apply Pontryagin's maximumprinciple, we start out by stating the current value Hamiltonian:

$$H = R(K) - c(I) - \frac{\alpha}{2}v^2 + \lambda_1 (I - \delta K) + \lambda_2 v.$$

Maximization of the Hamiltonian with respect to the control variable v gives:

$$v = \frac{1}{\alpha}\lambda_2. \tag{4}$$

Further application of Pontryagin's maximumprinciple and taking into account (1) and (2), leads to the following dynamic system:

$$
\begin{aligned}
\dot{K} &= I - \delta K, \\
\dot{I} &= v \;=\; \tfrac{1}{\alpha}\lambda_2, \\
\dot{\lambda}_1 &= (\rho+\delta)\lambda_1 - R'(K), \\
\dot{\lambda}_2 &= \rho\lambda_2 - \lambda_1 + c'(I).
\end{aligned} \tag{5}
$$

3.2 Stability Analysis

In a steady state it holds that $\dot{K} = \dot{I} = \dot{\lambda}_1 = \dot{\lambda}_2 = 0$. Now, it is straightforward to see that $\lambda_2 = 0$ and $I = \delta K$ is required for being in a steady state. Due to the dynamic equations for λ_1 and λ_2 it can be concluded that, additionally to $\lambda_2 = 0$ and $I = \delta K$, possible steady states have to fulfill $\lambda_1 = c'(\delta K) = \frac{1}{(\rho+\delta)} R'(K)$; confer Figure 1.

================ Figure 1 about here ====================

Here we see that in case of a convex cost function $c(I)$, at most one steady state exists if the revenue function $R(K)$ is concave, as is the case in the basic capital accumulation models. However, due to the convex segment for intermediate values of the capital stock, the revenue function that we consider has a concave – convex – concave shape. Depending upon the choice of the parameters for the functions $R(K)$ and $c(I)$ there are one, two or three different steady states. We continue with the investigation of the situation of three different steady states $\mathcal{K}_i = (K_i, I_i, v_i, \lambda_{1i}, \lambda_{2i}), i = 1, 2, 3$, where $\mathcal{K}_1$

corresponds to the steady state with the smallest capital stock, and the steady state with the largest capital stock value is denoted by $\mathcal{K}_3$.

Since this two state variable model with convex segment in the revenue function is too difficult to analyze analytically we rely on numerical methods. To this end we have to specify the functional forms. Concerning the revenue function a function with a convex segment for intermediate values of the capital stock has to be found. It turns out that a specification like

$$R\,(K) = k_1 \sqrt{K} - \frac{K}{1 + k_2\, K^4}. \tag{6}$$

satisfies this aim. For the investment cost function we adopt the quadratic specification imposed by, (e.g. Barucci, 1998):

$$c\,(I) = c_1\, I + \frac{c_2}{2}\, I^2. \tag{7}$$

To analyze the stability behavior of the dynamic system around the steady state, we determine the Jacobian and Dockner's K (see Dockner, 1985). The Jacobian equals (for reasons of surveyability we will not plug in specification (6) yet)

$$\det J(K) = \begin{vmatrix} -\delta & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \\ -R"(K) & 0 & \rho+\delta & 0 \\ 0 & c_2 & -1 & \rho \end{vmatrix} = \frac{1}{\alpha}\left[\delta\,(\rho+\delta)\,c_2 - R"\,(K)\right].$$

According to Figure 1 (second derivative of R) it holds that the Jacobian is positive in the first and the third steady state, $\mathcal{K}_1$ and $\mathcal{K}_3$. From the same figure we can conclude that in the second steady state, $\mathcal{K}_2$, it holds that

$$R\text{''}(K) > \delta\,(\rho + \delta)\, c_2,$$

so that the Jacobian is negative in $\mathcal{K}_2$. This implies that $\mathcal{K}_2$ is unstable, except for a one-dimensional manifold (see Feichtinger et al., 1994, Figure 1). Dockner's K equals

$$\kappa = \begin{vmatrix} -\delta & 0 \\ -R\text{''}(K) & \rho+\delta \end{vmatrix} + \begin{vmatrix} 0 & \frac{1}{\alpha} \\ c_2 & \rho \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = -\delta\,(\rho+\delta) - \frac{c_2}{\alpha} < 0.$$

Given the fact that the Jacobian is positive and Dockner's K is negative in $\mathcal{K}_1$ and $\mathcal{K}_3$, we know that both steady states are saddle points. Now, if

$$\frac{\kappa^2}{4} - \det J(K) = \frac{1}{4\alpha^2}\left[(\delta\,(\rho+\delta)\,\alpha - c_2)^2 + 4R\text{''}(K)\alpha\right] \geq 0,$$

then the steady state is a saddle point with monotonic motions. Otherwise saddle point convergence occurs with transient oscillations (damped cycles). This brings us to the following proposition:

Proposition 1 *The stability analysis in case of existence of three steady states gives the following results:*

- *The invariant manifold of steady state $\mathcal{K}_2$ is one dimensional.*
- *The invariant manifolds of the steady states $\mathcal{K}_1$and $\mathcal{K}_3$ are two dimensional.*
- *The steady state $\mathcal{K}_i, i = 1, 3,$ is a saddle point with transient oscillations, if*

$$[\alpha\,\delta\,(\rho+\delta) - c_2]^2 + 4R\text{''}(K_i)\alpha < 0,$$

otherwise it is a saddle point with monotonic motions.

The aim of the paper is to determine the location of the DNS-curve numerically. To do so the parameter values are fixed as follows:

$$\rho = \frac{4}{100};\ k_1 = 2;\ k_2 = \frac{3}{256};\ c_1 = \frac{3}{4};\ c_2 = \frac{10}{4};\ \alpha = 12;\ \delta = \frac{1}{4}. \tag{8}$$

The values of the steady states are numerically computed by Mathematica (Version 4.0.1.0, ©Wolfram Research, Inc.). They are listed in Table 1. For the following considerations it is enough that we retain the K, I, v - values of the first and of the third steady state, $\mathcal{K}_1 = \{0.58, 0.15, 0.0\}$ and $\mathcal{K}_3 = \{4.1, 1.0, 0.0\}$. These steady states are saddle points with transient oscillations.

================ Table 1 about here ===================

3.3 Stable Manifolds

================ Figure 2 about here ===================

================ Figure 3 about here ===================

Figure 2 and Figure 3 should help us to get an imagination how the stable manifolds of steady state $\mathcal{K}_1$ and $\mathcal{K}_3$ look like. We have determined them by computing the eigenvectors of the linearization of the canonical system in the steady states. Using real and imaginary parts of those eigenvectors, which correspond to the eigenvalues with negative real parts, we define ellipses in ε – neighbourhoods of the steady states with ε less than 10^{-4}. If ε is sufficiently small these ellipses are practically placed on the linearization of the stable manifolds which we want to compute. Choosing starting points on the ellipses and going back in time we compute numerically step by step the stable manifolds.

Figure 2 and Figure 3 show projections of higher dimensional surfaces onto the three dimensional state – control space, thus onto the (K, I, v) space[2]. In Figure 2 the shape of the stable manifold of steady state $\mathcal{K}_1$ is characterized. Steady state $\mathcal{K}_3$ is delineated by a bold face bar across the steady state orthogonal onto the (K, I) plane. This bar should help us both to find orientation in the figure and to get an imagination of the position of steady state $\mathcal{K}_3$ and stable manifold of steady state $\mathcal{K}_1$ to one another. In Figure 3 the shape of the stable manifold of steady state $\mathcal{K}_3$ is characterized. Now steady state $\mathcal{K}_1$ is delineated by a bold face bar across the steady state orthogonal onto the state plane.

Figure 2 depicts a piece of the stable manifold of steady state $\mathcal{K}_1$. We can see several trajectories converging to the steady state, which one can find in the northeast corner of the figure. However, the stable manifold does not range to steady state $\mathcal{K}_3$. It folds back far away from steady state $\mathcal{K}_3$. Figure 3 depicts a piece of the stable manifold of steady state $\mathcal{K}_3$. We can see several trajectories converging to the steady state, which

[2]In the following discussion of the figures we omit the phrase "...projection of...onto the (K, I, v) space".

one can find in the southwest corner of the figure. Again, the stable manifold does not range to the other steady state $\mathcal{K}_1$. It folds back far away from steady state $\mathcal{K}_1$.

3.4 DNS-curve

=============== Figure 4 about here ===================

The initial state values of our problem are given. Choosing a trajectory that converges to a steady state is equivalent to complement the given initial state values with values for the co-state and/or control variables in a way that the resulting initial point lies on the stable manifold of that steady state, where the trajectory should converge to. But thereby the value of the objective functional corresponding to this trajectory can be determined immediately. We just have to evaluate the Hamiltonian with the values of the initial point and to divide the result by the discount factor. Computing the stable manifold is the job. After finishing this job we can easily compute the value of the objective functional in case of trajectories converging to the steady state.

As both stable manifolds do not range out to the other steady state, it can never be optimal to start in one steady state and to converge to the other steady state. Under the assumption that optimal trajectories have to converge to a steady state, it is evident that there must be a division of the state space into two different basins of attraction for steady states $\mathcal{K}_1$ and $\mathcal{K}_3$. A one dimensional manifold in the state space separates these basins of attraction. Let us assume that this manifold is a smooth curve and let us call it DNS-curve (Dechert – Nishimura – Skiba curve). On this DNS-curve the firm is indifferent between converging to either one of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$, which implies that the values of the objective functional of these two trajectories are exactly the same there.

Next, we compute this curve of indifference between converging to steady state $\mathcal{K}_1$ and $\mathcal{K}_3$. We have already numerically computed the shape of the stable manifolds. And we have seen that it is not really a problem to compute the value of the objective functional in case of converging to steady state $\mathcal{K}_1$ or $\mathcal{K}_3$. It is evident that the indifference curve is situated in an area of the state space, where it is possible - for a given set of initial state values - to choose a trajectory starting with these initial state values and converging to steady state $\mathcal{K}_1$ and to choose another trajectory starting with these initial state values and converging to steady state $\mathcal{K}_3$.

As we can see in Figure 4 the state space ($(K, I)-$ plane) is split into three regions I (Peacock Blue), II (Cyan and Yellow) and III (Light Cadmium Yellow). Given initial values in the Peacock Blue region there are no trajectories converging to steady state $\mathcal{K}_3$ and given initial values in the Light Cadmium Yellow region it is not possible to find trajectories converging to steady state $\mathcal{K}_1$. It is obvious that steady state $\mathcal{K}_1$ is situated in the Peacock Blue region and steady state $\mathcal{K}_3$ in the Light Cadmium Yellow region.

Voids are dyed Geranium Lake Red, as we can see in the northeast corner. On voids numerical computation failed to compute trajectories either converging to steady state $\mathcal{K}_1$ or to steady state $\mathcal{K}_3$. Above we have added two figures depicting the stable manifolds. However, we have to clear up that with numerical methods we get only a partial information on the stable manifold. We put on each square centimeter of the state space a grid with 80 times 80 not necessarily equidistant points. For each point of the grid we tried numerically to find trajectories converging to steady state $\mathcal{K}_1$ and $\mathcal{K}_3$.

Computing both stable manifolds allow us to compare the objective functional for trajectories converging to steady state $\mathcal{K}_1$ and for trajectories converging to steady state $\mathcal{K}_3$. We dye the part of region II in Figure 4, where it is better (in the sense of maximizing the objective functional) to converge to steady state $\mathcal{K}_1$ with Cyan and the other, where it is better to converge to steady state $\mathcal{K}_3$, with Yellow. We denote the Cyan colored region with II a, the Yellow colored region with II b; by accident the unstable steady state $\mathcal{K}_2$ is situated in the Cyan colored region. The barrier between the Cyan and the Yellow region is the indifference curve between steady state $\mathcal{K}_1$ and steady state $\mathcal{K}_3$. This curve is the DNS curve.

3.5 Economic Analysis

For the parameter values concerned (see (8)) the optimal solution consists of two steady states to which it is optimal to converge to in the long run. In the steady state with the larger capital stock the revenue is larger, but on the other hand more depreciation investments have to be undertaken to remain in this steady state, which implies that the adjustment costs are larger too.

Both steady states have their own basin of attraction. The boundary between these two basins of attraction is formed by the DNS-curve. The DNS-curve consists of all points in the state space for which converging to each of the steady states leads to exactly the same value of the objective. Converging to the larger steady state requires an increase of the investment rate, while investments have to decrease in case the firm starts to approach the smaller steady state. This implies that exactly on this curve the firm's policy function is discontinuous: the control $v(K, I)$ is positive on the trajectory that approaches the larger steady state, while $v(K, I)$ is negative on the trajectory that will converge to the smaller steady state. Due to the convex adjustment cost function for the rate of change of investment, the function $v(K, I)$ is continuous everywhere outside the DNS-curve.

It thus depends on the initial levels of the capital stock and the investment rate to which steady state it is optimal to converge to. From Figure 4 it can be concluded that the DNS-curve is decreasing in the (K, I)-plane. From an economic point of view this can be explained as follows. In Figure 1 we see that for capital stock values between, say, 0.6 and 2.0, marginal revenue is low compared to marginal investment costs. Therefore, if the firm starts out with a low capital stock value and sufficiently low investment rate it is not profitable to enter a growth phase that passes this interval of capital stock values. This implies that convergence to the lower steady state is optimal for low values of investment and capital stock. Convergence to the larger steady state is optimal for sufficiently large initial values of capital stock and investment. This is especially caused by the fact that (1) changing the investment rate is costly (also in a negative direction), and (2) marginal revenue is large compared to marginal investment costs for capital stock values around 3 (see Figure 1).

The situation in Figure 4 occurs for a relatively large value of α, which is the parameter in the adjustment cost function for the rate of change of investment. For lower values of α the costs of $\dot{\mathrm{I}}$ are less, so that vertical motions in the state space are punished less. Then it will turn out that only one of the stable steady states will remain optimal so that the DNS-curve disappears. For this model numerical experiments show that reducing α leads to an upward movement of the DNS-curve, so that for a larger domain of initial values of capital stock and investment it becomes optimal to converge to the lower steady state. If α is sufficiently low, it will never be optimal to end up in the larger steady state.

4 Concluding Remarks

This paper considers two main features. First, and most important, in a two state variable optimal control model with two long run equilibria the location of a so-called DNS-curve is numerically determined and the economic intuition is provided. The DNS-curve connects all points in the state space on which the decision maker is indifferent concerning to which of the two long run equilibria to converge to. To our knowledge this contribution is the first one in which a DNS-curve is designed.

Second, our paper contributes to the literature of capital accumulation models. Especially the effects of increasing returns to scale for an intermediate interval of capital stock values are investigated. As such the paper extends the analysis of Davidson and Harris (1981) to a two dimensional framework. Furthermore the concept of adjustment costs is refined by punishing changes in the investment rate.

Finally, we list some suggestions for future research. First, consider the work of Ladron-de-Guevara et al. (1999). Building upon some basic results on dynamic programming, Ladron-de-Guevara et al. present in their work a new procedure for the characterization of optimal trajectories. To make their proof more transparent, the authors first basically restrict themselves to a single state variable. In general, however, their method of analysis which is based upon the construction of a 'candidate' value function, applies to two-state models. In particular, Ladron-de-Guevara et al. (1999) analyze a two-state endogenous growth model with physical and human capital in which leisure enters the utility function. By assuming that human capital does not affect the quality of leisure, while it influences production and investment, the analysis leads to multiple balanced growth paths. Using their method the authors show that unstable balanced growth paths with complex roots are non-optimal. Moreover, they are able to study continuity and discontinuity of the optimal policy function. The method developed by the authors should be of interest in related non-concave optimization framework. It would be a useful task to apply this method also to the analysis of the model described in the present paper.

Second, using the approach followed in this paper it must be possible to extend the analysis of two dimensional problems with non-concavities. In this way, the economic knowledge concerning the effects of increasing returns to scale can be increased considerably.

References

Brock, W.A. and W.D. Dechert, 1983, The Generalized Maximum Principle, Working paper University of Wisconsin-Madison, SSRI 8316.

Barucci, E., 1998, Optimal Investments with Increasing Returns to Scale, International Economic Review 39, 789-808.

Davidson, R. and R. Harris, 1981, Non-Convexities in Continuous-Time Investment Theory, Review of Economic Studies 48, 235-253.

Dechert, W.D., 1983, Increasing Returns to Scale and the Reverse Flexible Accelerator, Economics Letters 13, 69-75.

Dechert, W.D. and K. Nishimura, 1983, A Complete Characterization of Optimal Growth Paths in an Aggregated Model with a Non-Concave Production Function, Journal of Economic Theory 31, 332-354.

Dockner, E., 1985, Local Stability in Optimal Control problems with Two State Variables, in: G. Feichtinger, eds., Optimal Control Theory and Economic Analysis, Vol. 2, (North Holland, Amsterdam) 89-103.

Eisner, R. and R.H. Strotz, 1963, Determinants of Business Investment, in: Impacts of Monetary Policy, (Prentice Hall, Englewood Cliffs, NJ) 60-223.

Feichtinger, G., A. Novak, and F. Wirl, 1994, Limit Cycles in Intertemporal Adjustment Models, Journal of Economic Dynamics and Control 18, 353-380.

Gould, J.P., 1968, Adjustment Costs in the Theory of Investment of the Firm, Review of Economic Studies 35, 47-55.

Jorgensen, S., and P.M. Kort, Optimal Dynamic Investment Policies under Concave-Convex Adjustment Costs, Journal of Economic Dynamics and Control 17, 153-180.

Ladron-de-Guevara, A., S. Ortigueira, M.S. Santos, 1999, Two-Sector Model of Endogenous Growth with Leisure, Review of Economic Studies 66, 609-631.

Lucas, R.E., 1967, Adjustment Costs and the Theory of Supply, Journal of Political Economy 75, 321-334.

Skiba, A.K., 1978, Optimal Growth with a Convex-Concave Production Function, Econometrica 46, 527-540.

Wirl, F., and G. Feichtinger, 1999, History Dependence due to Unstable Steady States in Concave Intertemporal Optimizations, Working paper University of Magdeburg/Vienna University of Technology.

List of Tables

	K	I	v	λ_1	λ_2
$\mathcal{K}_1$	0.6	0.1	0.0	1.1	0.0
$\mathcal{K}_2$	2.3	0.6	0.0	2.2	0.0
$\mathcal{K}_3$	4.1	1.0	0.0	3.3	0.0

Table 1: Numerical values of the steady states

List of Figures

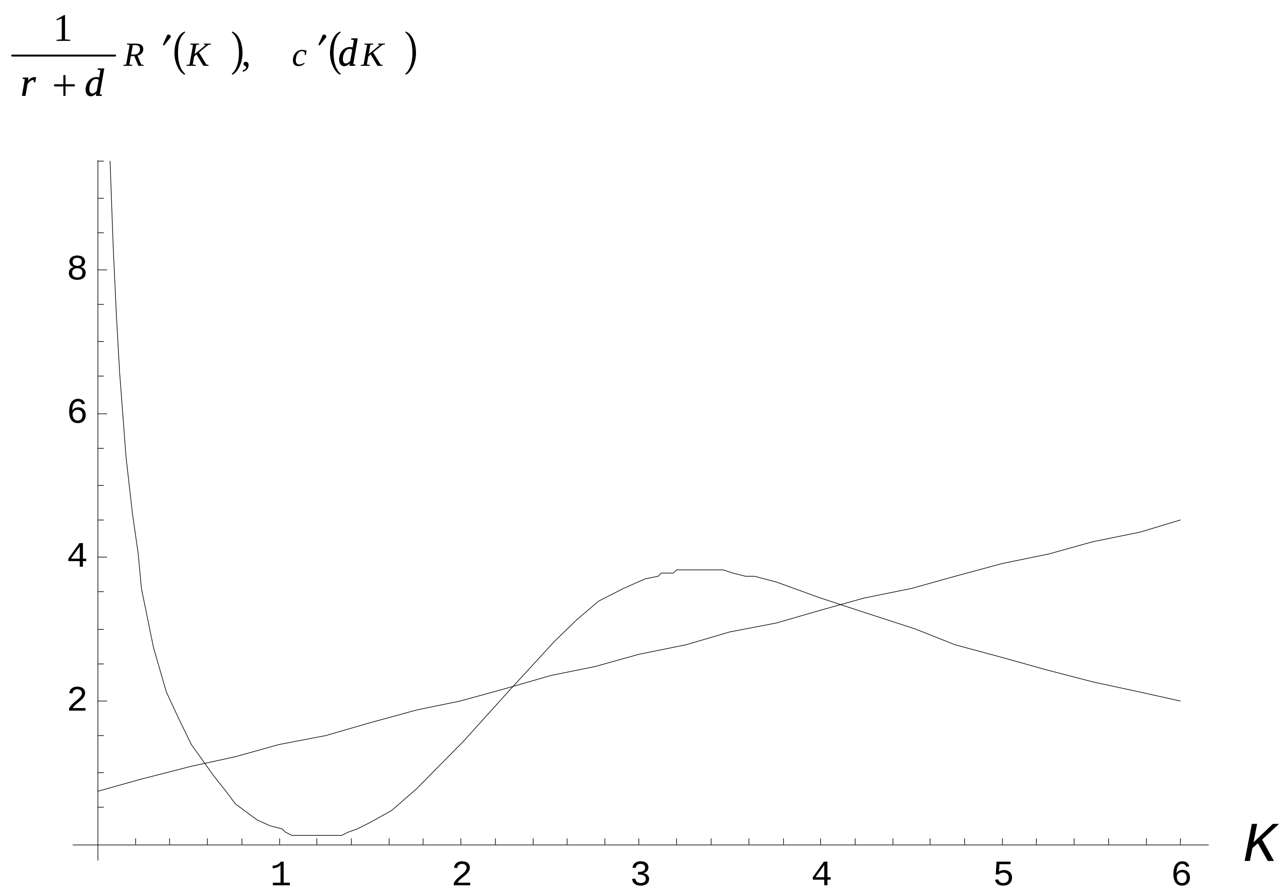

Figure 1

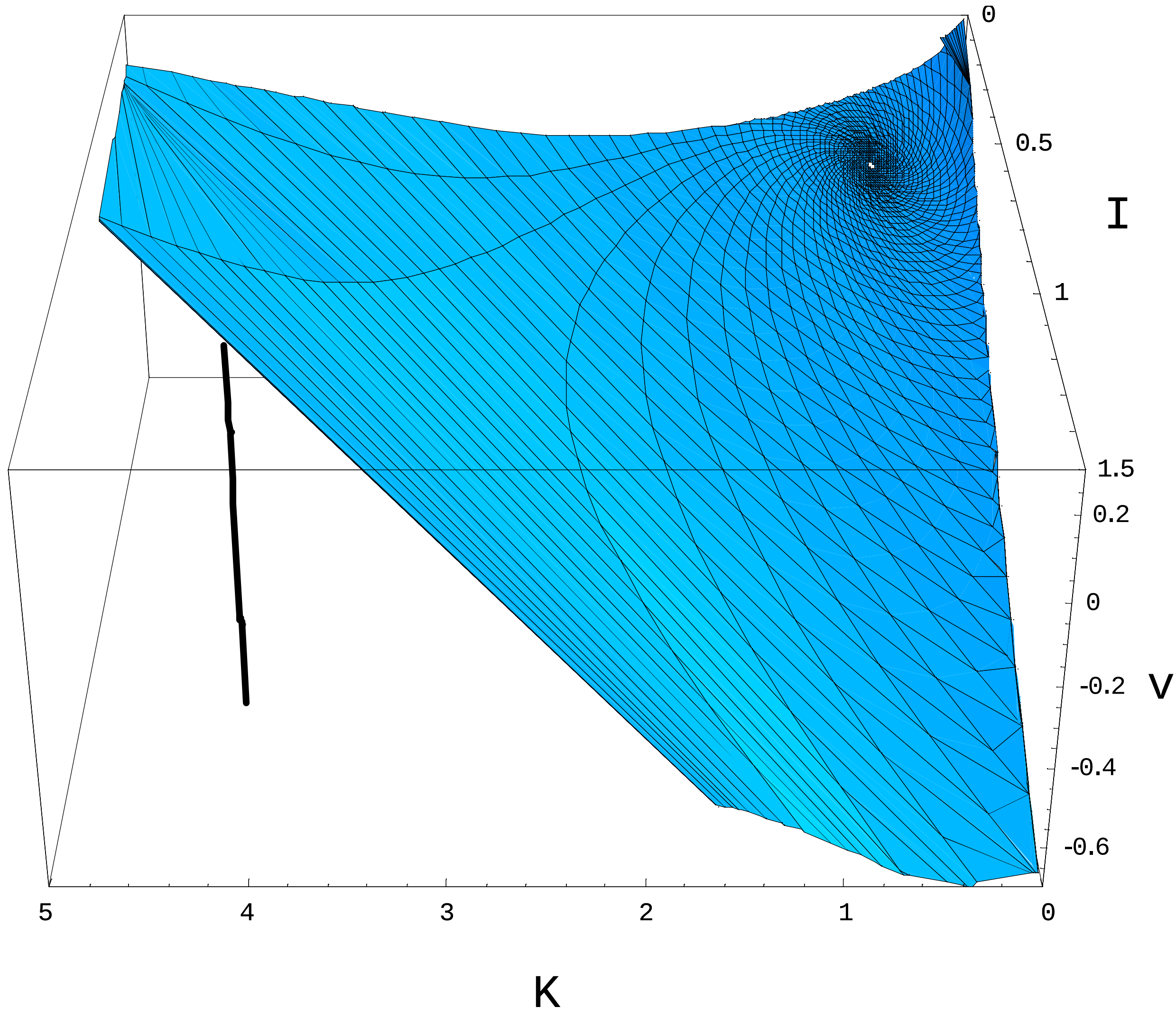

Figure 2

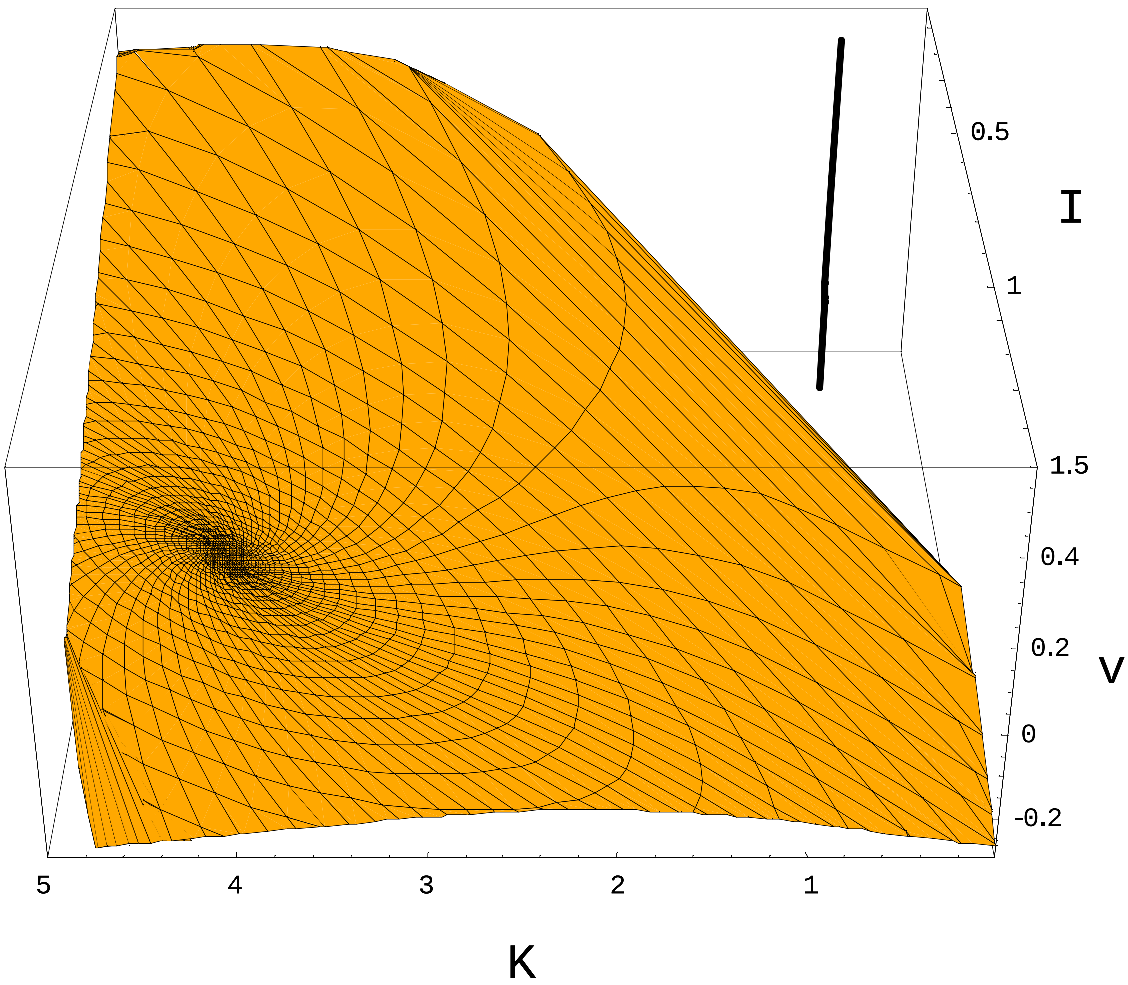

Figure 4

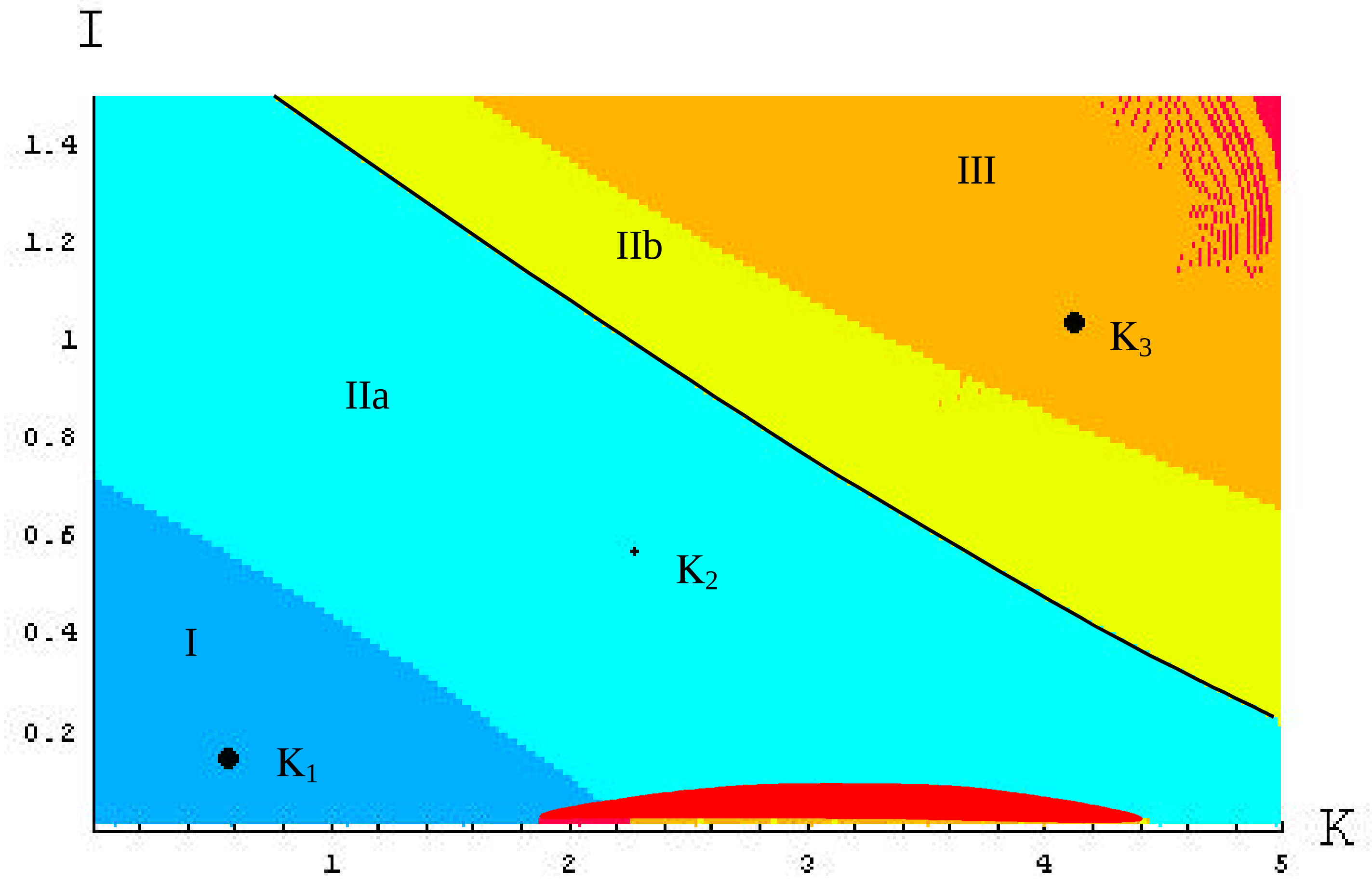

Figure 4