

# Cycles of Violence: A Dynamic Control Analysis <sup>1</sup>

*Jonathan P. Caulkins<sup>2</sup>, Gustav Feichtinger<sup>3</sup>, Gernot Tragler<sup>3</sup>, and Vladimir M. Veliov<sup>4</sup>*

## Abstract

We introduce and analyze a simple model of cycles of violence in which oscillations are generated when surges in lethal violence shrink the pool of active violent offenders. Models with such endogenously induced variation may help explain why historically observed trends in violence are generally not well correlated with exogenous forcing functions, such as changes in the state of the economy. The analysis includes finding the optimal dynamic trajectory of incarceration and violence prevention interventions. Those trajectories yield some surprising results, including situations in which myopic decision makers will invest more in prevention than will far-sighted decision makers.

## 1 Introduction

Levels of violence vary substantially over time, but the variations are not well correlated with indicators of "root causes" of violence. E.g., homicide rates surged in the United States during the relatively prosperous 1960s, ebbed during the stagflation of the 1970s, and rose sharply in the late 1980s amidst Reagan-era prosperity. Both the high amplitude of variation and the lack of correlation with traditional macro explanatory variables suggests the possibility of interesting internal dynamics.

A variety of hypotheses have been advanced, including that violence toward children prompts those children to become violent offenders (Widom and Maxfield, 2001 [8]) and that variation in the availability of abortion affects the proportion of children who grow up to be violent offenders (Donohue and Levitt, 2001 [4]). Another hypothesis is that violence is inherently self-limiting because crime is most often committed within rather than across social groups. In particular,

---

<sup>1</sup>This research was partly supported by the Austrian Science Foundation (FWF) under grant # 14060-OEK. Support was provided also in part by a grant from the National Consortium on Violence Research (NCOVR), headquartered at the Heinz School of Public Policy and Management at Carnegie Mellon University. NCOVR is supported under grant # SBR 9513040 from the National Science Foundation in partnership with the US Department of Housing and Urban Development and the National Institute of Justice.

<sup>2</sup>Carnegie Mellon University, H. John Heinz III School of Public Policy and Management; 5000 Forbes Ave., Pittsburgh, PA 15213 and RAND Drug Policy Research Center

<sup>3</sup>Institute for Econometrics, Operations Research and Systems Theory, Vienna University of Technology, Argentinierstrasse 8, A-1040 Vienna, Austria

<sup>4</sup>Institute for Econometrics, Operations Research and Systems Theory, Vienna University of Technology, Argentinierstrasse 8, A-1040 Vienna, Austria and Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

lethal violence may be perpetrated most often against people who are themselves criminally involved<sup>5</sup>. In this vision of the world, epidemics of violence burn themselves out by decimating the population of violent offenders. This paper proposes a formal model that captures this dynamic and explores some policy implications of trying to control optimally the resulting cycles of violence.

In a continuous time dynamic model, cycles cannot emerge with a single state variable, so the number of violent offenders must be augmented by some other quantity for non-monotonic variation to emerge. One possibility is that lethal interactions between violent offenders are much more likely within a general climate or expectation of violence than when the expectation is one of amicable co-existence and that this climate of violence is driven by recollection of actual instances of violence, not merely by counts of the current number of potential offenders. In other words, it is possible for a large number of potentially violent offenders to co-exist without killing each other in great numbers if there is a tradition of non-lethal interactions, but the expectation of violent interactions could itself be a violence promoter.

An expectation of violence can be a self-fulfilling prophecy for any of several reasons. For example, if a violence-prone individual perceives the world to be very risky, the individual might be more likely to carry a weapon, which in turn increases the likelihood confrontations will escalate to lethal violence (cf. Blumstein et al. [3]). Likewise, if one expects an adversary to attack, that increases the incentive to commit a preemptive first strike. Also, high levels of background violence might affect behavioral norms. In a society in which violence is rare, an individual might respond to an insult or challenge with words or fists, believing that more aggressive actions are uncalled for.

This suggests analyzing a model with two states: the number of potentially violent offenders ( $x$ ) and the perceived climate or expectation of violence ( $y$ ). One of the flows out of the population of potentially violent individuals is proportional to the square of the number of such individuals (as in predator-prey interactions, except that the predator and prey are the same species in this case) and the background level or reputation of the society for violence. Every such exit (representing a death) also generates an inflow to the reputation for the level of violence. Violent interactions between the offenders ( $x$ ) and the general public (whose population is assumed to be constant) are also assumed to be both promoted by and contribute to this general climate of violence ( $y$ ).

To focus attention on this dynamic, the rest of the model will be kept quite simple, with just a simple growth model for the offenders and exponential decay of the reputation for violence. In particular, individuals join the pool of violent offenders at a constant rate of  $k$  individuals per unit time, perhaps because a fixed proportion of every birth cohort has the capacity to be very violent, and there is a constant outflow rate per person per unit time ( $\mu$ ) due to aging, death for other reasons, being incarcerated, etc. The exponential decay of the reputation for the level of violence is just a forgetting term, analogous to that in Behrens et al. [1].

Formally the uncontrolled model can thus be written as:

---

<sup>5</sup>This possibility was suggested to the authors by Alfred Blumstein, personal communications.

$$\dot{x} = k - \mu x - \alpha x^2 y \tag{1}$$

$$\dot{y} = \alpha x^2 y + \beta xy - \delta y, \tag{2}$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are positive constants.

## 2 The Uncontrolled Model

In this section we establish some basic properties of the descriptive (uncontrolled) model (1),(2), where  $k, \mu, \alpha, \beta, \delta$  are positive parameters. In particular, we investigate the dependence of the steady state of the system on these parameters, whose base values (denoted by  $\bar{k}, \dots, \bar{\delta}$ ) are identified in Subsection 2.2.

### 2.1 Qualitative properties

**Proposition 1** *A unique solution of (1),(2) starts from every initial point in the first quadrant, the solution stays bounded, therefore is extendible to infinity, and remains in the first quadrant. The claim is true also for time-dependent (measurable in  $t$ ) parameters, provided that  $\mu$  and  $\delta$  are uniformly positive (do not approach zero).*

The invariance of the first quadrant is an obvious consequence of the classical Nagumo theorem. To prove boundedness we take time-dependent (measurable in  $t$ ) parameters such that  $\mu(t) \geq \mu_1 > 0$  and  $\delta(t) \geq \delta_1 > 0$  and a trajectory  $(x(t), y(t))$  starting from the first quadrant. Then obviously  $x(t) \leq \max\{x(0), k/\mu_1\}$  for any  $t \geq 0$ .

To prove boundedness of  $y$ , we consider the function  $V(t) = x + y + qx^2$  along a trajectory of (1),(2), where  $q$  is a sufficiently large number. Then we have

$$\dot{V} = k - \mu x + \beta xy - \delta y + 2qx(k - \mu x - \alpha x^2 y) = k - \mu x + 2qkx - 2q\mu x^2 - (\delta - \beta x + 2q\alpha x^3)y.$$

The number  $q$  can be fixed so large that the multiplier of  $y$  in the last expression is not smaller than  $\delta_1/2$ . Since  $x(t)$  is bounded,  $\dot{V}(t)$  becomes negative if  $y(t)$  is too large. Therefore  $V(t)$  is bounded, and from the invariance of the first quadrant we conclude that  $y(t)$  is bounded, too.

For any fixed positive values of the parameters, system (1),(2) has at most two equilibria in the first quadrant:

$$\hat{x}_0 = \frac{k}{\mu}, \quad \hat{y}_0 = 0$$

and

$$\hat{x} = \frac{-\beta + \sqrt{\beta^2 + 4\alpha\delta}}{2\alpha}, \quad \hat{y} = \frac{k - \mu\hat{x}}{\alpha\hat{x}^2}.$$

The second one, however, is situated in the interior of the first quadrant if and only if the inequality

$$\mu \hat{x} < k \tag{3}$$

holds. Notice that  $\hat{x}$  does not depend on  $k$  and  $\mu$ . Therefore, the above inequality fails for small  $k$  or big  $\mu$ . That is, if the inflow of offenders is small enough or the incarceration rate is big enough, then the system has only one equilibrium in the first quadrant, and it has a reputation of violence  $y = 0$ .

The Jacobian matrix, determining the local behaviour of the system around the equilibrium, is

$$\begin{pmatrix} -\mu - 2\alpha xy & -\alpha x^2 \\ 2\alpha xy + \beta y & \alpha x^2 + \beta x - \delta \end{pmatrix}.$$

Calculating the eigenvalues we come to the following conclusion:

**Proposition 2** (i) *If (3) is fulfilled, then the equilibrium  $(\hat{x}, \hat{y})$  belongs to the interior of the first quadrant and is stable. It is a focus (that is, the trajectories approach it oscillating around it) if*

$$(2\alpha \hat{x} \hat{y} + \mu)^2 < 4\alpha \hat{x}^2 (2\alpha \hat{x} \hat{y} + \beta \hat{y}). \tag{4}$$

*and a node (that is, the trajectories approach it tangentially to a fixed line) if the inverse inequality holds. The equilibrium  $(\hat{x}_0, \hat{y}_0)$  has a one-dimensional stable invariant manifold  $y = 0$ .*

(ii) *If (3) is not fulfilled, then  $(\hat{x}_0, \hat{y}_0)$  is the only equilibrium in the first quadrant. It is a stable node.*

Notice that the inequality (3) fails for all sufficiently small  $\alpha$  and for all sufficiently large  $\mu$ . In other words, if the incarceration rate  $\mu$  is large enough, or the "reputation of violence" is sufficiently small, then whatever is the initial state, the reputation of crime converges to zero, and the number of criminals converges to  $k/\mu$  (notice that the last value depends on the incarceration rate  $\mu$ , but not on  $\alpha$ ).

Since  $\hat{x}$  is monotone increasing with  $\delta$ , starting from  $\hat{x} = 0$  at  $\delta = 0$ , one may conclude that in the case (i) the inequality (4) fails for small  $\delta$ . It fails also for  $\delta$  so large that  $\hat{x}$  approaches the value  $k/\mu$ . However, it could be satisfied for some intermediate values of  $\delta$ . In particular, it holds for the base values of the parameters defined in the next subsection, so we encounter *damped oscillations* toward  $(\hat{x}, \hat{y})$  in this case.

## 2.2 Parameter identification.

Parameters are estimated by assuming that steady state quantities match what is currently observed. This could be done for any jurisdiction because we are primarily interested in the

qualitative nature of the solution, not jurisdiction-specific details. For convenience we model the "high-rate" offenders in California studied by Greenwood et al. [5].

Greenwood et al. [5] distinguish between low- and high-rate offenders, the latter representing about 20% of all offenders and having offense rates that are 17.52 times those of the low-rate offenders. There are an estimated 20,838 new high-rate offenders each year (p. 68), and there were 203,309 high-rate offenders active on the street at the time of the study (p.69). Measuring the population in millions, this suggests  $\bar{k} = 0.020838$  and the steady state  $\bar{x} = 0.203309$ .

Greenwood et al. (p. 67) report there are 778,741 low-rate offenders on the street, therefore  $203,309 \times 17.52 / (203,309 \times 17.52 + 778,741) = 82.0596\%$  of all offences are committed by high-rate offenders. This suggests that in steady state the rate at which fear of violence is generated by interactions among high-rate offenders ( $\alpha x^2 y$ ), and the rate at which fear of violence is generated by interactions with others ( $\beta xy$ ) are related as follows:

$$B \stackrel{def}{=} \frac{\beta xy}{\alpha x^2 y} = \frac{1 - 0.820596}{0.820596}. \quad (5)$$

In order to determine the values of  $\alpha$  and  $\beta$  from the above one needs to know what proportion of homicide victims are themselves high-rate offenders. A. Blumstein (personal communication with the first author) suggested assuming that the proportion of all victims of violent crime who are high-rate offenders is the same as the proportion of all victimizations that are committed by high-rate offenders. There are two arguments for making such an assumption. First, most crime is committed intra- not intergroup. If high-rate offenders only interacted with high-rate offenders and low-rate offenders only interacted with low-rate offenders, then the above assumption would necessarily be true. One could argue, however, that the society may be more segregated along traditional group definitions (e.g. age, income, and ethnicity) than it is by offense rates.

The second justification is that we are interested not in all crime but in violent crime, particularly lethal crime. Much such crime is a product not of a single interaction, but is embedded within an ongoing conflictual relationship, and the final, lethal acts may often be retributive, retaliatory, or preemptive acts taken against a perpetrator. Since 82% of perpetrators are high-rate offenders, this suggests that 82% of victims of retribution, retaliation, and preemptive strikes may be high-rate offenders.

There were 4,158 homicides in California in Greenwood et al.'s base year 1992 ([5, p. 56]). That suggests an outflow rate due to homicides of high-rate offenders of  $0.82 \times 4,158 = 3,412$ , but we should augment this to account for high-rate offenders who are incarcerated as a result of those homicides (typically for fairly long terms). In 1994, for the US as a whole, there were 23,330 murders and non-negligent manslaughters ([6, p. 261]) and 12,168 felony convictions for murder/manslaughter ([6, p. 421]). So we might guess that there were about  $0.82 \times 4,158 \times 12,168 / 23,330 = 1,780$  high-rate offenders incarcerated in California in that year because of homicides, for a total outflow of 5,191.6223. Hence we complement (5) with the relation

$$A \stackrel{def}{=} \alpha x^2 y = 0.0051916223. \quad (6)$$

$k$	$\mu$	$\alpha$	$\beta$	$\delta$	$\hat{x}$	$\hat{y}$	$\alpha\hat{x}^2\hat{y}$
0.020838	0.076959	9.926266	0.441213	0.5	0.203309	0.0126533	0.0051916

Table 1: Base values of the parameters.

The remaining piece of information that is needed is  $\delta$ , the rate of forgetting, for which empirical information seems to be missing. As a base value we take  $\bar{\delta} = 0.5$  so the half-life of perceptions of violence is between one and two years).

Solving (5),(6) together with the equations  $\dot{x} = \dot{y} = 0$  we obtain

$$\begin{aligned}\bar{\mu} &= (\bar{k} - A)/\bar{x}, \\ \bar{y} &= (1 + B)A/\bar{\delta}, \\ \bar{\beta} &= AB/(\bar{x}\bar{y}), \\ \bar{\alpha} &= \delta/((1 + B)\bar{x}\bar{y}).\end{aligned}$$

The implicit numerical values of the parameters are given in Table 1.

### 2.3 Analysis of the parameterized uncontrolled model

The qualitative picture is given in figures 1a-1c below. The isoclines  $\dot{x} = 0$  and  $\dot{y} = 0$  (dashed lines) and three trajectories are plotted in each case. The base values of the parameters are used in the first plot. The value of  $\delta$  is changed from 0.5 to 0.09 in the second plot, and the stable focus becomes a stable node. The value of  $\mu$  is 1.5 times its base value  $\bar{\mu}$  in the third plot and case (ii) arises. A similar case and plot (omitted) arise if  $\alpha = 0.5\bar{\alpha}$ .

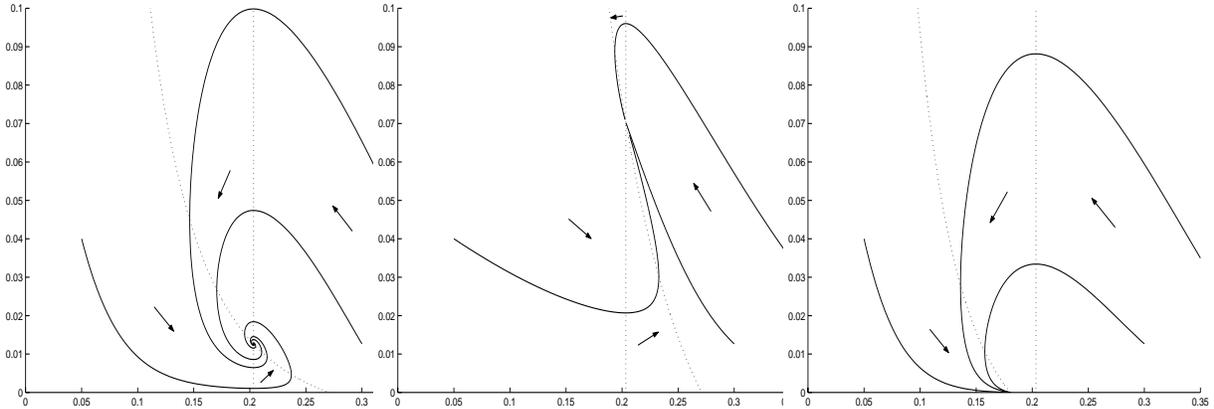


Figure 1: a)  $\delta = 0.5$ ; b)  $\delta = 0.09$ ; c)  $\mu = 1.5\bar{\mu}$ .

The dependence of the stable equilibrium in the first quadrant on the parameters  $k$ ,  $\mu$ ,  $\alpha$ ,  $\beta$  and  $\delta$  can be investigated analytically. The conclusions are summarized in Table 2.

	$(\hat{x}_0, \hat{y}_0)$	$\hat{x}_0$	$\hat{y}_0$	$(\hat{x}, \hat{y})$	$\hat{x}$	$\hat{y}$
$k$	$[0, 0.0158]$	$\uparrow$	0	$(0.0158, \dots)$	0.2033	$\uparrow$
$\mu$	$[0.1016, \dots]$	$\downarrow$	0	$(0, 0.1016]$	0.2033	$\downarrow$
$\alpha$	$[0, 5.1617]$	0.2708	0	$(5.1617, \dots)$	$\downarrow$	$\uparrow$
$\beta$	$\emptyset$	—	—	$(0, \dots]$	$\downarrow$	$\uparrow$
$\delta$	$[0.85, \dots]$	0.2708	0	$(0, 0.85]$	$\uparrow$	$\downarrow$

Table 2: Dependence of the stable equilibrium on the parameters.

The second and fifth columns in Table 2 indicate the range in the corresponding parameter’s value for which each equilibrium is the unique stable equilibrium in the first quadrant. For the base case parameter values, conditions (3) and (4) hold, so the equilibrium of interest is  $(\hat{x}, \hat{y})$ . The arrows indicate how changes in the parameter affect the equilibrium value. Where numerical values appear instead of arrows, the equilibrium state value is not affected by changes in the corresponding parameter.

We see from the first row that decreasing the inflow of violent offenders ( $k$ ) reduces the equilibrium level of violence, but the resulting reduction in homicides exactly offsets the decreased inflow of offenders and so does not affect the equilibrium number of offenders until after the equilibrium level of violence becomes zero and we shift to the other type of equilibrium. Increasing the rate at which offenders are removed ( $\mu$ ) has similar effects.

Increasing  $\alpha$  amplifies the key dynamic through which a history of past violence promotes further lethal interactions. Reasonably enough, this increases equilibrium perceptions of violence, which in turn suppresses the population of offenders through lethal interactions. Amplified violence with the general population (increasing  $\beta$ ) and stronger memories of past violence (smaller  $\delta$ ) have similar effects.

Plotting the trajectories over time (not shown) starting with initial conditions that differ from the equilibrium values by 10% shows considerable oscillation. It is intuitively plausible that exogenous shocks, such as a recession or "baby boom", could push the system 10% or more off the equilibrium, suggesting that the mechanism modeled here might play a role in observed oscillations in violence. These oscillations can be dampened considerably by adding a simple linear feedback rule as follows: a 1% increase in the number of offenders leads to a 1% increase in the incarceration rate  $\mu$  and a 1% increase in the reputation for violence leads to a 1% decrease in  $\alpha$ . The power of even such simple feedback suggests that investigating the optimal dynamic control paths will be of interest, as we will see next.

### 3 Optimal Control of the Violence Model

Governments take an active interest in trying to control crime and violence, because both high average levels and sharp variations in crime rates over time are costly. In this section we explore

how a decision maker might intervene to control or shape the course of a violence epidemic governed by the dynamics described above.

### 3.1 Formulation of the Optimization Problem

All of the parameters of the model (1)–(2) considered above can be influenced by some policy intervention. However, we focus on the parameters  $\mu$  and  $\alpha$ , which admit clear interpretations. Increasing  $\mu$  corresponds to enhancing criminal justice sanction generally to remove high-rate offenders from the active population. Reducing  $\alpha$  corresponds to reducing the probability that lethal violence emerges from interactions among violent offenders. This could be accomplished through interventions such as conflict resolution programs, stop and frisk policies, and gun control measures that discourage carrying of firearms. Therefore, we consider  $\alpha$  and  $\mu$  as control functions. As an objective function for minimization we chose

$$\int_0^{+\infty} e^{-rt} [ax(t) + b\alpha(t)x^2(t)y(t) + c\beta x(t)y(t) + u(\alpha(t)) + v(\mu(t))] dt, \quad (7)$$

which will be interpreted as the total cost to society of crime committed by high-rate offenders (the first three terms) plus the cost of the control policies (the last two terms). Thus  $a$ ,  $b$ , and  $c$  are nonnegative constants representing the social costs (damages) caused per active high-rate offender per unit of time, by the homicide of a high-rate offender, and by the homicide of someone who is not a high-rate offender, respectively. Greenwood et al. [5, p. 56] estimate that high-rate offenders commit an average of 4.13 serious crimes per year and that averting such crimes through expanded incarceration costs \$12,000 – 16,000 per crime (p.26). Multiplying these figures together suggests a cost of about \$58,000 per person per year. Since the population is measured in millions, with  $\bar{a} = 58$  the objective function is measured in billions of dollars. The average social cost per homicide (including intangible "quality of life" costs) has been estimated to be \$2.94 million ([7]). Since only  $23,330/(23,330 + 12,168) = 0.65722$  of the quantities  $\alpha x^2 y$  and  $\beta xy$  represent murdered people, we obtain values  $\bar{b} = \bar{c} = 1,932,647$ .

It is common in societal benefit-cost studies to value all lives equally, particularly when information about annual earnings is not part of the analysis, but some may prefer to consider assigning a lower cost to the death of a violent criminal than to the death of someone else. Hence, in sensitivity analysis below we consider particularly variations in which  $\bar{b}$  is less than  $\bar{c}$ .

The functions  $u(\alpha)$  and  $v(\mu)$  represent the costs of maintaining value  $\alpha$  of the frequency with which interactions between high-rate offenders lead to death, and value  $\mu$  of the exit rate of high-rate offenders, respectively. The nature of these two parameters suggests that  $u(\alpha)$  is monotone decreasing,  $v(\mu)$  is monotone increasing, and both are convex (diminishing returns). We investigate two pairs of such functions:

$$u(\alpha) = -\frac{1}{\rho_\alpha} \ln \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1}, \quad v(\mu) = -\frac{1}{\rho_\mu} \ln \frac{\mu_2 - \mu}{\mu_2 - \mu_1} \quad (8)$$

$r$	$a$	$b$	$c$	$\rho_\alpha$	$\alpha_2$	$\rho_\mu$	$\mu_1$
0.04	58	1932	1932	0.2	19.85252	8000	0.038479

Table 3: Base values of the objective function and control parameters.

or

$$u(\alpha) = \frac{\rho_\alpha}{2}(\alpha_2 - \alpha)^2, \quad v(\mu) = \frac{\rho_\mu}{2}(\mu - \mu_1)^2, \quad (9)$$

where  $\alpha_1, \alpha_2, \rho_\alpha, \mu_1, \mu_2, \rho_\mu$  are parameters (presumably positive). In the first choice of  $u$  and  $v$  the values of  $\alpha$  and  $\mu$  are bounded in the intervals  $(\alpha_1, \alpha_2]$  and  $[\mu_1, \mu_2)$ , respectively, while in the second choice the bounds are  $[0, \alpha_2)$  and  $[\mu_1, +\infty)$ , respectively.

There is no literature suggesting appropriate values for these control function parameters. We choose values such that in the solution the controls are used aggressively enough to have an influence but not so heavily as to be far into the region of diminishing returns. These are admittedly subjective criteria, and we shy away from drawing conclusions about whether intervening with  $\alpha$  or  $\mu$  is likely to be most cost-effective, focusing instead on the qualitative effects of such interventions and how the aggressiveness of such interventions should vary over a cycle of violence, assuming they are to be used at all. With all plausible parameter choices we considered, however, the bounds mentioned above would not be active for the optimal solutions.

As in Greenwood et al. (1994) we discount future costs at an annual rate of  $r = 0.04$ .

The values of the objective function and control parameters are summarized in Table 3. The latter correspond to the cost functions (9). All results presented below are obtained with these cost functions, but our experiments with (8) give qualitatively similar results.

### 3.2 Sensitivity of the Optimal Steady State

The "current-value" Hamiltonian of the optimal control problem (1),(2),(7), with the adjoint variables denoted by  $\xi$  and  $\eta$  reads as

$$H(x, y, \xi, \eta, \alpha, \mu)$$

$$\stackrel{def}{=} \xi(k - \mu x - \alpha x^2 y) + \eta(\alpha x^2 y + \beta xy - \delta y) - (ax + b\alpha x^2 y + c\beta xy + u(\alpha) + v(\mu)).$$

The adjoint equations of Pontryagin's maximum principle are

$$\dot{\xi} = (\mu + 2\alpha xy + r)\xi - (2\alpha xy + \beta y)\eta + a + 2b\alpha xy + c\beta y, \quad (10)$$

$$\dot{\eta} = \alpha x^2 \xi + (\delta - \alpha x^2 - \beta x + r)\eta + b\alpha x^2 + c\beta x. \quad (11)$$

Maximizing the Hamiltonian with respect to  $\alpha$  and  $\mu$ , and supposing that the maximizers stay in the interior of the admissible control regions (we do not make this a priori supposition when

	$x$	$y$	$\alpha x^2 y$	$\alpha$	$u(\alpha)$	$\mu$	$v(\mu)$	Costs
$k$	0.68	↑ 0.62 ↓	↑ 0.49 ↓	-1.49	5.00	0.41	1.38	1.50
$\beta$	-0.05	0.14	-0.04	-0.08	0.28	0.06	0.19	0.11
$\delta$	0.58	-4.26	-3.16	↓ -0.09 ↑	↑ 0.29 ↓	↑ 0.02 ↓	↑ 0.05 ↓	-0.51
$r$	0.07	0.10	0.09	-0.15	0.50	-0.08	-0.28	0.03
$a$	-0.15	0.67	0.69	0.32	-1.09	0.02	0.57	0.30
$b$	0.59	-4.57	-4.69	-1.30	4.29	0.30	1.00	0.24
$c$	0.09	-0.73	-0.74	-0.19	0.62	0.05	0.18	0.04
$\rho_\alpha$	-0.60	3.08	3.20	1.31	-3.38	↑ 0.00 ↓	↑ -0.01 ↓	0.03
$\rho_\mu$	↑ 0.07 ↓	1.55	1.54	↓ -0.15 ↑	↑ 0.50 ↓	-0.36	-0.20	0.40

Table 4: Sensitivity of the OSS at the base values.

solving the problem numerically, but the numerically found solutions happen to be always admissible) we obtain the relations

$$u'(\alpha) = (\eta - \xi - b)x^2 y, \quad v'(\mu) = -\xi x, \quad (12)$$

which determine  $\alpha$  and  $\mu$  as functions of  $x, y, \xi, \eta$  and  $x, \xi$ , respectively. Then the canonical system of the problem consists of the equations (1),(2),(10),(11), with  $\alpha$  and  $\mu$  determined from (12). The functions  $u(\alpha)$  and  $v(\mu)$  are given either by (8) or by (9).

Around the base values of the parameters the canonical system has a unique equilibrium with  $x \geq 0$  and  $y > 0$ , and it has a two-dimensional stable invariant manifold. This equilibrium, therefore, is the only candidate for optimal steady state (OSS). The OSS determines the corresponding control values  $\alpha$  and  $\mu$ , and instantaneous total cost  $ax + b\alpha x^2 y + c\beta xy + u(\alpha) + v(\mu)$ .

The OSS for the base parameter values is calculated numerically to be  $x = 0.1836748$ ,  $y = 0.0079036$ ,  $\alpha = 12.4186302$ ,  $\mu = 0.0954226$ . Comparing with the steady state for the base parameters (see Table 1) we observe that traditional criminal justice control is exercised aggressively, leading to about 10% fewer offenders and substantially lower levels of both the general perception of violence (37% lower) and of lethal violence between high-rate offenders (36% lower) in OSS.

Table 4 summarizes numerical results concerning the sensitivity of the OSS to variation in the parameter values. The numbers in the table represent the elasticity of the OSS-component with respect to the parameter. E.g., the upper-leftmost number indicates that the OSS number of offenders  $\hat{x}$  would increase by 0.68% if the inflow of such offenders  $k$  increased by 1%. Elasticity information is local. Sometimes the relationship changes when the parameter moves a modest amount away from its base values. When the sign of the dependence of the OSS-component on the parameter changes if the parameter is varied by less than 20%, that is indicated by an arrow. E.g., the downward-pointing arrow to the right of the lower-leftmost number in the table indicates that if the parameter  $\rho_\mu$  increases by 20%, then the OSS-component  $\hat{x}$  becomes

decreasing rather than increasing in the parameter. In most cells there are no arrows. We refer to those as definite cases. They are dependencies about which relatively robust conclusions can be drawn from the sensitivity analysis.

For ease of exposition, we will refer to efforts to reduce  $\alpha$  as violence prevention interventions, although it should be remembered that they represent a range of policy interventions (education, policing, legal changes, etc.).

More myopic decision makers (those with higher values of  $r$ ) invest somewhat less in traditional criminal justice interventions such as incarceration that increase  $\mu$  and substantially more in violence prevention. Naturally this leads to more high-rate offenders in the long run. Interestingly, it also increases the OSS reputation for violence  $y$  and the level of violence  $\alpha x^2 y$ .

This result may seem odd at first. One typically finds that short-sighted policy makers prefer incarceration to prevention, but the apparent paradox is easily explained. In this case, prevention does not imply long lags before benefits accrue. Prevention in this model reduces violence immediately, but in the long run it carries a cost because more offenders remain to commit crime in the future.

Not surprisingly, a higher forgetting rate,  $\delta$ , leads to a lower reputation for violence (notice the very high negative elasticity of  $y$  – more than -4%), but the number of offenders at OSS is larger because there are fewer lethal interactions. Effects on control spending are not definite.

Decision makers who place a lower value on offenders' lives,  $b$ , spend significantly less on violence prevention (elasticity of 4.29%). This creates a much higher ambient sense of violence (elasticity of  $y$  is -4.57%) and level of violence (elasticity of  $\alpha x^2 y$  is -4.69%), so the OSS number of offenders declines even though efforts to remove such offenders through traditional interventions is reduced. In effect, decision makers who do not care whether violent offenders kill each other will allow them to do so in order to reduce the amount of crime suffered by others. Similarly, decision makers who value highly preventing general crime committed by these offenders (high  $a$ ) will spend more on traditional criminal justice measures and less on violence prevention.

When  $b = c$ , increases in the proportion of youth who becomes high-rate offenders (increases in  $k$ ) have predictable effects on the OSS number of offenders, spending on traditional criminal justice interventions, violence prevention efforts, and total social costs (all increase), but the effects on OSS levels of perceived violence are less obvious, increasing at first, but decreasing again for large increases in  $k$ . However, when decision makers place a lower value on offenders' lives ( $b < c$ ), increasing the inflow of high-rate offenders can actually decrease the OSS number of offenders because the decision maker will curtail spending on violence prevention and allow the violence to escalate to the point that offenders are killing each other in large numbers.

When either  $a$  (cost of general crime) or  $c$  (cost of homicides of non-offenders) increase, spending on traditional criminal justice interventions ( $\mu$ ) increases, but not so spending on violence prevention. Increasing concern about violent victimization (increasing  $c$ ) leads to greater spending on violence prevention ( $u(\alpha)$ ) and lower levels of perceived violence ( $y$ ). However, increased concern about other types of crime (increasing  $a$ ) has the opposite effect.

### 3.3 Oscillating Optimal Behaviour

As we mentioned in Section 2.3 even a simple linear feedback can damp the oscillations considerably. The optimal control, however, need not possess this damping property, in general. We found numerically a number of Hopf bifurcations for many of the parameters of the canonical system (consisting of the equations (1),(2),(10),(11),(12)). All of them are subcritical, giving rise, therefore, to unstable periodic solutions. We mention that the parameter values for which we detect the possibility of cyclic optimal behaviour are not close to the base values.

Figure 2 represents a trajectory, which starts very close to an unstable cycle. Despite the fact that it converges to the stable equilibrium, it is highly oscillating. The discount rate has the nonrealistic value  $r = 0.407$  in this example.

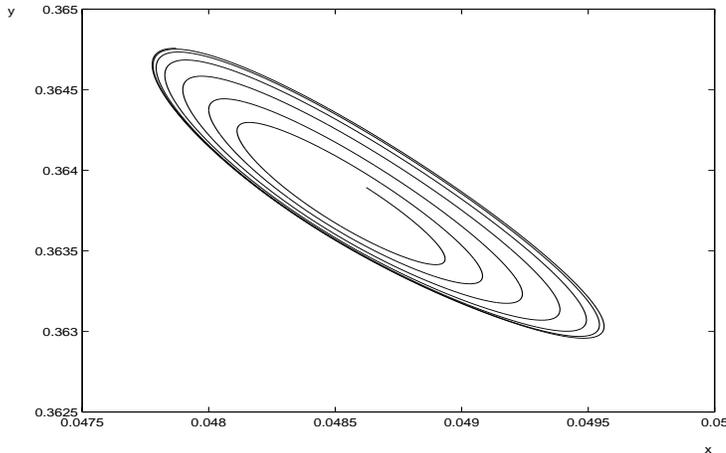


Figure 2: Oscilating optimal trajectory starting near the unstable cycle.

The corresponding time paths are plotted on Figure 3. For better representation, the values of  $x$ ,  $y$ ,  $\alpha$  and  $\mu$  are normalized in the intervals  $[0, 0.7]$ ,  $[0, 0.6]$ ,  $[0, 1]$ , and  $[0, 0.9]$ , respectively. The time interval of 40 years is somewhat longer than the period, which is about 31 years.

### 3.4 From Infinite to Finite Horizon

The sensitivity analysis above pertained only to the steady state values. Next we turn to the analysis of the optimal trajectories whose sensitivity may vary over time.

In order to cope with the optimal control problem (7),(1),(2),(9) numerically, we shall approximate it with a problem on a finite time horizon  $[0, T]$  with objective function

$$e^{-rT} g(x(T), y(T)) + \int_0^T e^{-rt} [ax(t) + b\alpha(t)x^2(t)y(t) + c\beta x(t)y(t) + u(\alpha(t)) + v(\mu(t))] dt. \quad (13)$$

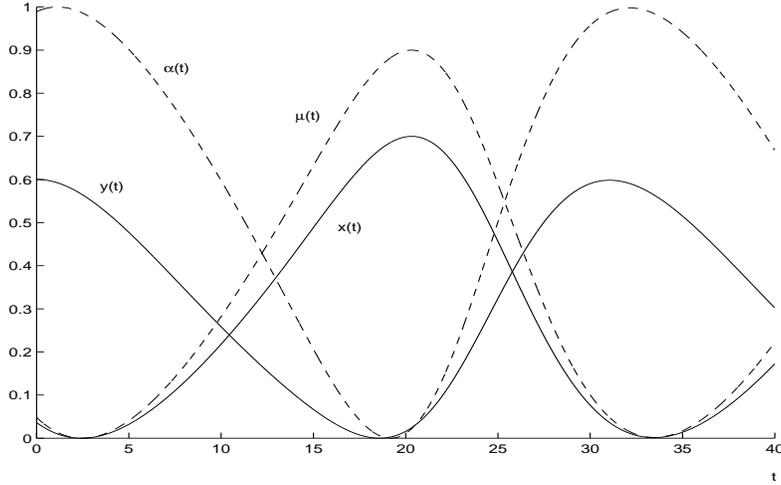


Figure 3: Periodic optimal paths (normalized).

The idea for an appropriate salvage function  $g$  is the following. Denote by  $E(\alpha, \mu)$  the stable equilibrium of (1),(2) in the first quadrant, corresponding to control values  $\alpha$  and  $\mu$ . From the expression for the equilibrium  $(\hat{x}, \hat{y})$  and the stability condition (3) it is easy to see that the image of the positive values of  $\alpha$  and  $\mu$  covers the region  $D = \{(x, y) : x < \delta/\beta, y < k/(\delta - \beta x)\}$ . Thus the set  $D$  consists of stable equilibria only, and for any  $(x, y) \in D$  the inverse  $E^{-1}(x, y)$  is uniquely defined.

Now we may pose the following optimal control problem on  $[0, T]$ : starting from a given equilibrium point  $(x_0, y_0) \in D$ , shift the system to a new equilibrium point  $(x_T, y_T) \in D$  (not specified in advance) in such a way that the cost of the transition to  $(x_T, y_T)$  plus the cost of staying at  $(x_T, y_T)$  on  $(T, +\infty)$  is minimal. Since staying at  $(x_T, y_T)$  means applying control  $E^{-1}(x_T, y_T)$ , one can easily calculate the second component of the total cost (that is, the first summand in (13)) as

$$g(x, y) = \frac{1}{r}q(x, y, E^{-1}(x, y)) \quad (14)$$

where  $q(x, y, \alpha, \mu) = ax + b\alpha x^2 y + c\beta xy + u(\alpha) + v(\mu)$ .

**Proposition 3** *Let  $J_\infty$  be the optimal value of the infinite horizon problem (7),(1),(2),(9), and  $J_T$  be the optimal value of the problem (13),(1),(2),(9) on  $[0, T]$  with  $g$  given by (14). Then*

(i)  $J_{T_1} \geq J_{T_2} \geq J_\infty$  for all sufficiently large  $T_1 \leq T_2$ ;

(ii) if the infinite horizon problem has an optimal trajectory that converges to a steady state belonging to  $D$ , then

$$\lim_{T \rightarrow +\infty} e^{rT}(J_T - J_\infty) = 0.$$

The first claim is obvious, since the finite horizon problem differs from the infinite horizon one only with the additional restriction that the control has a constant value on  $[T, +\infty)$ , determined by its history until  $T$  (namely, the value  $E^{-1}(x_T, y_T)$ ). The proof of the second claim is also simple (thanks to the invertibility of  $E$ ), but somewhat technical, and we skip it.

The essence of the second claim is that the convergence of  $J_T$  to  $J_\infty$  is *faster* than the convergence of the discount factor  $e^{-rT}$  to zero.

Our computer experiments with the violence model showed deviation of  $J_T$  from  $J_\infty$  well fitted by the exponent  $33e^{-0.39T}$ , which is significantly faster convergence than the discount factor  $e^{-0.04T}$ . In particular, for  $T = 20$  the deviation  $J_T - J_\infty$  is less than 0.04% of  $J_\infty$ . Therefore, in the numerical sensitivity analysis in the next subsection we employ the problem (13) on a 20-years time horizon.

### 3.5 Dynamic Sensitivity Analysis of the Optimal Policies

Starting with base case parameter values and initial conditions  $x(0) = \bar{x}$ ,  $y(0) = \bar{y}$  (the steady state of the uncontrolled model) the optimal solution to the finite time horizon approximation to the full problem smoothly approaches the OSS. (Figure omitted.) The decision maker initially employs both controls aggressively, and both the number of offenders and the climate of violence declines. Control spending is eased back and the decline in the number of offenders plateaus after about ten years, but the climate of violence continues to ebb for the full twenty years. There is little overshoot in the optimal trajectories with the base case parameters. I.e., the OSS is approached almost monotonically, in contrast with the approach to the uncontrolled steady state depicted in Figure 1a.

We investigated numerically the sensitivity of the optimal dynamic solution (controls, trajectories, shadow prices, and total cost) of the finite horizon problem with respect to the parameters. This analysis is complementary to the sensitivity analysis of the OSS, and in most cases simply affirms the earlier findings. E.g., increasing the discount rate  $r$ , making the decision maker more myopic, in most cases simply shifts trajectories in the direction indicated by Table 3 without changing their shape. In particular, spending on violence prevention is higher, spending on traditional criminal justice interventions is lower, and the number of offenders is higher at each point in time over the 20 year planning horizon. The only exception is that with a higher discount rate the optimal climate of violence is lower for the first approximately 18 years, not higher, as it is beyond 18 years, including at the OSS as is indicated by Table 3. Similarly, increasing the inflow of offenders  $k$  increases the OSS climate of violence, but it may reduce the optimal level of violence in the earlier years.

Two interesting exceptions pertain to the valuation of the life of a high rate offender,  $b$ , and the rate of forgetting,  $d$ . Decision makers who place a relatively low value on offenders' lives ( $b < c$ ) will reduce spending on violence control, increase spending on traditional criminal justice interventions, and reduce the number of offenders throughout the planning horizon in a fairly

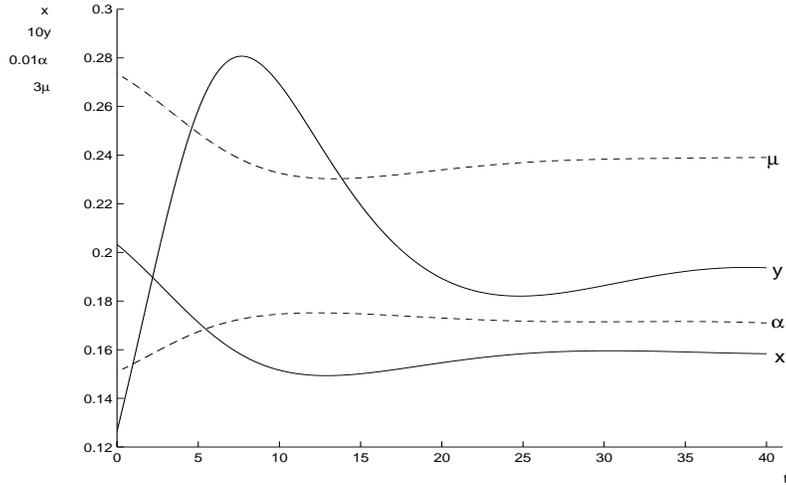


Figure 4: The optimal controls and trajectories for  $b = 1332$

smooth manner. They will likewise tolerate higher levels of violence among offenders throughout the planning period, but the trajectory is not monotonic. With modest ( $\approx 30\%$ ) reduction in  $b$ , the climate of violence is allowed to spike sharply around year 6 or 7 before ebbing. Interestingly, when one examines a slightly longer time horizon, one sees evidence of oscillation, which is not present with the base case parameters. (See Figure 4, which scales the trajectories so they can be displayed together.)

It is interesting to observe how well the linear feedback  $\mu = 1/3x + 0.025$  approximates the optimal control  $\mu$ . (This corresponds to shifting the graph of  $x$  up by 0.075 units since Figure 4 already includes the  $1/3$  scaling.) It thus seems plausible that a simple policy of linearly increasing incarceration rates (per offender per unit time, not just per unit time) when there are many offenders might be a reasonable approximation to the optimal policy. There is no similar relation between  $\alpha$  and  $y$ .

Changes in the forgetting rate,  $\delta$ , have more complicated effects, as is foreshadowed by the number of entries in Table 3 for  $\delta$  that are not definite. Increasing  $\delta$  simply shifts most of the trajectories (although it introduces non-monotonicity in the trajectory for the number of offenders), but the longer memories associated with smaller values of  $\delta$  invite "investment" in dramatic changes in the early years, which are then carried forward by the stability of perceptions of violence when memories are long. These patterns are illustrated in Figures 5 and 6.

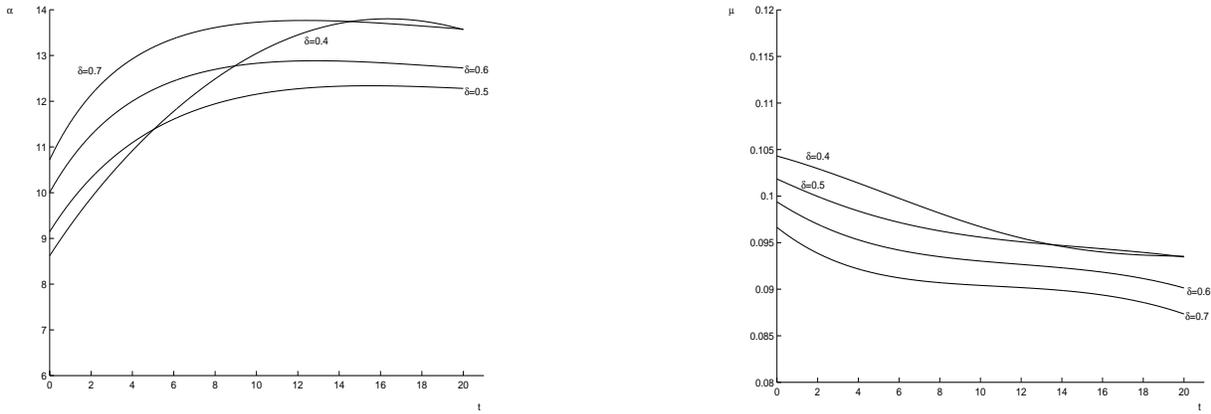


Figure 5: The optimal controls  $\alpha(t)$  (left) and  $\mu(t)$  (right) for different  $\delta$ .

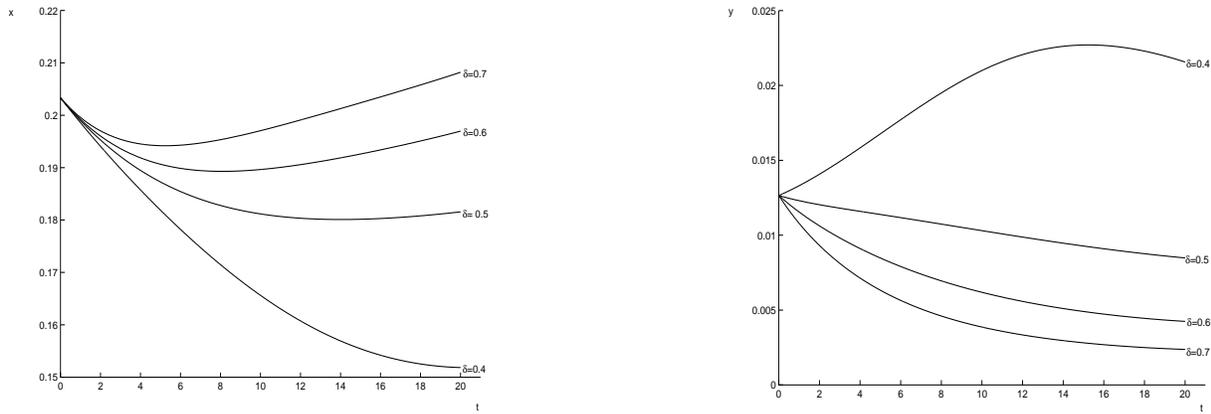


Figure 6: The optimal trajectories  $x(t)$  (left) and  $y(t)$  (right) for different  $\delta$ .

## 4 Conclusions

Violence varies over time in ways that are not well explained by exogenous factors such as changes in the unemployment rate, suggesting that there may be internal dynamics at work. One suggestion for such a dynamic is that violence begets violence, in the sense that a general climate of violence promotes hair trigger responses, preemptive strikes, and carrying of more powerful weapons, but this positive feedback is ultimately contained in a very direct manner. Since most crime occurs within rather than between groups, most homicides are of people who themselves have criminal proclivities and high rates of lethal violence may reduce the very population that perpetrates such violence.

We formalized this conjecture in a simple dynamic model that indeed does display oscillations in numbers of offenders, levels of violence, and perceptions of violence. For our base case parameter values the oscillations are damped, suggesting that this dynamic alone could not generate variation in perpetuity. However, the determinants of violence are clearly many, and this

dynamic in conjunction with other factors could explain the historical conundrum of substantial temporal variation that is not significantly correlated with the socio-demographic and economic factors theory suggests ought to act as exogenous shocks. According to this dynamic, when such exogenous shocks occur, they create a non-monotonic response in behavior. Inasmuch as statistical investigations of the historical relationship between these exogenous factors and violence rates have sought not only monotonic but actually linear relationships among the variables, this dynamic could account for the failure of those statistical studies to find clear evidence of links.

The controlled version of this model presents a Faustian dilemma. The killing of a high-rate offender is clearly bad in and of itself, but it does bring a perverse sort of "benefit" by averting some offenses. Hence, one way to protect the public from victimization is to refrain from interventions that reduce the likelihood that interactions among offenders turn lethal. Policy makers who view homicides as equally costly, regardless of the victim's criminal tendencies, would not exercise this option, but less egalitarian decision makers could, in a purely mathematical sense, improve their definition of social welfare by allowing such killings. Interestingly, this strategy generates oscillatory behavior, whereas with the base case parameter values control dampens oscillations.

The prescription concerning traditional criminal justice interventions that remove offenders is more straightforward. The optimal removal rate, per offender per unit time, increases (almost linearly) in the number of offenders. (And, hence, the optimal rate of imprisonment, per unit time, grows supra-linearly in the number of offenders.)

A natural extension would be to relax the assumption that the flow into the population of high-rate offenders is constant. In particular, it would be interesting to observe the conditions under which this deterministic but nonlinear model does and does not generate relationships between a forcing function (such as the inflow of offenders) and modeled rates of crime or violence that would be easy to detect with linear statistical models.

## References

- [1] D.A. Behrens, J.P. Caulkins, G. Tragler, and G. Feichtinger. Why present-oriented societies undergo cycles of drug epidemics. to appear in *Journal of Economic Dynamics and Control*, 2001
- [2] A. Blumstein, J. Canela-Cacho, and J. Cohen. Filtered Sampling from Populations with Heterogeneous Event Frequencies. *Management Science*, 39(7):886–899, 1993.
- [3] A. Blumstein, F.P. Rivara, and R. Rosenfeld. The rise and decline of homicide—and why. *Annu. Rev. Public Health*, 21:505–541, 2000.
- [4] J.J. Donohue III and S.D. Levitt. The impact of legalized abortion on crime. *Quarterly Journal of Economics*. 116(2), 2001.

- [5] P.V. Greenwood et al. *Three Strikes and You're Out: Estimated Benefits and Costs of California's New Mandatory Sentencing Law*. MR-509-RC, RAND, Santa Monica, CA, 1994.
- [6] K. Maguire and A.L. Pastore (eds.). *Sourcebook of Criminal Justice Statistics, 1997*. USGPO, Washington, DC, 1998.
- [7] T.R. Miller, M.A. Cohen, and B. Wiersma. *Victim Costs and Consequences: A New Look*. National Institute of Justice Research Report, 155282, Feb., 1994.
- [8] C.S. Widom and M.G. Maxfield. *An Update on the "Cycle of Violence"*. National Institute of Justice Research in Brief, USGPO, Washington, DC, 2001.