Optimality Conditions for Age-Structured Control Systems[1]

Gustav Feichtinger[2], *Gernot Tragler*[2], *and Vladimir M. Veliov*[3]

Abstract

We consider a fairly general model (extension of the Gurtin-MacCamy model of population dynamics) of an age structured control system with nonlocal dynamics and nonlocal boundary conditions. A necessary optimality condition is obtained in the form of Pontryagin's maximum principle, which is applicable to a number of practically meaningful models where the previously known results fail. We discuss such models (an epidemic control, and a capital accumulation model) as illustrations.

Keywords: age-structured systems, population dynamics, Gurtin-MacCamy equation, optimal control of distributed systems, Pontryagin's maximum principle.

AMS (MOS) subject classification: 49K20,49K22,92D25,90A16.

Abbreviated title: *Age-Structured Control Systems*

Address for correspondence: the Vienna address of the third author

[1]This research was partly supported by the Austrian Science Foundation under contract N0. 14060-OEK

[2]Institute for Econometrics, Operations Research and Systems Theory, Vienna University of Technology, Argentinierstrasse 8/119, A-1040 Vienna, Austria, e-mail: or@e119ws1.tuwien.ac.at, tragler@e119ws1.tuwien.ac.at

[3]Institute for Econometrics, Operations Research and Systems Theory, Vienna University of Technology, Argentinierstrasse 8/119, A-1040 Vienna, Austria and Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria, e-mail: vveliov@eos.tuwien.ac.at

1 Introduction

Optimal control problems for age-structured systems are of interest for many areas of application, as harvesting [18, 19, 26, 23, 2], birth control [8, 9, 4], epidemic disease control and optimal vaccination [21, 16, 24, 25], investment economic models [12, 5, 28, 6, 14], and for a variety of models in the social area [20, 1]. Many of the above papers present optimality conditions for particular models, usually in the form of a *maximum principle* of Pontryagin's type. A general maximum principle for nonlinear McKendric-type systems is obtained in [7]. However, a number of extensions of the McKendric and Gurtin-MacCamy [17] models arose recently, where the existing optimality conditions are not applicable. Below we sketch two such extensions, which are simplified versions (aimed just to outline the critical points) of models investigated in [24, 1, 15].

The first one arises in epidemic/vaccination control problems, as well as in the social area, where contagious factors are responsible for the evolution (viruses, fashion, drug consumption, crime, etc.).

The principle equations are of the form

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y(t,a) = S(t,a)\Phi(p(t,a)) - u(t,a)\Theta(y(t,a)) \tag{1}$$

$$p(t,a) = \int_0^\omega m(a,a')y(t,a')\,\mathrm{d}a', \tag{2}$$

with the following meaning: $y(t,a)$ is the number of the individuals of age a who belong to a specific group (the group of infected individuals, in the epidemic context; the group of drug users, or offenders, in a social context) at time t. The quantity $p(t,a)$ measures the total impact towards getting infected, or becoming a drug user, etc., to which a non-member individual of age a is exposed at time t. The function m measures, therefore, the impact of a member of the group of age a' on a nonmember of age a. Individuals of age more than ω are disregarded. The "reputation" function Φ transforms the total impact $p(t,a)$ to a *rate* at which the non-member individuals of age a ($S(t,a)$ is their number at time t) become infected (users). The function $\Theta(y)$ represents a state-feedback factor, $u(t,a)$ is an age-specific control (treatment) that is to be chosen to minimize the objective function

$$\int_0^T \int_0^\omega [d(a)y(t,a) + c(u(t,a))]\,\mathrm{d}a\,\mathrm{d}t, \tag{3}$$

possibly subject to control constraints. Here $d(a)$ is the damage caused by a member of age a, while the second term represents the cost of the control. Budgetary control constraints of the form

$$\int_0^\omega u(t,a)\,\mathrm{d}a \le B(t), \tag{4}$$

also often arise.

The known optimality conditions are not applicable to this problem at least because of the age-dependency of the integral (nonlocal) quantity $p(t,a)$. Kernels $m(a,a')$ depending in a nonseparable way on a and a' are typical: the contacts between 10 years old children are much more intensive than between 10 and 20 years old; the impact of 17 years old on 15 years old individuals (with respect to fashion, behavior, etc.) is much higher than that of the 50 years old individuals (the latter can be even with a negative sign).

Another specific case of an age-dependent kernel $m(a,a')$ (where m is an indicator function) arises when the fertility/mortality rate at an age a depends on the number of individuals younger (and/or older) than a ([10]). One of our goals in the paper is to cope with such situations (also in presence of budgetary control constraints).

The second example is a simplified version of a capital accumulation model presented in [15]. Below, $K(t,a)$ is the stock of capital goods of age a at time t, $u(t,a)$ is the intensity of buying machines of age a at time t, $v(t)$ is the purchase rate of new machines, $\delta(a)$ is the depreciation rate, $Q(t)$ is the output at time t, $I(t)$ is the total purchase rate of old machines at t. The basic equations are

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)K(t,a) = -\delta(a)K(t,a)+u(t,a), \tag{5}$$

$$Q(t) = \int_0^\omega m(t-a)d(a)K(t,a)\,\mathrm{d}a, \tag{6}$$

$$I(t) = \int_0^\omega u(t,a)\,\mathrm{d}a,$$

with initial and boundary conditions

$$K(0,a)=K_0(a), \quad K(t,0)=v(t).$$

Here $d(a)$ is the productivity of the capital goods of age a, corrected by the factor $m(t-a)$ reflecting the technological progress (the a years old machines at time $t+\tau$ are more productive than the a years old machines at time t), i.e. $m'>0$. The maximal age of the capital goods is ω, r is the discount rate. The objective function to be maximized is

$$\int_0^T e^{-rt}\{R(Q(t))-\alpha_0 v(t)-\beta v(t)^2-\gamma_i I(t)^2-\int_0^\omega[\alpha(a)u(t,a)+\gamma_e(a)u(t,a)^2]\,\mathrm{d}a\}\,\mathrm{d}t,$$

where $R(Q)$ is the revenue (which is usually assumed to be either a linear or a quadratic function of Q, depending on whether a competitive or monopolistic economy is considered), $\alpha_0 v$ and αu are the acquisition costs of machines, βv^2 is the adjustment cost for new machines, $\gamma_i I^2$ and $\gamma_e u^2$ are adjustment costs and implementation costs for old machines.

Two are the specific features for which the general result in [7] is not applicable. The first is the presence of a boundary control, $v(t)$. The second is more essential: the distributed control $u(t,a)$ appears both in the differential equation and in the equation for I, where the integration is with respect to a. There is a deep obstacle to treat such systems with the abstract approach from [7]. The measure theoretical constructions needed to implement this approach require that the controls in the differential equation, and in the equations where integration in a is involved, are *independent*; in particular, they cannot coincide or be subjected to joint constraints.

Notice also, that it makes sense to consider $K_0(a)$ as another control, if for example, a new enterprise is to be created. Therefore we include initial value control in the general model below.

In the present paper we consider the following general optimal control problem:

$$\text{minimize } \int_0^\omega l(a,y(T,a))\,\mathrm{d}a+\int_0^T\int_0^\omega L(t,a,y(t,a),p(t,a),q(t),u(t,a),v(t),w(a))\,\mathrm{d}a\,\mathrm{d}t, \tag{7}$$

subject to the equations

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y(t,a) = f(t,a,y(t,a),p(t,a),q(t),u(t,a)), \tag{8}$$

$$p(t,a) = \int_0^\omega g(t,a,a',y(t,a'),u(t,a'))\,\mathrm{d}a', \tag{9}$$

$$q(t) = \int_0^\omega h(t,a,y(t,a),p(t,a),q(t),u(t,a))\,\mathrm{d}a, \tag{10}$$

the initial condition

$$y(0,a) = y^0(a,w(a)), \tag{11}$$

the boundary condition

$$y(t,0) = \varphi(t,q(t),v(t)), \tag{12}$$

and the control constraints

$$u(t,a) \in U, \quad v(t) \in V, \quad w(a) \in W. \tag{13}$$

Here t is the time, running in a given interval $[0,T]$, $a \in [0,\omega]$ is a scalar variable, $y(t,a) = (y_1(t,a),\ldots,y_m(t,a)) \in \mathbf{R}^m$, $p(t,a) = (p_1(t,a),\ldots,p_n(t,a)) \in \mathbf{R}^n$, and $q(t) = (q_1(t),\ldots,q_r(t)) \in \mathbf{R}^r$ are the states of the system, $u(t,a) \in U$, $v(t) \in V$ and $w(a) \in W$ are distributed, boundary and initial controls, respectively, U, V and W are subsets of finite dimensional linear normed spaces, $l, L, f, g, h, y^0, \varphi$ are given functions, ω and T are given positive numbers. The strict formulation and the suppositions are given in the next section.

The above system could be interpreted as follows. (We refer to [27, 11] for more detailed discussion of the age-structured systems.) A "population" consists of m groups. Each group is characterized by some specific qualitative and/or quantitative conditions, such as (in a human context) sex, education, social status, income, level of consumption of a specific good, etc. For example, in the vaccination models at least three groups are considered: susceptible, infected, and recovered individuals. At any instant of time t each individual member is characterized by the group to which it belongs, as well as by its "age"; the latter could be the elapsed time since it entered for the first time the "society" (that is, since its "birth"), or the elapsed time since the individual entered its current group, etc. The value $y_i(t,a)$ is the number of individuals in the i-th group, of age a at time t. The right-hand side in (8) represents the transition intensities between the groups as well as the outflow (due to mortality) and the distributed inflow. These intensities are age-specific, but also may depend on *age-specific* integral (nonlocal) factors, represented by $p(t,a)$ and $q(t)$. Examples of such are the total population $q(t)$ in the classical Gurtin-MacCamy model, or the total impact $p(t,a)$ in (2).

The boundary condition (12) represents the inflow at age zero in each group. It may also depend on nonlocal factors represented by $q(t)$.

We consider initial, boundary, and distributed controls, w, v, and u, respectively, but some of the components of u may be age-independent, or u may be subjected to budgetary constraints.

The main goal of the paper is to obtain a necessary optimality condition (maximum principle) for the problem (7)–(13), in a form readily applicable to the numerous models of this type that arise. Since the system is nonlinear and nonconvex, and since, on the other hand, our aim is to obtain a *global* maximum principle, the general approach from [13] is appropriate, in principle. However, the results from [13] cannot be directly applied, at least since the trajectory $t \longrightarrow y(t,\cdot)$ (considered as an element of $C([0,T];L_1(0,\omega))$ is not differentiable with respect to "needle variations", because of the nonlocal (integral) terms. We still apply the approach (in fact, the classical Weierstrass method in the calculus of variations) but choosing more specific variations and utilizing precise sensitivity estimates (Proposition 1).

The paper is organized as follows. In Section 2 we give a precise formulation of the problem and the suppositions. Section 3 provides some properties of system (8)–(10) that are needed for the proof of the maximum principle. Of particular importance is Proposition 1 estimating the impact of perturbations on the solution. The maximum principle for the problem (7)–(13) is formulated and discussed in Section 4 and proven in Section 5. Then in Section 6 we consider two different types of control constraints: nondistributed control, and budgetary control constrains as in (4). The maximum principle is appropriately modified in these cases. In the last section we briefly analyze the two examples presented above, utilizing the obtained results.

2 Problem Statement and Suppositions

We denote by $D \stackrel{def}{=} [0,T] \times [0,\omega]$ the domain in which we consider the problem (7)–(13). Thus

$$y : D \mapsto \mathbf{R}^m, \quad p : D \mapsto \mathbf{R}^n, \quad q : [0,T] \mapsto \mathbf{R}^r, \quad u : D \mapsto U, \quad v : [0,T] \mapsto V, \quad w : [0,\omega] \mapsto W,$$

$$l : [0,\omega] \times \mathbf{R}^m \mapsto \mathbf{R}, \quad L : D \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^r \times U \times V \times W \mapsto \mathbf{R},$$

$$f : D\times\mathbf{R}^m\times\mathbf{R}^n\times\mathbf{R}^r\times U \mapsto \mathbf{R}^m, \quad g : D\times[0,\omega]\times\mathbf{R}^m\times U \mapsto \mathbf{R}^n, \quad h : D\times\mathbf{R}^m\times\mathbf{R}^n\times\mathbf{R}^r\times U \mapsto \mathbf{R}^r,$$

$$y^0 : [0,\omega] \times W \mapsto \mathbf{R}^m, \quad \varphi : [0,T] \times \mathbf{R}^r \times V \mapsto \mathbf{R}^m.$$

Standing Suppositions. The sets U, V and W are compact, V and W are convex. The functions $l, L, f, g, h, y^0, \varphi$ are Carathéodory (that is, measurable in t, a, a' and continuous in the rest of the variables), locally essentially bounded, differentiable in (y,p,q,v,w), with locally Lipschitz partial derivatives, uniformly with respect to $u \in U$ and $(t,a) \in D$, $a' \in [0,\omega]$. The i-th component of h is independent of q_j for each $i = 1,\ldots,r$ and $j \geq i$.

Admissible control is any triple (u,v,w) with measurable functions $u : D \mapsto U$, $v : [0,T] \mapsto V$ and $w : [0,\omega] \mapsto W$.

For a fixed such triple we shall define the notion of solution to the system (8)–(10). Our definition is equivalent to that in [4] (see Lemma 1 below) and, essentially, also to those in [27, 7]. Since in the next sections we shall need this notion for systems with different side conditions, we consider now the following somewhat more general problem for the equations (8)–(10). Let $\Gamma \subset D$ be an arbitrary continuous curve joining the points $(0,\omega)$ and $(T,0)$, and such that every characteristic line $t - a = \text{const}$ intersects Γ at a single point. Instead of the initial condition (11) and the boundary condition (12) we consider the condition

$$y(\gamma) = \bar{y}(\gamma), \quad \gamma \in \Gamma, \tag{14}$$

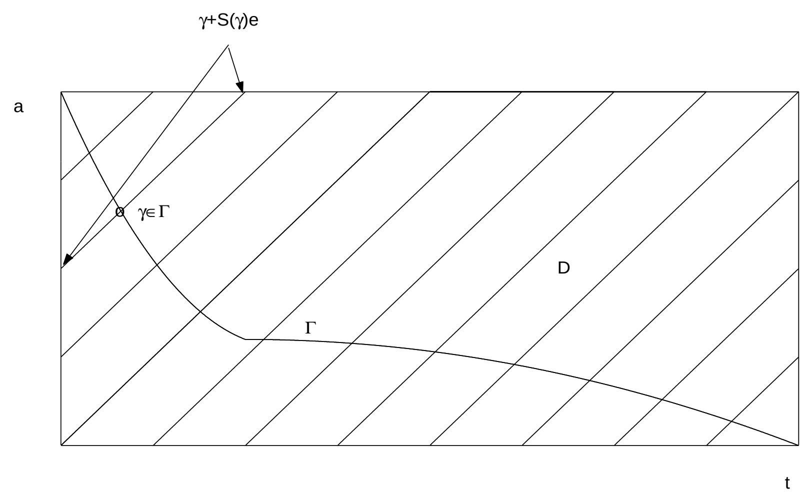

Figure 1: The "side" conditions are given on Γ

where $\bar{y}$ is a given measurable bounded function on Γ. Let us denote by e the vector $(1,1)$. For $\gamma \in \Gamma$ we denote by $S(\gamma)$ the interval of all values s such that $\gamma + se \in D$ (see Figure 1).

Definition 1. Solution of (8)–(10),(14) is any triple of measurable and bounded functions y, p, q on D ($[0,T]$, respectively) such that the equality

$$y(\gamma + se) = \bar{y}(\gamma) + \int_0^s f(\gamma + \tau e, y(\gamma + \tau e), p(\gamma + \tau e), q(t_0 + \tau), u(\gamma + \tau e))\, \mathrm{d}\tau \tag{15}$$

holds for a.e. $\gamma = (t_0, a_0) \in \Gamma$ and a.e. $s \in S(\gamma)$, (9) holds for a.e. $(t,a) \in D$, and (10) holds for a.e. $t \in [0,T]$.

Obviously every function that differs on a set of measure zero from a solution is also a solution, therefore the solution can be considered as an element of $L_\infty(D; \mathbf{R}^m) \times L_\infty(D; \mathbf{R}^n) \times L_\infty([0,T]; \mathbf{R}^r)$. Similarly, the controls u, v and w can be considered as elements of $L_\infty(D; U)$, $L_\infty([0,T]; V)$ and $L_\infty([0,\omega]; W)$, respectively.

Equation (15) implies that for a.e. $\gamma \in \Gamma$ the solution y is (equivalent to) an absolutely continuous function on the characteristic line through γ.

Remark 1 As a consequence of the last fact one easily obtains that for any solution $(y,p,q) \in L_\infty(D; \mathbf{R}^m) \times L_\infty(D; \mathbf{R}^n) \times L_\infty([0,T]; \mathbf{R}^r)$ and for any continuous curve Υ that intersects each characteristic line at most finitely many times, the restriction $y_{|\Upsilon}$ is an well-defined (trough the values of the absolutely continuous representatives along the characteristic lines) element of $L_\infty(\Upsilon; \mathbf{R}^m)$ and $\|y_{|\Upsilon}\|_{L_\infty(\Upsilon)} \le \|y\|_{L_\infty(D)}$.

In particular, $y(t,\cdot)$ is a well defined L_∞-function for *every* $t \in [0,T]$, therefore the terminal term in (7) makes sense.

Lemma 1 *If* $(y, p, q) \in L_\infty(D; \mathbf{R}^m) \times L_\infty(D; \mathbf{R}^n) \times L_\infty([0, T]; \mathbf{R}^r)$ *is a solution of (8)–(10),(14), then the mapping*

$$[0, T] \ni t \longrightarrow y(t, \cdot) \in L_1([0, \omega])$$

is Lipschitz continuous.

Proof. We have

$$y(t + \varepsilon, a) - y(t, a) = [y(t + \varepsilon, a + \varepsilon) - y(t, a)] + [y(t + \varepsilon, a) - y(t + \varepsilon, a + \varepsilon)].$$

The first term in the right-hand side is proportional to ε thanks to the absolute continuity of y along the characteristic lines, and the boundedness of all functions involved. The second term, integrated on $[0, \omega - \varepsilon]$ is also proportinal to ε since y is bounded. This implies in an obvious way the claim. Q.E.D.

3 Some Basic Properties of the Age-Structured System

In this section we present some properties of system (8)–(12), that will be used in the proof of the maximum principle.

Theorem 1 *For every compact set* $Y \subset \mathbf{R}^m$ *there exists a compact set* $Z \supset Y$ *and a number* $\bar{t} > 0$ *such that for every admissible* u *and* v *and for every initial condition* $y(0, \cdot) \in L_\infty([0, \omega]; Y)$ *the problem (8)–(10),(12) has a unique solution in the domain* $[0, \bar{t}] \times [0, \omega]$ *and* $y(t, a) \in Z$ *almost everwhere in this domain.*

A general existence theorem is proved in [27], but formally it does not apply to (8)–(12), since it concerns time invariant systems, and claims existence in L_1. Existence in L_∞ is proven in [7], but for a more restricted class of equations. However, the proof of the above theorem is very similar to that in [7] (uses a standard contraction mapping argument), therefore we omit it.

The next lemma claims that for fixed control functions u, v and w, the equations (8)–(10) represent a (nonstationary) dynamical system. The proof is obvious.

Lemma 2 *Let* $0 < t_1 < t_2 \le T$. *Let* $z_1 = (y_1, p_1, q_1)$ *be a solution of (8)–(12) in the domain* $[0, t_1] \times [0, \omega]$. *Let* $z_2 = (y_2, p_2, q_2)$ *be a solution of (8)–(10) in the domain* $[t_1, t_2] \times [0, \omega]$ *with initial condition* $y_2(t_1, a) = y_1(t_1, a)$, $a \in [0, \omega]$ *and boundary condition (12) in* $[t_1, t_2]$. *Then the time-concatenation of* z_1 *and* z_2 *is a solution of (8)–(12) in* $[0, t_2] \times [0, \omega]$.

Corollary 1 *There is a maximal number* $T_e \in (0, T]$ *such that the solution of (8)–(12) exists in the domain* $[0, T_e) \times [0, \omega]$. *If* $T_e < T$, *then*

$$\lim_{t \to T_e} \|y(t, \cdot)\|_{L_\infty([0,\omega])} = +\infty.$$

The following proposition plays a key role in the proof of the optimality condition in Section 5. Since we consider the problem with the initial and boundary conditions (11)–(12), the set Γ consists of the left and the lower bounds of D. We shall use the notation $D' \stackrel{def}{=} D \times [0, \omega]$.

Proposition 1 *Suppose that the solution $\hat{y}, \hat{p}, \hat{q}$, of (8)–(12), corresponding to admissible control functions $\hat{u}$, $\hat{v}$ and $\hat{w}$ exists in the domain D. Let v and w be given admissible control functions such that the solution $\tilde{y}, \tilde{p}, \tilde{q}$ of (8)–(12), corresponding to the triple $(\hat{u}, v, w)$ also exists in the domain D. For an arbitrary $u \in L_\infty(D;U)$ we define*

$$\begin{aligned}
\Delta_f(t,a) &\stackrel{def}{=} |f(t,a,\hat{y}(t,a),\hat{p}(t,a),\hat{q}(t),u(t,a)) - f(t,a,\hat{y}(t,a),\hat{p}(t,a),\hat{q}(t),\hat{u}(t,a))|,\\
\Delta_g(t,a,a') &\stackrel{def}{=} |g(t,a,a',\hat{y}(t,a'),u(t,a')) - g(t,a,a',\hat{y}(t,a'),\hat{u}(t,a'))|,\\
\Delta_h(t,a) &\stackrel{def}{=} |h(t,a,\hat{y}(t,a),\hat{p}(t,a),\hat{q}(t),u(t,a)) - h(t,a,\hat{y}(t,a),\hat{p}(t,a),\hat{q}(t),\hat{u}(t,a))|,\\
\Delta_{y^0}(a) &\stackrel{def}{=} |y^0(a,w(a)) - y^0(a,\hat{w}(a))|,\\
\Delta_\varphi(t) &\stackrel{def}{=} |\varphi(t,\hat{q}(t),v(t)) - \varphi(t,\hat{q}(t),\hat{v}(t))|.
\end{aligned}$$

We introduce also the following quantities:

$$\Delta^1(u) \stackrel{def}{=} \|\Delta_f\|_{L_1(D)} + \|\Delta_g\|_{L_1(D')} + \|\Delta_h\|_{L_1(D)},$$

$$\Delta^1(v,w) \stackrel{def}{=} \|\Delta_{y^0}\|_{L_1([0,\omega])} + \|\Delta_\varphi\|_{L_1([0,T])},$$

$$\Delta_\infty(u) \stackrel{def}{=} \max\left\{\operatorname*{esup}_{\gamma\in\Gamma}\int_{S(\gamma)}\Delta_f(\gamma+se)\,\mathrm{d}s,\ \operatorname*{esup}_{t\in[0,T]}\int_0^\omega\Delta_h(t,a)\,\mathrm{d}a,\ \operatorname*{esup}_{(t,a)\in D}\int_0^\omega\Delta_g(t,a,a')\,\mathrm{d}a'\right\},$$

$$\Delta^\infty(v,w) \stackrel{def}{=} \|\Delta_{y^0}\|_{L_\infty([0,\omega])} + \|\Delta_\varphi\|_{L_\infty([0,T])},$$

where esup *means "essential supremum". Then there exist constants $\varepsilon > 0$ and C such that for every $u \in L_\infty(D;U)$ with $\Delta_\infty(u) \le \varepsilon$ the corresponding solution (y,p,q) of (8)–(12) exists in D and satisfies*

$$\|p-\hat{p}\|_{L_\infty(D)} + \|q-\hat{q}\|_{L_\infty([0,T])} \le C(\Delta_\infty(u) + \Delta^1(v,w)), \tag{16}$$

$$\|y-\hat{y}\|_{L_\infty(D)} \le C(\Delta_\infty(u) + \Delta^\infty(v,w)), \tag{17}$$

$$\begin{aligned}
&\|y(t,\cdot)-\hat{y}(t,\cdot)\|_{L_1([0,\omega])} + \|p-\hat{p}\|_{L_1(D)} + \|q-\hat{q}\|_{L_1([0,T])}\\
&\qquad \le C(\Delta^1(u) + \Delta^1(v,w)) \quad \forall t \in [0,T].
\end{aligned} \tag{18}$$

Proof. Let $Y \subset \mathbf{R}^m$ be a convex compact set such that

$$\hat{y}(t,a) + \mathbf{B} \subset Y, \quad \tilde{y}(t,a) + \mathbf{B} \subset Y \quad \text{for a.e. } (t,a) \in D,$$

where $\mathbf{B}$ is the unit ball in $\mathbf{R}^m$. Let Z and $\bar{t}$ be the compact set and the positive number from Theorem 1. Since U and V are bounded, equations (9),(10) imply that there exist compact sets $P \in \mathbf{R}^n$ and $Q \in \mathbf{R}^r$ with the following property: if a solution (y,p,q), corresponding to admissible controls, exists in some domain $[0,\theta]\times[0,\omega] \subset D$ and $y(t,a) \in Z$ in this domain, then $p(t,a) \in P$ and $q(t) \in Q$ almost everywhere (we remind the structural assumption for h).

Let M be a Lipschitz constant of f, g, h and φ with respect to $(y,p,q) \in Z\times P\times Q$, uniformly in $(t,a) \in D$, $u \in U$, $v \in V$ (respectively $(t,a,a') \in D'$ for the function g).

The number $\varepsilon > 0$ below will be considered as fixed, but its value will be specified later in the proof. Let us fix a control $u \in L_\infty(D;U)$, such that $\Delta_\infty(u) \le \varepsilon$. According to Corollary 1, there is a maximal interval $[0,T_e) \subset [0,T]$ of existence of the solution (y,p,q), corresponding to controls u,v,w, and if $T_e < T$, then $\|y(t,\cdot)\|_\infty$ escapes to infinity at T_e. We denote

$$t^* \stackrel{def}{=} \sup\{t' \le T_e : \ y(t,a) \in Z \quad \text{a.e. on } [0,t'] \times [0,\omega]\}.$$

According to Theorem 1 we have $t^* \ge \bar{t} > 0$.

Below $d_1, d_2..., C_1, C_2...$ denote constants depending only on M, T and ω. The following calculations are routine, but somewhat lengthy, therefore we skip some details.

Denoting

$$\Delta_y(t,x) \stackrel{def}{=} |y(t,x) - \hat{y}(t,x)|, \ \ \Delta_p(t,x) \stackrel{def}{=} |p(t,x) - \hat{p}(t,x)|, \ \ \Delta_q(t) \stackrel{def}{=} |q(t) - \hat{q}(t)|$$

we obtain successively

$$\Delta_p(t,a) \le M \int_0^\omega \Delta_y(t,a')\,\mathrm{d}a' + \int_0^\omega \Delta_g(t,a,a')\,\mathrm{d}a', \tag{19}$$

$$\Delta_q(t) \le d_1 \int_0^\omega \Delta_y(t,a')\,\mathrm{d}a' + d_2 \int_0^\omega \int_0^\omega \Delta_g(t,a',a'')\,\mathrm{d}a''\,\mathrm{d}a' + d_3 \int_0^\omega \Delta_h(t,a')\,\mathrm{d}a', \tag{20}$$

where the second inequality is obtained iteratively using the first one and the structural supposition about h. Using the definition of a solution to (8) first with $\gamma = (0,a_0)$ and then with $\gamma = (t_0,0)$ we obtain

$$\Delta_y(s,a_0+s) \le \Delta_{y^0}(a_0) + \int_0^s [M\Delta_y(\tau,a_0+\tau) + M\Delta_p(\tau,a_0+\tau) + M\Delta_q(\tau) + \Delta_f(\tau,a_0+\tau)]\,\mathrm{d}\tau \tag{21}$$

and

$$\Delta_y(t_0+s,s) \tag{22}$$

$$\le \Delta_\varphi(t_0) + M\Delta_q(t_0) + \int_0^s [M\Delta_y(t_0+\tau,\tau) + M\Delta_p(t_0+\tau,\tau) + M\Delta_q(t_0+\tau) + \Delta_f(t_0+\tau,\tau)]\,\mathrm{d}\tau.$$

Then utilizing (19) and (20), changing appropriately the variables and reordering the terms we come up with the following inequality, valid for a.e. $(t,a) \in [0,t^*] \times [0,\omega]$:

$$\Delta_y(t,a) \le \Delta_0(t,a) + \int_{\theta(t-a)}^t \Big[M\Delta_y(s,a-t+s) + d_2 \int_0^\omega \Delta_y(s,a')\,\mathrm{d}a' + \Delta_f(s,a-t+s) \tag{23}$$

$$+ M \int_0^\omega \Delta_g(s,a-t+s,a')\,\mathrm{d}a' + M \int_0^\omega \Delta_h(s,a')\,\mathrm{d}a' + M^2 \int_0^\omega \int_0^\omega \Delta_g(s,a',a'')\,\mathrm{d}a'\,\mathrm{d}a'' \Big]\,\mathrm{d}s$$

where

$$\theta(\alpha) \stackrel{def}{=} \max\{0,\alpha\},$$

$$\Delta_0(t,a) \stackrel{def}{=} \begin{cases} \Delta_{y^0}(a-t) & \text{if } a > t, \\ \Delta_\varphi(t-a) + \int_0^\omega [d_3\Delta_y(t-a,a') + M\Delta_h(t-a,a') & \\ + \int_0^\omega M^2 \Delta_g(t-a,a',a'')\,\mathrm{d}a''] \, da' & \text{if } a < t. \end{cases}$$

Integrating (23) with respect to a in $[0, \omega]$ and changing the order of integration where appropriate, we obtain an integral inequality for the function

$$\delta(t) \stackrel{def}{=} \int_0^\omega \Delta_y(t, a)\, \mathrm{d}a,$$

of the form

$$\delta(t) \le \|\Delta_{y^0}\|_{L_1([0,\omega])} + \|\Delta_\varphi\|_{L_1([0,t^*])}$$
$$+ d_4 \int_0^t [\delta(s) + \|\Delta_f(s,\cdot)\|_{L_1([0,\omega])}\, \mathrm{d}s + \|\Delta_g(s,\cdot,\cdot)\|_{L_1([0,\omega]\times[0,\omega])} + \|\Delta_h(s,\cdot)\|_{L_1([0,\omega])}]\, \mathrm{d}s.$$

According to Lemma 1, δ is a continuous function and one can apply the Grunwall inequality to obtain a constant C_1 such that

$$\int_0^\omega \Delta_y(t, a)\, \mathrm{d}a \le C_1(\Delta^1(u) + \Delta^1(v, w)),$$

where the quantity $\Delta^1(u, v, w)$ is taken on $[0, t^*]$ instead of $[0, T]$. For each $t \in [0, t^*]$ we obtain the estimation for the first summand in (18). Then integrating in t the above inequality and (20) and integrating in (t, a) the inequality (19), we obtain (18) (in the domain $[0, t^*] \times [0, \omega]$, so far).

The next step will be to prove that (17) is fulfilled in the domain $[0, t^*] \times [0, \omega]$. Obviously

$$\Delta^1(u) + \Delta^1(u, v) \le C_2(\Delta_\infty(u) + \Delta^\infty(v, w)).$$

Then from (19) and (20) we obtain (16) on $[0, t^*] \times [0, \omega]$. The substitution of this estimate in (21) gives

$$\Delta_y(s, a_0 + s) \le \Delta_{y^0}(a_0) + \int_0^s [M\Delta_y(\tau, a_0 + \tau) + MC_3(\Delta_\infty(u) + \Delta^1(v, w)) + \Delta_f(\tau, a_0 + \tau)]\, \mathrm{d}\tau.$$

Since the function $s \longrightarrow \Delta_y(s, a_0 + s)$ is absolutely continuous we can apply the Grunwall inequality to obtain

$$\Delta_y(t, a_0 + t) \le C_4[\Delta_{y^0}(a_0) + \Delta_\infty(u) + \Delta^1(v, w)]$$

for a.e. $a_0 \in [0, \omega]$ and for a.e. $t \in [0, \omega - a_0]$. Similarly we obtain from (22) that

$$\Delta_y(t_0 + a, a) \le C_5[|\Delta_\varphi(t_0)| + \Delta_\infty(u) + \Delta^1(v, w)]$$

for a.e. $t_0 \in [0, t^*]$ and for a.e. $a \in [0, \min\{\omega, t^* - t_0\}]$. Combining these two inequalities we obtain

$$\Delta_y(t, a) \le C_6[\Delta_\infty(u) + \Delta^\infty(v, w)],$$

which means that (17) is fulfilled almost everywhere in the domain $[0, t^*] \times [0, \omega]$ (instead of D).

Since the definition of the sets Y, P and Q (therefore of the constant M) is symmetric with respect to $(\tilde{y}, \tilde{p}, \tilde{q})$ and $(\hat{y}, \hat{p}, \hat{q})$ we may apply the above estimation to the difference between $(\tilde{y}, \tilde{p}, \tilde{q})$ and (y, p, q). Since the initial and the boundary controls coincide for $\tilde{y}$ and y we obtain

$$\|y - \tilde{y}\|_{L_\infty([0,t^*]\times[0,\omega])} \le C_6 \Delta_\infty(u). \tag{24}$$

Let ε be chosen in advance such that $C_6\varepsilon < 1$. Then the function $y(t^*, \cdot)$, which is an well defined element of $L_\infty([0, \omega])$, according to Remark 1, satisfies $y(t^*, a) \in Y$ on $[0, \omega]$. (Otherwise we easily come to a contradiction with (24) thanks to the absolute continuity of y on the characteristic lines.) Then Theorem 1 and the choise of t^* imply $t^* = T$. Q.E.D.

4 The Necessary Optimality Condition

The issue of existence of an optimal solution of the problem (7)–(13) is complicated, even for much less general problems (see the recent paper [4] for an existence result for the Gurtin-MacCamy system). Therefore, we make the following supposition.

Condition E: There exists an optimal solution $(\hat{y}, \hat{p}, \hat{q}, \hat{u}, \hat{v}, \hat{w}) \in L_\infty(D; \mathbf{R}^m) \times L_\infty(D; \mathbf{R}^n) \times L_\infty([0,T]; \mathbf{R}^r) \times L_\infty(D; U) \times L_\infty([0,T]; V) \times L_\infty([0,\omega]; W)$.

Let us introduce the following

Notational Convention: we abridge the notations by skipping those arguments of functions that are fixed at a value with a "hat". For example

$$f(t,a) \overset{def}{=} f(t,a,\hat{y}(t,a),\hat{p}(t,a),\hat{q}(t),\hat{u}(t,a)),$$

while

$$f(t,a,u) \overset{def}{=} f(t,a,\hat{y}(t,a),\hat{p}(t,a),\hat{q}(t),u),$$

etc. Especially, by definition

$$g(t,a,a',u) \overset{def}{=} g(t,a,a',\hat{y}(t,a'),u),$$

and similarly for $g(t,a,a')$.

Below ∇_z denotes differentiation with respect to the variable z.

We introduce the following *adjoint system* for the *adjoint functions* (ξ, η, ζ), considered as row-vector functions (while y, p and q are column-vectors) from $L_\infty(D; \mathbf{R}^m) \times L_\infty(D; \mathbf{R}^n) \times L_\infty([0,T]; \mathbf{R}^r)$:

$$\begin{aligned} -\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \xi(t,a) \;&=\; \nabla_y L(t,a) + \xi(t,a)\nabla_y f(t,a) + \zeta(t)\nabla_y h(t,a) \qquad (25)\\ &\quad + \int_0^\omega \eta(t,a')\nabla_y g(t,a',a)\,\mathrm{d}a', \quad \xi(T,a) = \nabla_y l(a,\hat{y}(T,a)), \quad \xi(t,\omega) = 0,\\ \eta(t,a) \;&=\; \nabla_p L(t,a) + \xi(t,a)\nabla_p f(t,a) + \zeta(t)\nabla_p h(t,a), \qquad (26)\\ \zeta(t) \;&=\; \xi(t,0)\nabla_q \varphi(t) + \int_0^\omega [\nabla_q L(t,a) + \xi(t,a)\nabla_q f(t,a) + \zeta(t)\nabla_q h(t,a)]\,\mathrm{d}a. \quad (27) \end{aligned}$$

Remark 2 The notion of solution to the adjoint system in D is a particular case of Definition 1, where Γ consists of the the right and upper bounds of Γ.

For the solution $(\hat{y}, \hat{p}, \hat{q})$, and the corresponding solution (ξ, η, ζ) of the adjoint system (we shall prove that it exists) we define the initial, boundary, and distributed Hamiltonians:

$$\begin{aligned} H_0(a,w) &\overset{def}{=} \xi(0,a)y^0(a,w) + \int_0^T L(s,a,w)\,\mathrm{d}s,\\ H_b(t,v) &\overset{def}{=} \xi(t,0)\varphi(t,v) + \int_0^\omega L(t,b,v)\,\mathrm{d}b,\\ H(t,a,u) &\overset{def}{=} L(t,a,u) + \xi(t,a)f(t,a,u) + \int_0^\omega \eta(t,a')g(t,a',a,u)\,\mathrm{d}a' + \zeta(t)h(t,a,u). \end{aligned}$$

Theorem 2 *(*Pontryagin's Maximum Principle*). Under Supposition* **E** *the adjoint system (25)–(27) has a unique solution ξ, η, ζ and for a.e. $t_0 \in [0,T]$, $a_0 \in [0,\omega]$ and $(t,a) \in D$*

$$\begin{aligned}
&\frac{\partial H_0}{\partial w}(a_0, \hat{w}(a_0))(w - \hat{w}(a_0)) \geq 0 \quad \forall w \in W, \\
&\frac{\partial H_b}{\partial v}(t_0, \hat{v}(t_0))(v - \hat{v}(t_0)) \geq 0 \quad \forall v \in V, \\
&H(t,a,u) - H(t,a,\hat{u}(t,a)) \geq 0 \quad \forall u \in U. \qquad (28)
\end{aligned}$$

Notice that the maximum principle is local with respect to the side controls and global with respect to the distributed control. The reason is the discontinuity of the operator "*side control* $\longrightarrow$ *solution*" considered in the spaces $L_1 \longrightarrow L_\infty$ (c.f. (17))

5 Proof of the Maximum Principle

Proposition 2 *The adjoint system (25)–(27) has a unique solution in D.*

Proof. For $\lambda \stackrel{def}{=} (\xi, \eta, \zeta)$ we denote by $F_2(\lambda)$ and $F_3(\lambda)$ the right-hand sides of (26) and (27), respectively. For $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ and $s \in S(\gamma) = [-s(\gamma), 0]$ we define

$$F_1(\lambda)(\gamma + se) \stackrel{def}{=} \bar{\xi}(\gamma) + \int_s^0 [\nabla_y L(\gamma + \tau e) + \xi(\gamma + \tau e)\nabla_y f(\gamma + \tau e) + \zeta(\gamma_1 + \tau)\nabla_y h(\gamma + \tau e)$$
$$+ \int_0^\omega \eta(\gamma_1 + \tau, a')\nabla_y g(\gamma_1 + \tau, a', \gamma_2 + \tau)\, \mathrm{d}a']\, \mathrm{d}\tau,$$

where $\bar{\xi}(\gamma)$ is the side condition in (25) (here Γ consists of the upper and the right sides of D). We shall prove existence of a fixed point of the operator $\lambda \longrightarrow (F_1(\lambda), F_2(\lambda), F_3(\lambda))$ by the Banach contraction mapping theorem. For this purpose we introduce the following norms in the spaces $L_\infty(D; \mathbf{R}^m)$, $L_\infty(D; \mathbf{R}^n)$, $L_\infty([0,T]; \mathbf{R}^r)$, and in their product, respectively:

$$\begin{aligned}
\|\xi\| &\stackrel{def}{=} \operatorname*{esup}_{(t,x)\in D} e^{-Mt}|\xi(t,x)|, \\
\|\eta\| &\stackrel{def}{=} \operatorname*{esup}_{(t,x)\in D} d_p e^{-Mt}|\eta(t,x)|, \\
\|\zeta\| &\stackrel{def}{=} \operatorname*{esup}_{t\in[0,T]} d_q e^{-Mt}|\zeta(t)|, \\
\|\lambda\| &\stackrel{def}{=} \|\xi\| + \|\eta\| + \|\zeta\|,
\end{aligned}$$

where M, d_p and d_q are constants that will be appropriately fixed later.

Since F is an affine operator, it is enough to prove that $\|F(\cdot) - F(0)\| < 1$. We have

$$\|(F_1(\lambda) - F_1(0)\| \leq \operatorname*{esup}_{\gamma\in\Gamma,\, s\in S(\gamma)} e^{-M(\gamma_1+s)} \int_s^0 [\|\nabla_y f\|_\infty |\xi(\gamma + \tau e)|$$

$$+\|\nabla_y h\|_\infty |\zeta(\gamma_1+\tau)| + \int_0^\omega \|\nabla_y g\|_\infty |\eta(\gamma_1+\tau,a')| \,\mathrm{d}a']\mathrm{d}\tau$$
$$\le \operatorname*{esup}_{\gamma\in\Gamma,\, s\in S(\gamma)} Ce^{-M(\gamma_1+s)} \int_s^0 e^{M(\gamma_1+\tau)} \left[\|\xi\| + d_q^{-1}\|\zeta\| + \omega d_p^{-1}\|\eta\|\right] \mathrm{d}\tau$$
$$\le C\left[\frac{1}{M}\|\xi\| + \frac{1}{d_q M}\|\zeta\| + \frac{\omega}{d_p M}\|\eta\|\right]$$

where C majorizes all the coefficients $\|\nabla_y f\|_\infty$, ..., $\|\nabla_q h\|_\infty$ in (25)–(27).

Similarly we obtain

$$\|F_2(\lambda) - F_2(0)\| \le C\left[d_p\|\xi\| + \frac{d_p}{d_q}\|\zeta\|\right].$$

Finally, using Remark 1 and the structural condition for the function h we obtain

$$\|F_3(\lambda) - F_3(0)\| \le C_1 d_q \|\xi\|,$$

where $C_1 = (1+\omega)C(1+\omega C + \ldots (\omega C)^{q-1})$. It is easy to verify that one can chose the constants d_q, d_p and M (in this order) in such a way that all the coefficients multiplying the norms of ξ, η and ζ in the above estimations for F_1, F_2 and F_3 are less than $1/3$. This means $\|F(\cdot) - F(0)\| < 1$ and implies the claim of the proposition. Q.E.D.

Lemma 3 *Let y and ξ be solutions in D to the equations*

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) y(t,a) = F(t,a) \quad \text{and} \quad -\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \xi(t,a) = \Xi(t,a),$$

respectively, where $F, \Xi \in L_\infty(D)$. Then the function

$$\nu(t) \stackrel{def}{=} \int_0^\omega \xi(t,a) y(t,a) \,\mathrm{d}a$$

is Lipschitz continuous and for a.e. $t \in [0,T]$

$$\dot\nu(t) = \xi(t,0)y(t,0) - \xi(t,\omega)y(t,\omega) + \int_0^\omega [-\Xi(t,a)y(t,a) + \xi(t,a)F(t,a)] \,\mathrm{d}a.$$

Proof. The Lipschitz continuity follows from Lemma 1 and the boundedness of y and ξ. Changing the variable $a = t - x$ we have

$$\nu(t) \stackrel{def}{=} \int_{t-\omega}^t \xi(t,t-x) y(t,t-x) \,\mathrm{d}x.$$

Here one can differentiate in t, since for a.e. t and for a.e. $x \in [t-\omega, t]$ both y and ξ are differentiable in the direction e. Differentiating and changing back the variable x we obtain the claim. Q.E.D.

Now we proceed with the proof of Theorem 2. Below we shall use the notations introduced in Proposition 1, with the difference that Δ_y, Δ_p, ..., Δ_φ are defined without taking the norms. According to Supposition **E** and Proposition 1 there exist compact sets $Y \times P \times Q$ and $e > 0$ such that for every admissible u, v, w with $\Delta_\infty(u) + \Delta^\infty(v, w) \le \varepsilon$ the corresponding solution (y, p, q) exists in D and its values belong to $Y \times P \times Q$. The standing supposition ensures that the partial derivatives of l, L, f, g, h, φ are Lipschitz continuous in $Y \times P \times Q$, uniformly with respect to $(t, a) \in D$ (resp. $(t, a, a') \in D'$ and $(u, v, w) \in U \times V \times W$).

From (9) and (10) we obtain

$$\Delta_p(t,a) = \int_0^\omega [\nabla_y g(t,a,a')\Delta_y(t,a') + \Delta_g(t,a,a') + e_p(t,a,a')] \, \mathrm{d}a', \tag{29}$$

$$\Delta_q(t) = \int_0^\omega [\nabla_y h(t,a)\Delta_y(t,a) + \nabla_p h(t,a)\Delta_p(t,a) + \nabla_q h(t,a)\Delta_q(t) + \Delta_h(t,a) + e_q(t,a)] \, \mathrm{d}a, \tag{30}$$

where the functions e_p, e_q can be estimated (almost everywhere) using Proposition 1 as follows:

$$\begin{aligned} |e_p(t,a,a')| &\le C(\Delta_\infty(u) + \Delta^\infty(v,w))(|\Delta_y(t,a')| + \Theta_g(u;t,a,a')), \\ |e_q(t,a)| &\le C(\Delta_\infty(u) + \Delta^\infty(v,w))(|\Delta_y(t,a)| + |\Delta_p(t,a)| + |\Delta_q(t)| + \Theta_h(u;t,a)), \end{aligned}$$

where

$$\begin{aligned} \Theta_g(u;t,a,a') &= |\nabla_y g(t,a,a',u(t,a')) - \nabla_y g(t,a,a')|, \\ \Theta_h(u;t,a) &= \max\{|\nabla_y h(t,a,u(t,a)) - \nabla_y h(t,a)|, \, |\nabla_p h(t,a,u(t,a)) - \nabla_p h(t,a)|, \\ &\qquad |\nabla_q h(t,a,u(t,a)) - \nabla_q h(t,a)|\}. \end{aligned}$$

Here and below C stays for a constant, independent of u, v, w. Similarly,

$$\begin{aligned} \delta(t,a) &\stackrel{def}{=} L(t,a,y(t,a),p(t,a),q(t),u(t,a),v(t),w(a)) - L(t,a) \\ &= \Delta_L(t,a) + \nabla_y L(t,a)\Delta_y(t,a) + \nabla_p L(t,a)\Delta_p(t,a) + \nabla_q L(t,a)\Delta_q(t) + e_L(t,a), \end{aligned} \tag{31}$$

where $\Delta_L(t,a) \stackrel{def}{=} L(t,a,u(t,a),v(t),w(a)) - L(t,a)$. Also

$$\Delta_l(a) \stackrel{def}{=} l(a, y(T,a)) - l(a, \hat{y}(T,a)) = \nabla_y l(a, \hat{y}(T,a))\Delta_y(T,a) + e_l(a),$$

where

$$\begin{aligned} |e_L(t,a)| &\le C(\Delta_\infty(u) + \Delta^\infty(v,w) + \|w - \hat{w}\|_{L_\infty([0,\omega])} + \|v - \hat{v}\|_{L_\infty([0,T])}) \times \\ &\qquad (|\Delta_y(t,a)| + |\Delta_p(t,a)| + |\Delta_q(t)| + \Theta_L(u;t,a)) \end{aligned} \tag{32}$$

$$|e_l(a)| \le C(\Delta_\infty(u) + \Delta^\infty(v,w))|\Delta_y(T,a)|, \tag{33}$$

where $\Theta_L(u;t,a)$ is defined similarly as $\Theta_h(u;t,a)$ but with the function L instead of h. (For the final part of the proof it is crucial to keep track of all the remainders e and the way they are integrated below.)

Utilzing Lemma 3 (and the notations there) we obtain

$$\int_0^\omega [\xi(T,a)y(T,a) - \xi(0,a)y(0,a)] \, \mathrm{d}a = \nu(T) - \nu(0) = \int_0^T \dot{\nu}(t) \, \mathrm{d}t$$

$$= \int_0^T [\xi(t,0)y(t,0) - \xi(t,\omega)y(t,\omega) + \int_0^\omega (-\Xi(t,a)y(t,a) + \xi(t,a)F(t,a))\,\mathrm{d}a]\,\mathrm{d}t.$$

Subtracting the same equality applied to $\hat{y}$ and using that $\xi(t,\omega) = 0$ we obtain

$$I \stackrel{def}{=} \int_0^\omega [\xi(T,a)\Delta_y(T,a) - \xi(0,a)\Delta_y(0,a)]\,\mathrm{d}a \tag{34}$$

$$= \int_0^T [\xi(t,0)\Delta_y(t,0) + \int_0^\omega (-\Xi(t,a)\Delta_y(t,a) + \xi(t,a)\Delta_F(t,a))\,\mathrm{d}a]\,\mathrm{d}t, \tag{35}$$

where

$$\begin{aligned}\Delta_F(t,a) &\stackrel{def}{=} f(t,a,y(t,a),p(t,a),q(t),u(t,a)) - f(t,a)\\ &= \nabla_y f(t,a)\Delta_y(t,a) + \nabla_p f(t,a)\Delta_p(t,a) + \nabla_q f(t,a)\Delta_q(t) + \Delta_f(t,a) + e_f(t,a)\end{aligned}$$

and, similarly as above,

$$|e_f(t,a)| \le C(\Delta_\infty(u) + \Delta^\infty(v,w))(|\Delta_y(t,a)| + |\Delta_p(t,a)| + |\Delta_q(t)| + \Theta_f(u;t,a)).$$

where $\Theta_f(u;t,a)$ is defined similarly as $\Theta_h(u;t,a)$ but with the function f instead of h. We represent also

$$\Delta_y(t,0) = \varphi(t,q(t),v(t)) - \varphi(t,\hat{q}(t),\hat{v}(t)) = \nabla_q\varphi(t)\Delta_q(t) + \Delta_\varphi(t) + e_y^0(t)$$

with

$$|e_y^0(t)| \le C(\Delta_\infty(u) + \Delta^\infty(v,w) + |v(t) - \hat{v}(t)|)|\Delta_q(t)|.$$

From the adjoint equation for ζ the first term in the right-hand side of (35) can be expressed as

$$\begin{aligned}\xi(t,0)\Delta_y(t,0) &= \xi(t,0)\nabla_q\varphi(t)\Delta_q(t) + \xi(t,0)\Delta_\varphi(t) + \xi(t,0)e_y^0(t)\\ &= \zeta(t)\Delta_q(t) - \int_0^\omega [\nabla_q L(t,a)\Delta_q(t) + \xi(t,a)\nabla_q f(t,a)\Delta_q(t) + \zeta(t)\nabla_q h(t,a)\Delta_q(t)]\,\mathrm{d}a\\ &\quad + \xi(t,0)\Delta_\varphi(t) + \xi(t,0)e_y^0(t).\end{aligned}$$

Also we add to the right-hand side of (35) the quantity $\eta(t,a)\Delta_p(t,a) - (\ldots)\Delta_p(t,a) = 0$, where $(\ldots)$ is the expression for $\eta(t,a)$ in the adjoint equation. Thus the right-hand side of (35) takes the form

$$\begin{aligned}I = \int_0^T \{&\zeta(t)\Delta_q(t) - \int_0^\omega [\nabla_q L(t,a)\Delta_q(t) + \zeta(t)\nabla_q h(t,a)\Delta_q(t)]\,\mathrm{d}a + \xi(t,0)\Delta_\varphi(t) + \xi(t,0)e_y^0(t)\\ &+ \int_0^\omega [-\nabla_y L\Delta_y(t,a) - \zeta(t)\nabla_y h\Delta_y(t,a) - \int_0^\omega \eta(t,a')\nabla_y g(t,a',a)\Delta_y(t,a)\,\mathrm{d}a'\\ &+ \xi\Delta_f(t,a) + \xi e_f(t,a) + \eta\Delta_p(t,a) - \nabla_p L\Delta_p(t,a) - \zeta(t)\nabla_p h\Delta_p(t,a)]\,\mathrm{d}a\}\,\mathrm{d}t\end{aligned} \tag{36}$$

Now we multiply (29) by $\eta(t,a)$ and (30) by $\zeta(t)$, and integrate on D and on $[0,T]$, respectively. After changing the order of integration in the triple integral we add the so two obtained equalities to (36) and obtain

$$
\begin{aligned}
I = \int_0^T \{&\xi(t,0)\Delta_\varphi(t) + \xi(t,0)e_y^0(t) \\
&+ \int_0^\omega [-\nabla_y L\Delta_y(t,a) - \nabla_p L\Delta_p(t,a) - \nabla_q L(t,a)\Delta_q(t)]\,\mathrm{d}a \\
&\quad +\xi\Delta_f(t,a) + \int_0^\omega \eta(t,a')\Delta_g(t,a',a)\,\mathrm{d}a' + \zeta(t)\Delta_h(t,a) \\
&\quad +\xi e_f(t,a) + \int_0^\omega \eta(t,a')e_p(t,a',a)\,\mathrm{d}a' + \zeta(t)e_q(t,a)]\,\mathrm{d}a\}\,\mathrm{d}t.
\end{aligned} \tag{37}
$$

From the optimality of $(\hat{u},\hat{v},\hat{w},\hat{y},\hat{p},\hat{q})$ and the definition of $\delta(t,a)$ in (31) we have

$$
0 \le \int_0^\omega \Delta_l(a)\,\mathrm{d}a + \int_0^T \int_0^\omega \delta(t,a)\,\mathrm{d}a\,\mathrm{d}t.
$$

Adding to this inequality the equality (37), subtracting (34), and taking into account the end-time condition for ξ we obtain

$$
\begin{aligned}
0 \le& \int_0^\omega \xi(0,a)\Delta_y(0,a)\,\mathrm{d}a + \int_0^T \xi(t,0)\Delta_\varphi(t)\,\mathrm{d}t \\
&+ \int_0^T \int_0^\omega [\Delta_L(t,a) + \xi(t,a)\Delta_f(t,a) + \int_0^\omega \eta(t,a')\Delta_g(t,a',a)\,\mathrm{d}a + \zeta(t)\Delta_h(t,a)]\,\mathrm{d}a\,\mathrm{d}t + e,
\end{aligned} \tag{38}
$$

where

$$
\begin{aligned}
e \overset{def}{=} &\int_0^\omega e_l(a)\,\mathrm{d}a + \int_0^T \xi(t,0)e_y^0(t)\,\mathrm{d}t \\
&+ \int_0^T \int_0^\omega [e_L(t,a) + \xi(t,a)e_f(t,a) + \int_0^\omega \eta(t,a')e_p(t,a',a)\,\mathrm{d}a' + \zeta(t)e_q(t,a)]\,\mathrm{d}a\,\mathrm{d}t].
\end{aligned}
$$

According to the estimations for each remainder and Proposition 1 we have

$$
\begin{aligned}
|e| \le \varepsilon \overset{def}{=} &C(\Delta_\infty(u) + \Delta^\infty(v,w) + \|w-\hat{w}\|_{L_\infty([0,\omega])} + \|v-\hat{v}\|_{L_\infty([0,T])}) \times \\
&(\Delta^1(u) + \Delta^1(v,w) + \mathrm{meas}\{(t,a)\in D:\ u(t,a)\ne\hat{u}(t,a)\}).
\end{aligned} \tag{39}
$$

Expressing the summands in (38) in terms of the Hamiltonians we obtain

$$
\begin{aligned}
0 \le& \int_0^\omega [H_0(a,w(a)) - H_0(a,\hat{w}(a))]\,\mathrm{d}a + \int_0^T [H_b(t,v(t)) - H_b(t,\hat{v}(t))]\,\mathrm{d}t \\
&+ \int_0^T \int_0^\omega [H(t,a,u(t,a)) - H(t,a,\hat{u}(t,a))]\,\mathrm{d}a\,\mathrm{d}t \\
&+ \int_0^T \int_0^\omega [(L(t,a,u(t,a),v(t),w(a)) - L(t,a,w(a))) - (L(t,a,u(t,a),v(t)) - L(t,a))]\,\mathrm{d}a\,\mathrm{d}t \\
&+ \int_0^T \int_0^\omega [(L(t,a,u(t,a),v(t)) - L(t,a,v(t))) - (L(t,a,u(t,a)) - L(t,a))]\,\mathrm{d}a\,\mathrm{d}t + \varepsilon
\end{aligned} \tag{40}
$$

Construction of needle variations.

Below we fix arbitrary representatives of all L_∞-functions involved. Let us prove the first variational inequality in Theorem 2. Let $a_0 \in (0,\omega)$ be a Lebesgue point of the functions $\frac{\partial H_0}{\partial w}(a,\hat{w}(a))$ and $\frac{\partial H_0}{\partial w}(a,\hat{w}(a))\hat{w}(a)$. We fix an arbitrary $w \in W$, and define for $h \in (0,1)$ (small enough, so that the interval $[a_0-h, a_0+h]$ is contained in $[0,\omega]$)

$$w(a) \stackrel{def}{=} \hat{w}(a) + h(w - \hat{w}(a))\chi_{[a_0-h,a_0+h]}(a),$$

where $\chi_{[a_1,a_2]}$ denotes the characteristic function of $[a_1,a_2]$. Then $w(\cdot)$ is an admissible control, since W is convex. We take also $u = \hat{u}$ and $v = \hat{v}$. Obviously for some constant d the nonsero terms in (39) can be estimated as

$$\Delta^\infty(v,w) \le dh, \quad \|w - \hat{w}\|_{L_\infty([0,\omega])} \le dh, \quad \Delta^1(v,w) \le dh^2.$$

Then $\Delta_\infty(u) + \Delta^\infty(v,w) \le dh < \varepsilon$ for all sufficiently small h (see the beginning of the proof) and (40) is fulfilled. The nonzero terms give

$$0 \le \int_0^\omega [H_0(a,w(a)) - H_0(a,\hat{w}(a))]\,\mathrm{d}a + Cd^2h^3.$$

Using the particular form of $w(\cdot)$ and the Lipschitz differentiability of H_0 in w we obtain

$$0 \le \frac{1}{h}\int_{a_0-h}^{a_0+h} \frac{\partial H_0}{\partial w}(a,\hat{w}(a))(w - \hat{w}(a))\,\mathrm{d}a + O(h),$$

which gives the desired inequality, since a_0 is a Lebesgue point of the integrand.

The proof of the second inequality in Theorem (2) is completely analogous.

The last inequality will be proven by contradiction. Assume that $H(t,a,\hat{u}(t,a)) > \inf\{H(t,a,u) : u \in U\}$ on a set of positive measure in D. Since H is continuous in u and U is compact, by a standard measure theoretic argument, there exist $\bar{u} \in U$, $\delta > 0$ and a set $\Omega \subset D$ of positive measure, such that

$$H(t,a,\bar{u}) - H(t,a,\hat{u}(t,a)) \le -\delta \tag{41}$$

for every $(t,a) \in \Omega$. Almost every point in the interior of D is a Lebesgue point of both $H(t,a,\bar{u})$ and $H(t,a,\hat{u}(t,a))$, therefore there exists such belonging to Ω. Denote it by $r = (\bar{t},\bar{a})$, and let $R(r,h)$ be the coordinate box of size $h \times h$ centered at r. For all sufficiently small $h > 0$ we have $R(r,h) \subset D$. Define the controls

$$u(t,a) = \chi_{R(r,h)}\bar{u} + (1 - \chi_{R(r,h)})\hat{u}(t,a), \quad v(t) = \hat{v}(t), \quad w(a) = \hat{w}(a).$$

Obviously for an appropriate constants C_1 and C_2 we have for the nonzero terms in (39)

$$\Delta_\infty(u) \le C_1 h, \quad \Delta^1(u) \le C_2 h^2, \quad \mathrm{meas}\{(t,a) \in D : \ u(t,a) \ne \hat{u}(t,a)\} \le h^2$$

Then (40) is fulfilled for all sufficiently small h and we obtain

$$0 \le \frac{1}{h^2}\int_{R(r,h)} [H(t,a,\bar{u}) - H(t,a,\hat{u}(t,a))]\,\mathrm{d}a\,\mathrm{d}t + CC_1C_2h,$$

which contradicts (41) since r is a Lebesgue point of both $H(t,a,\bar{u})$ and $H(t,a,\hat{u}(t,a))$. The proof of Theorem 2 is complete.

6 Additional Control Constraints

Below we consider two types of more specific control constraints that often arise in practice. The first is the case where some of the control components are age-independent, the second is the case of a "budgetary" constraint that has to be satisfied at each instant of time.

6.1 Nondistributed controls

Let us split the control function u into two parts: $u = (u_1, u_2)$, where u_2 is supposed to be nondistributed (independent of a). Correspondingly, the control constraint is assumed to be in the form

$$u_1(t,a) \in U_1, \quad u_2(t) \in U_2.$$

Instead of the Hamiltonian H here we introduce two Hamiltonians

$$\begin{aligned} H_1(t,a,u_1) &\stackrel{def}{=} L(t,a,u_1,\hat{u}_2(t)) + \xi(t,a) f(t,a,u_1,\hat{u}_2(t)) \\ &\quad + \int_0^\omega \eta(t,a') g(t,a',a,u_1,\hat{u}_2(t))\, \mathrm{d}a' + \zeta(t) h(t,a,u_1,\hat{u}_2(t)), \end{aligned}$$

$$\begin{aligned} H_2(t,u_2) &\stackrel{def}{=} \int_0^\omega [L(t,a,\hat{u}_1(t,a),u_2) + \xi(t,a) f(t,a,\hat{u}_1(t,a),u_2) \\ &\quad + \int_0^\omega \eta(t,a') g(t,a',a,\hat{u}_1(t,a),u_2)\, \mathrm{d}a' + \zeta(t) h(t,a,\hat{u}_1(t,a),u_2)]\, \mathrm{d}a. \end{aligned}$$

Theorem 3 *Under Supposition* E *and the above control specification the claim of Theorem 2 remains valid, with the condition (28) replaced with the following two conditions:*

(i) for a.e. $(t,a) \in D$

$$H_1(t,a,u_1) - H_1(t,a,\hat{u}_1(t,a)) \geq 0 \quad \forall u_1 \in U_1;$$

(ii) for a.e. $t \in [0,T]$

$$H_2(t,u_2) - H_2(t,\hat{u}_2(t)) \geq 0 \quad \forall u_2 \in U_2.$$

Proof. The only place in the previous sections where the class of admissible control functions plays a role is the "construction of a needle variation" in the final part of the proof of Theorem 2. In the present situation this part should be modified as follows. Since the constraints for u_1 and u_2 are independent, we may construct the variations for u_1 and u_2 independently. For u_1 the construction remains the same as before (with u_2 fixed at $\hat{u}_2$). For u_2 we apply a needle variation on an interval of length h in the t direction. Notice, that for the so obtained function u_2 we have $\Delta^1(u_2) \leq Ch$, $\Delta_\infty(u_2) \leq Ch$ and the proof can be completed similarly as that of Theorem 2. Q.E.D.

6.2 Budgetary control constraint

In this subsection we consider a specific set U, namely

$$U = \{u = (u_1, \dots, u_n) : \ 0 \le u_i \le 1\},$$

but we pose the additional constraint

$$\sum_{i=1}^{n} l_i \int_0^{\omega} u_i(t,a)\,\mathrm{d}a \le d(t), \tag{42}$$

which has to be satisfied for a.e. t. Here $l = (l_1, \dots, l_n)$ is a given row-vector with positive components, $d(\cdot)$ is a given measurable function (the budget at time t) with positive values.

The result below does not change if $d(\cdot)$ depends also on q.

The necessary optimality condition below involves the following notations:

$$\Theta_i^0(t) \stackrel{def}{=} \{a \in [0,\omega] : \ \hat{u}_i(t,a) > 0\},$$

$$\Theta_i^1(t) \stackrel{def}{=} \{a \in [0,\omega] : \ \hat{u}_i(t,a) < 1\}.$$

These two sets depend on the particular representative of $\hat{u} \in L_\infty(D;U)$, but different representatives lead to sets $\Theta_i^0(t)$ and $\Theta_i^1(t)$ that coincide for a.e. t, modulo a set of measure zero in the a-space, which does not affect the theorem below.

Theorem 4 *Under Supposition* **E**, *suppose that* $l_i > 0$, $i = 1, \dots, n$, *and that* $d(\cdot) : [0,T] \mapsto (0,+\infty)$ *is measurable. Suppose in addition, that* f, g, h *and* L *are continuously differentiable in* u, *uniformly with respect to the rest of the variables in a compact set.*

Then the claim of Theorem 2 remains valid with the condition (28) replaced with the following:

(a) for a.e. $t \in [0,T]$ *for which* $\hat{u}(t,\cdot)$ *satisfies the budgetary constraint (42) as a strict inequality, and for a.e.* $a \in [0,\omega]$, *condition (28) is fulfilled;*

(b) for a.e. $t \in [0,T]$ *for which* $\hat{u}(t,\cdot)$ *satisfies (42) as equality, and for every* $i,j = 1, \dots, n$,

$$\operatorname*{einf}_{a \in \Theta_i^1(t)} \frac{1}{l_i} \frac{\partial H}{\partial u_i}(t,a,\hat{u}(t,a)) \ge \operatorname*{esup}_{a \in \Theta_j^0(t)} \frac{1}{l_j} \frac{\partial H}{\partial u_j}(t,a,\hat{u}(t,a)). \tag{43}$$

The maximum principle in Theorem 2 allows, at least in principle, to determine the optimal control u, provided that the function H is known. We shall see, that the same holds, to a certain extent, also in the case of a budgetary constraint. Namely, let us take a point t where the budgetary constraint is active, so that the necessary optimality condition is (43). Let us denote

$$\mu(\alpha;t) \stackrel{def}{=} \sum_{i=1}^{n} l_i \ \mathrm{meas}\{a \in [0,\omega] : \ \frac{1}{l_i} \frac{\partial H}{\partial u_i}(t,a,\hat{u}(t,a)) \le \alpha\}.$$

Since $\mu(-\infty;t) = 0$, $\mu(\alpha;t) = \omega(l_1 + \ldots + l_n) \ge d(t)$ for a sufficiently large α (we presuppose the last inequality, but if it is not fulfilled then the budgetary constraint cannot be active), and the function $\mu(\cdot;t)$ is upper semicontinuous, there exists a first number $\hat{\alpha}(t)$ such that

$\mu(\hat{\alpha}(t);t) \geq d(t)$. We distinguish the following two cases: (i) *regular* case, if $\mu(\hat{\alpha}(t);t) = d(t)$ and $\mu(\alpha;t) > d(t)$ for $\alpha > \hat{\alpha}(t)$; (ii) *irregular* is the complementary case. In the regular case we have

$$\hat{u}_i(t,a) = \begin{cases} 1 & \text{if } \frac{1}{l_i}\frac{\partial H}{\partial u_i}(t,a,\hat{u}(t,a)) \leq \hat{\alpha}(t), \\ 0 & \text{else.} \end{cases}$$

Indeed, suppose that

$$\frac{1}{l_i}\frac{\partial H}{\partial u_i}(t,a,\hat{u}(t,a)) \geq \hat{\alpha}(t) + \varepsilon > \hat{\alpha}(t), \tag{44}$$

but nevertheless $\hat{u}_i(t,a) > 0$, and this happens for a in a set of positive measure. For these a we have $a \in \Theta_i^0$, from where we conclude that the right-hand side of (43) is at least as big as $\hat{\alpha}(t) + \varepsilon$, therefore, the left-hand side is such, as well. Thus (44) is satisfied for every $j = 1, \ldots, n$ (instead of i) and for a.e. a for which $\hat{u}_j(t,a) < 1$. Hence, if

$$\frac{1}{l_j}\frac{\partial H}{\partial u_j}(t,a,\hat{u}(t,a)) \leq \hat{\alpha}(t) + 0.5\varepsilon,$$

then $\hat{u}_j(t,a) = 1$. According to the regularity we have $\mu(\hat{\alpha}(t) + 0.5\varepsilon; t) > d(t)$, which contradicts the budgetary constraint that $\hat{u}$ satisfies.

Similarly we prove the claim concerning $\hat{u}_i(t,a) = 1$, where we use the fact that $\hat{u}$ satisfies the budgetary constraint as an equality at t.

Thus allocation of the budget at moments t where the regular case holds is of "bang-bang" type: for any age a and any control component u_i, either the maximal allowed amount is invested ($u_i = 1$), or nothing.

The above consideration is useful mainly for Hamiltonians H which are linear in u. In the nonlinear case, "irregularity" typically arises and $\hat{u}(t,a)$ can take values in the interior of $[0,1]$. In this case, the determination of $\hat{u}(t,a)$ is facilitated by the following observation (we consider the case $n = 1$). If $I(t)$ denotes the set of points a where $\hat{u}(t,a) \in (0,1)$ then $I(t) \subset \Theta^1(t) \cap \Theta^0(t)$ which implies

$$\frac{\partial H}{\partial u}(t,a,\hat{u}(t,a)) = \frac{\partial H}{\partial u}(t,a',\hat{u}(t,a')) \quad \text{for a.e.} \quad a, a' \in I(t).$$

Then for every fixed t the function $\hat{u}(t,\cdot)$ solves on $I(t)$ the equation

$$\frac{\partial H}{\partial u}(t,a,\hat{u}(t,a)) = \text{const}. \tag{45}$$

The constant should be determined in such a way that the budgetary constraint be satisfied as equality. (Clearly, const $= 0$ if there is no budgetary constraint.) Equation (45) will be applied to the second example in the next section.

Proof of Theorem 4. For those t for which (42) is a strict inequality one can apply the same construction of a needle variation for u as in the proof of Theorem 2, and obtain (a).

In the proof of (43) it will be convenient to fix an arbitrary representative of each L_∞-function involved. In particular, we may assume that the representative of $\hat{u}$ is fixed in such a way that $0 \leq \hat{u}(t,a) \leq 1$ and (42) hold everywhere.

Assume that (42) is satisfied as equality on a set of positive measure, but (43) is violated on a subset of positive measure for some i and j. By a standard measure theoretic argument in $\mathbf{R}^2$, there is a subset $\Omega \subset (0,T)$ of positive measure, positive numbers δ, α and μ, and measurable (in set-valued sense [3]) maps $\Xi^k : \Omega \Rightarrow [0,\omega]$, $k=0,1$ such that for every $\bar t \in \Omega$, $\bar a_0 \in \Xi^0(\bar t)$ and $\bar a_1 \in \Xi^1(\bar t)$

(i) $\mathrm{meas}(\Xi^k(\bar t)) \geq \mu$, $k=0,1$;

(ii) $\hat u_i(\bar t, \bar a_1) \leq 1-\alpha$, $\hat u_j(\bar t, \bar a_0) \geq \alpha$;

(iii)

$$\frac{1}{l_i}\frac{\partial H}{\partial u_i}(\bar t, \bar a_1, \hat u(\bar t, \bar a_1)) \leq \frac{1}{l_j}\frac{\partial H}{\partial u_j}(\bar t, \bar a_0, \hat u(\bar t, \bar a_0)) - \delta. \tag{46}$$

Denote $e_k = (0,\ldots,1,\ldots,0)$, with 1 at the k-th place, and $\lambda \stackrel{def}{=} \min\{l_s\}$. We fix points $\bar t \in \Omega$, $\bar a_k \in \Xi^k(\bar t)$, $\bar a_0 \neq \bar a_1$ in such a way that $\bar r_k = (\bar t, \bar a_k)$ is a Lebesgue point for each of the functions $H(t,a,\hat u(t,a)+\varepsilon e_i/l_i)$ and $H(t,a,\hat u(t,a)-\varepsilon e_j/l_j)$ with $\varepsilon \in [0,\lambda\alpha]$, $\bar r_k$ is a density point of the set graph$\,(\Xi^k)$, and $\bar a_k$ is a density point of $\Xi^k(\bar t)$. Notice that the Hamiltonian $H(t,a,\cdot)$ is Lipschitz continuous, uniformly with respect to $(t,a)\in D$ thanks to the uniform continuous differentiabillity condition and the compactness of U. This fact implies the existence of a Lebesgue point independent of ε. By definition, $\bar r_k$ is a density point of graph$\,\Xi^k$ if $h^{-2}\mathrm{meas}(\mathrm{graph}\,(\Xi^k) \cap R(\bar r, h)) \longrightarrow 1$ (as before $R(r,h)$ stays for the coordinate box of size $h \times h$ centered at r). Similarly, $h^{-1}\mathrm{meas}([\bar a_k - 0.5h, \bar a_k + 0.5h] \cap \Xi^k(\bar t)) = 1 + \theta(h)$ with $\theta(h) \longrightarrow 0$.

Now we shall modify the construction of a needle variation in the final part of the proof of Theorem 2 in the following way. For $\varepsilon \in (0,\lambda\alpha]$ we define the control

$$u(t,a) \stackrel{def}{=} \hat u(t,a) + \varepsilon\tau\frac{e_i}{l_i}\mathcal{X}^1_h(t,a) - \varepsilon\frac{e_j}{l_j}\mathcal{X}^0_h(t,a),$$

where $\tau \in (0,1)$ and $\mathcal{X}^k_h$ is the indicator function of $R(\bar r_k, h) \cap \mathrm{graph}\,(\Xi^k)$. Clearly, the values $u(t,a)$ belong to U thanks to the choice of ε and property (ii). Moreover, if $t \in \Omega \cap [\bar t - 0.5h, \bar t + 0.5h]$ (for other values of t we have $u(t,\cdot) = \hat u(t,\cdot)$), then

$$\int_0^\omega l u(t,a)\,\mathrm{d}a = \int_0^\omega l\hat u(t,a)\,\mathrm{d}a$$

$$+\varepsilon\tau\mathrm{meas}([\bar a_1 - 0.5h, \bar a_1 + 0.5h] \cap \Xi^1(\bar t)) - \varepsilon\mathrm{meas}([\bar a_0 - 0.5h, \bar a_0 + 0.5h] \cap \Xi^0(\bar t))$$

$$\leq 1 + \varepsilon\tau h(1+\theta(h)) - \varepsilon h(1-\theta(h)) \leq 1$$

if h is so small that $(1+\tau)\theta(h) \leq 1-\tau$. For such h (supposing also that h is so small that $R(\bar r_k, h) \subset D$ and $R(\bar r_1, h) \cap R(\bar r_0, h) = \emptyset$) u is an admissible control. Then (40) holds with $w = \hat w$ and $v = \hat v$. We have

$$\begin{aligned}
&\int_D H(t,a,u(t,a))\,\mathrm{d}a\,\mathrm{d}t \\
&= \int_{R(\bar r_1,h)\cap \mathrm{graph}\,(\Xi^1)} H(t,a,\hat u(t,a) + \varepsilon\tau\frac{e_i}{l_i})\,\mathrm{d}a\,\mathrm{d}t + \int_{R(\bar r_0,h)\cap \mathrm{graph}\,(\Xi^1)} H(t,a,\hat u(t,a) - \varepsilon\frac{e_j}{l_j})\,\mathrm{d}a\,\mathrm{d}t \\
&= \int_{R(\bar r_1,h)} H(t,a,\hat u(t,a) + \varepsilon\tau\frac{e_i}{l_i})\,\mathrm{d}a\,\mathrm{d}t + \int_{R(\bar r_0,h)} H(t,a,\hat u(t,a) - \varepsilon\frac{e_j}{l_j})\,\mathrm{d}a\,\mathrm{d}t + h^2\theta_1(h) \\
&= h^2 H(\bar t, \bar a_1, \hat u(\bar t, \bar a_1) + \varepsilon\tau\frac{e_i}{l_i}) + h^2 H(\bar t, \bar a_0, \hat u(\bar t, \bar a_0) - \varepsilon\frac{e_j}{l_j}) + h^2\theta_2(h),
\end{aligned}$$

where $\theta_1(h)$ and $\theta_2(h)$ tend to zero with h. Then, as in the proof of Theorem 2, we obtain from (40)

$$0 \leq H(\bar{t}, \bar{a}_1, \hat{u}(\bar{t}, \bar{a}_1) + \varepsilon\tau e_i/l_i) - H(\bar{t}, \bar{a}_1, \hat{u}(\bar{t}, \bar{a}_1)) + H(\bar{t}, \bar{a}_0, \hat{u}(\bar{t}, \bar{a}_0) - \varepsilon e_j/l_j) - H(\bar{t}, \bar{a}_0, \hat{u}(\bar{t}, \bar{a}_0)).$$

Taking the limit in ε we obtain

$$0 \leq \frac{\tau \partial H}{l_i \partial u_i}(\bar{t}, \bar{a}_1, \hat{u}(\bar{t}, \bar{a}_1)) - \frac{\partial H}{l_j \partial u_j}(\bar{t}, \bar{a}_0, \hat{u}(\bar{t}, \bar{a}_0)).$$

Since $\tau \in (0, 1)$ is arbitrary, the inequality holds also for $\tau = 1$. This contradicts (46) and completes the proof. Q.E.D.

7 Some Applications

In this section we apply the maximum principle to the two "stylized" models discussed in the introduction. For the first model we obtain some qualitative features of the optimal control, while for the second model we outline an analytic representation of the optimal controls. More detailed analysis and numerical implementations based on the maximum principle for models in the social area (drug control and investment models) will be presented in [1, 15] and elsewhere.

1. First we consider the model (1),(2),(3). We apply Theorem 2 with the specifications

$$L(a, y, u) = d(a)y + c(u), \quad l = 0,$$

$$f(t, a, y, p, u) = S(t, a)\Phi(p) - u\Theta(y), \quad g(a, a', y) = m(a, a')y.$$

The adjoint system becomes

$$\begin{aligned} -(\xi_t + \xi_a) &= d(a) - \xi(t, a)u(t, a)\Theta'(y(t, a)) + \int_0^\omega \eta(t, a')m(a', a)\,\mathrm{d}a', \\ &\xi(T, a) = 0, \quad \xi(t, \omega) = 0, \\ \eta(t, a) &= \xi(t, a)S(t, a)\Phi'(p(t, a)), \end{aligned}$$

and the minimization condition of the Hamiltonian gives

$$c'(u) = \xi(t, a)\Theta(y(t, a)).$$

Further we simplify the consideration taking $\Theta(y) = 1$. Moreover, the following natural conditions will be supposed: $d(a)$, $m(a, a')$, $S(t, a)$ are nonnegative and twice differentiable, $\Phi(\cdot)$ is monotone increasing, $c(\cdot)$ is strictly convex, $d(\cdot)$ and $m(a, \cdot)$ are concave. Then the adjoint equation becomes

$$-(\xi_t + \xi_a) = d(a) + \int_0^\omega \xi(t, b)S(t, b)\Phi'(p(t, b))m(b, a)\,\mathrm{d}b.$$

Since $c'(\cdot)$ is strictly monotone increasing and differentiable, the following properties of the optimal control can be deduced from the same properties of $\xi(t, a)$:

(i) for every fixed t the optimal distribution $u(t,\cdot)$ is a concave differentiable function; in particular it has a single peak $a(t) \in [0,\omega]$;

(ii) the magnitude $u(t,a(t))$ of the peak decreases with t.

It is not straightforward that ξ has the above two properties. The proof is technical, therefore we just sketch the main points. First, from the adjoint equation and the boundary conditions one can prove that ξ satisfies the equation

$$\xi_t + \xi_a = \xi_a(t,\omega) - [\xi_{ta}(t,\omega) + \xi_{aa}(t,\omega)](\omega - a) - \int_a^\omega \int_{a'}^\omega \beta(t,b)\,\mathrm{d}b\,\mathrm{d}a',$$

where

$$\beta(t,a) = d''(a) + \int_0^\omega \xi(t,b)S(t,b)\Phi'(p(t,b))m_2''(b,a)\,\mathrm{d}b$$

(m_2'' denotes the second derivative with respect to the second argument of m). The suppositions imply $\beta(t,a) \le 0$. The above differential equation has the explicit solution

$$\int_t^{\min\{t+\omega-a,T\}} \left[-\xi_a(s,\omega) + (\xi_{ta}(s,\omega) + \xi_{aa}(s,\omega))(\omega - a + t - s) + \int_{a-t+s}^\omega \int_{a'}^\omega \beta(t,a'')\,\mathrm{d}a''\,\mathrm{d}a' \right] \mathrm{d}s. \tag{47}$$

From here one can calculate all derivatives of ξ that are of interest. In particular

$$\xi_{aa}(t,a) = \begin{cases} \int_t^T \beta(s,a-t+s) & \text{for } a \in [0,\omega+t-T] \\ \xi_{aa}(t+\omega-a,\omega) + \int_t^{t+\omega-a} \beta(s,a-t+s) & \text{for } a \in (\omega+t-T,\omega]. \end{cases}$$

It is easy to check that the above function is continuous (as well as ξ_a). Since the values of β are nonpositive, to prove concavity of ξ it remains to show that $\xi_{aa}(t+\omega-a,\omega) \le 0$. This can be done making use of the boundary condition and the inequality $\xi_a(t,\omega) \le 0$, which also follows from the boundary condition $\xi(t,\omega) = 0$ together with $\xi(t,a) \le 0$.

The second conclusion (ii) follows from

$$\frac{d}{dt}\xi(t,a(t)) = \xi_t(t,a(t)) + \xi_a(t,a(t))\dot{a}(t) = \xi_t(t,a(t)) = \xi_t(t,a(t)) + \xi_a(t,a(t)) \le 0.$$

Using the representation (47) one can obtain also other qualitative properties of ξ, therefore also of the optimal control.

2. Now let us apply Theorem 2 to the vintage capital model described in the introduction. Here we have the specifications $y = K$, $q = (Q,I)$, p and w are missing,

$$L(t,a,K,Q,I,u,v) = -e^{-rt}\{\frac{1}{\omega}[R(Q) - \alpha_0 v - \beta v^2 - \gamma_i I^2] - [\alpha(a)u + \gamma_e u^2]\}, \quad l = 0,$$

$$f(t,a,K,u) = -\delta(a)K + u, \quad h_1(t,a,K) = m(t-a)d(a)K, \quad h_2(u) = u, \quad \varphi(v) = v.$$

The adjoint system becomes

$$\begin{aligned} -(\xi_t + \xi_a) &= -\xi(t,a)\delta(a) + \zeta_1(t)m(t-a)d(a), \quad \xi(T,a) = 0, \;\; \xi(t,\omega) = 0, \\ \zeta_1(t) &= -e^{-rt}R'(Q(t)), \\ \zeta_2(t) &= 2e^{-rt}\gamma_i I(t). \end{aligned}$$

Maximizing the Hamiltonians corresponding to u and v and substituting ζ_2 from the above equation we obtain

$$u(t,a) = -\frac{1}{2\gamma_e(a)}[e^{rt}\xi(t,a) + 2\gamma_i I(t) + \alpha(a)], \tag{48}$$

$$v(t) = -\frac{1}{2\beta}(e^{rt}\xi(t,0) + \alpha_0). \tag{49}$$

Integrating (48) on $[0,\omega]$ one can express $I(t)$ as

$$I(t) = -\frac{1}{2(\gamma_e + \omega\gamma_i)} \int_0^\omega [e^{rt}\xi(t,a) + \alpha(a)]\,\mathrm{d}a. \tag{50}$$

Substituting $\lambda = -e^{rt}\xi$, the equation for ξ becomes

$$\lambda_t + \lambda_a = \lambda(t,a)(\delta(a) + r) - R'(Q(t))m(t-a)d(a), \lambda(T,a) = 0, \;\; \lambda(t,\omega) = 0.$$

A routine calculation (we suppose $T \geq \omega$) gives

$$\lambda(t,a) = m(t-a) \int_t^{\min\{t+\omega-a,T\}} e^{-\int_a^{a-t+s}(\delta(\theta)+r)\,\mathrm{d}\theta} R'(Q(s))d(a-t+s)\,\mathrm{d}s. \tag{51}$$

Notice that the solution λ, therefore also ξ, is negative. The quantity $-\lambda(t,a)$ represents the current shadow price at time t of a unit of capital of age a.

If R is linear (as in the case of nonmonopolistic economy) then $\lambda(t,a)$ is explicitly found, in particular the trace $\lambda(t,0)$ needed to determine $v(t)$. In this case the problem allows an analytical solution. In the monopolistic case (quadratic R) the above equation is linear in $Q(\cdot)$, therefore the optimal controls are linear functionals of $Q(\cdot)$. Similarly as for the adjoint equation, one can solve analytically (5), expressing the solution as a linear functional of $Q(\cdot)$. Substituting in (6) we obtain a Fredholm integral equation for the function $Q(t)$. Then (51),(50),(48) and (49) give analytic expressions for the optimal control.

Now let us see how the solution is modified in presence of "budgetary" constraints

$$\int_0^\omega u(t,a)\,\mathrm{d}t \leq b(t).$$

Let us denote by u_b the optimal control in the constrained problem. We shall keep the same notations as before for the state and costate variables.

If for some t the control $u(t,\cdot)$ determined above satisfies the budgetary constraint then we have $u_b(t,\cdot) = u(t,\cdot)$. If the budgetary constraint is violated, then we apply (45) with a constant $C(t)$ in the right-hand side and obtain

$$u_b(t,a) = \frac{1}{2\gamma_e(a)}[C(t) + \lambda(t,a) - 2\gamma_i I(t) - \alpha(a)].$$

Integrating on $[0,\omega]$ we exclude $I(t)$ as above:

$$I(t) = \frac{1}{2(\gamma_e + \omega\gamma_i)}[\omega C(t) + \int_0^\omega [\lambda(t,a) - \alpha(a)]\,\mathrm{d}a.$$

Then we determine the constant $C(t)$ so that the budgetary constraint is satisfied as an equality and substitute in the expression for $u_b(t,\cdot)$ obtaining finally that

$$u_b(t,a) = \frac{1}{2\gamma_e(a)}\left[\frac{1}{\omega}\int_0^\omega (-\lambda(t,a') + \alpha(a'))\,\mathrm{d}a' + \frac{2\gamma_e(a)}{\omega}b(t) - \alpha(a) + \lambda(t,a)\right].$$

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