Why Politics Makes Strange Bedfellows: A Dynamic Model with DNS Curves

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Abstract

We analyze a two dimensional system which has three equilibria in the uncontrolled version. After adding a control variable, two more equilibria occur and Skiba curves can be analyzed. In this model it is possible to derive under what conditions each of the different equilibria is a saddle point, a node, or a focus. In particular, for certain parameter ranges all five equilibria have real eigenvalues. In this case, the Skiba curves can be computed in a more straightforward way than usual.

1 Introduction

It is often said that "politics makes strange bedfellows" meaning that people who originally seem to hold quite different beliefs can end up adopting similar positions. Likewise, the notion that "all's fair in love, war, and politics" suggests that another politician coming from a similar perspective is not a guarantee that one can count on that individual's support down the road. Translating these aphorisms into mathematical terms, they imply a complicated, nonlinear, and even discontinuous mapping between the space describing politicians' "original" or "true" beliefs and their ultimate positions.

It is not easy to explain such behavior if the "issue space" is one-dimensional. The first issue is the median voter theorem which was first presented by Hotelling in 1929 (Ref. 1) and became part of the new Public Choice theory through the work of Black 1948 (Ref. 2).The median voter theorem prescribes that in a two-party system the parties should move on the political spectrum (e.g., from liberal to conservative) toward the position of the median voter, and the expectation is that voters will support the party that most closely represents their view. E.g., all voters who are more conservative than the most conservative of the two parties will support that party. Such a model is simple and continuous in its behavior.

Introducing a second issue makes it easier to explain why these aphorisms are apropos. For example, "log-rolling" or vote trading can lead to "unholy alliances". Someone may vote against their conscience on one bill in exchange for getting someone else to vote with them on a second bill. This is a partially but not entirely satisfying explanation because there is a sense that politics makes strange bedfellows not just with respect to votes on particular bills but also with respect to general positions taken and held over time.

This paper seeks to augment the "log-rolling" explanation for why politics can lead to counter-intuitive mappings between original and final positions with a dynamic story that describes the evolution of positions politicians take on a two-dimensional issue space. The final positions allowed in the model are very simple: for an issue, against it, or a middling position. But the trajectories leading to those positions are anything but simple. The model solution involves multiple DNS curves that sharply separate the two-dimensional issue space into sets of original beliefs that lead to each final position. Furthermore, those DNS curves take the form of spirals that draw some people from each quadrant in the original belief space into each of the three final positions.

From a mathematical point of view, we analyze a two dimensional dynamical system which has three equilibria in the uncontrolled version, two of which are centers and one of which is a saddle point. After adding a control variable, two more equilibria can occur.

In this model it is possible to derive under what conditions each of the different equilibria is a saddle point, a node, or a focus. In particular, for certain parameter ranges all five equilibria have real eigenvalues. In this case the DNS curves (also called Skiba curves) can be computed in a more straightforward way than usual. They are simply the projection of the one-dimensional stable manifolds in the two new "threshold"-equilibria into the state space.

For other parameter values, the two new "threshold"-equilibria have complex eigenvalues and the stable manifolds of the three "stable" equilibria overlap. In this case the DNS curves could only be analyzed by evaluating the objective function value of both candidates in the overlap region.

2 The Model

Politicians take positions simultaneously on many issues, but since we are seeking a parsimonious explanation for the notion that "politics makes strange bedfellows" we consider a simple two-dimensional issue space. The vertical dimension (y) denotes how the individual is perceived by voters on a general spectrum ranging from conservative ($y < 0$) to liberal ($y > 0$). The horizontal dimension (x) denotes how the individual is perceived with respect to a particular "litmus test" issue. Without loss of generality, suppose the left side of the spectrum ($x < 0$) is associated with liberal views, and the right side ($x > 0$) is associated with conservative views. Suppose further that there are three generally recognized positions with respect to this "litmus test" issue: for, against, or middle of the road. Code the "liberal" position as $x = -1$, the conservative position as $x = +1$, and the moderate position as $x = 0$. (Examples of issues might include abortion: pro-choice, pro-life, and moderate; drug policy: hawkish, pro-legalization, and favoring reform within a prohibition; war: hawkish, pacifist, and "support the troops" but not necessarily the policy; etc.)

Consistent with the median voter theorem, we suppose that politicians seek to minimize the extent to which they are generally perceived of as being "too conservative" or "too liberal" (i.e., y different than 0), and the extent to which they must exercise "spin control" over their positions on litmus test issues. The notion of spin control here is effort above and beyond what politicians routinely do to manage public perceptions of their positions.

Our image of the "natural" evolution of public perception of a politician's position has two parts, corresponding to the two dimensions of our issue space. First, we assume that people expect politicians with a generally liberal (conservative) outlook to "move left" (right) on important litmus test issues. I.e.,

$$\dot{x} = -y.$$

The evolution of perceptions is more complicated with respect to litmus test issues. If the politician is perceived as having moderate support for the liberal (conservative) view on the litmus test issue, the politician's general reputation for being liberal (conservative) will rise. I.e.,

$$\text{for } |x| < 1 \text{ we have } Sgn(\dot{y}) = Sgn(-x).$$

However, if the public labels the politician as an extremist on the litmus test issue, the politician will generally seek to avoid being marginalized at the edge of the general (y) political spectrum by adopting contrary positions in other areas. E.g., someone seen as favoring a radical environmental agenda may take pains to be business friendly in other arenas, and someone adamantly opposed to gun control may aggressively court "Soccer Moms" in other ways. To the extent that this is true, it implies that

$$\text{for } |x| > 1 \text{ we have } Sgn(\dot{y}) = Sgn(x).$$

A simple functional form that captures this flipping behavior is $\dot{y}$, i.e.

$$\dot{y} = x^3 - x = (x+1)\,x\,(x-1)\,.$$

Figure 1 shows typical trajectories (computed-numerically) of the uncontrolled system:

$$\begin{aligned} \dot{x} &= -y, & (1) \\ \dot{y} &= (x+1)\,x\,(x-1)\,. & (2) \end{aligned}$$

There are three equilibria at $(x, y) = (-1, 0)$, $(0, 0)$, and $(1, 0)$. The first and third are centers, and the origin is a saddle point.

Insert Figure 1 here

3 The Control Problem

So far we have not introduced any control variables. In the interest of simplicity, we assume there is just one control - effort exerted (e.g., through media spots or mass mailings), v, that seeks to position the candidate in a moderate place on the general political spectrum. The politician's objective is assumed to be to reach and maintain such a moderate perception while expending the least possible effort on these expensive outreach campaigns, subject to the usual discounting to reflect time preference. Hence, our model of the politician's problem is to

$$\max_{v} \int_0^\infty e^{-rt}\left(-y^2 - kv^2\right) dt$$

s.t.

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= (x+1)\,x\,(x-1) + v, & (3) \\ x(0) &= x_0, \\ y(0) &= y_0. \end{aligned}$$

where r is the discount rate and k is a positive constant which represents the relative cost of expending effort vs. being perceived of as out of the mainstream.

Since the objective function penalizes deviations from the three equilibria and control efforts, clearly staying in one of the three equilibria is costless, so these remain equilibria in the controlled problem. It is not surprising that these will be saddle point stable.

We now analyze this OC-problem (3) using the maximum principle; see, e.g., Feichtinger and Hartl (Ref. 3).

To do this, we write down the Hamiltonian as

$$H = -y^2 - \frac{1}{2}kv^2 - \lambda y + \mu\left((x+1)\,x\,(x-1) + v\right).$$

Since H is strictly concave in the control and there are no control constraints, the Hamiltonian maximizing condition is $H_v = -kv + \mu$, i.e.

$$v = \frac{\mu}{k}.$$

The adjoint equations are

$$\begin{aligned} \dot{\lambda} &= r\lambda - D_x H = r\lambda - \mu\left(3x^2 - 1\right), \\ \dot{\mu} &= r\mu - D_y H = r\mu + 2y + \lambda. \end{aligned}$$

and the canonical system is therefore

$$\begin{aligned} \dot{x} &= -y, & (4) \\ \dot{y} &= (x+1)\,x\,(x-1) + \frac{\mu}{k}, & (5) \\ \dot{\lambda} &= r\lambda - \mu\left(3x^2 - 1\right), & (6) \\ \dot{\mu} &= r\mu + 2y + \lambda. & (7) \end{aligned}$$

In the next section, we will look for equilibria of the canonical system.

4 The Equilibria and their Stability

From (4) to (7) we conclude that all equilibria must satisfy

$$\begin{aligned} y &= 0, & (8) \\ (x+1)\,x\,(x-1) + \frac{\mu}{k} &= 0, & (9) \\ r\lambda &= \mu\left(3x^2 - 1\right), & (10) \\ r\mu &= -\lambda. & (11) \end{aligned}$$

On the one hand, we have the three old equilibria (of the uncontrolled system) (1), $x = 0$, $x = 1$, and $x = -1$ which are more precisely:

$$\begin{aligned} x &= -1,\ 0,\ \text{or}\ 1, & (12) \\ y &= 0, \quad \lambda = 0, \quad \mu = 0. & (13) \end{aligned}$$

On the other hand, if x is not 0 or ± 1, then μ is not zero; cf. (9) which, by (11) implies $\lambda \neq 0$. Then (10) and (11) yield

$$-r^2 = 3x^2 - 1. \qquad (14)$$

Thus, for $r < 1$, there are two new equilibria:

$$\begin{aligned} x &= \sqrt{\frac{1-r^2}{3}},\ y = 0, & (15) \\ \mu &= \frac{k}{3}\left(2 + r^2\right)\sqrt{\frac{1-r^2}{3}}, & (16) \\ \lambda &= -\frac{kr}{3}\left(2 + r^2\right)\sqrt{\frac{1-r^2}{3}} & (17) \end{aligned}$$

(note that $(x+1)\,x\,(x-1) = -\frac{1}{3}\left(2+r^2\right)\sqrt{\frac{1-r^2}{3}}$ here), and

$$\begin{aligned} x &= -\sqrt{\frac{1-r^2}{3}},\ y=0, &(18)\\ \mu &= -\frac{k}{3}\left(2+r^2\right)\sqrt{\frac{1-r^2}{3}}, &(19)\\ \lambda &= \frac{kr}{3}\left(2+r^2\right)\sqrt{\frac{1-r^2}{3}}. &(20) \end{aligned}$$

In the next section, we will investigate the stability properties of the "old" and "new" equilibria.

The Jacobian of the canonical system (4) to (7) is

$$i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 3x^2-1 & 0 & 0 & \frac{1}{k} \\ -6\mu x & 0 & r & 1-3x^2 \\ 0 & 2 & 1 & r \end{bmatrix}, \tag{21}$$

which we can evaluate in the 5 equilibria.

4.1 The Three Old Equilibria

4.1.1 Equilibrium $x = 0$

The eigenvalues of (21) which simplifies to

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & \frac{1}{k} \\ 0 & 0 & r & 1 \\ 0 & 2 & 1 & r \end{bmatrix} \tag{22}$$

are

$$\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{1+k \pm \sqrt{1+2k+k^2r^2}}{k}}, \tag{23}$$

and we derive:

Proposition 1. *For all reasonable parameter values (i.e. for $r < 1$), this equilibrium $x = 0$ has two positive and two negative real eigenvalues.*

Proof. Clearly $1 + 2k + k^2r^2$ is always positive and

$$\frac{r^2}{4} + \frac{1+k-\sqrt{1+2k+k^2r^2}}{k} > 0$$

is equivalent to

$$\begin{aligned} kr^2+4+4k-4\sqrt{(1+2k+k^2r^2)} &> 0 \Longleftrightarrow \\ \left(kr^2+4+4k\right)^2 - 16\left(1+2k+k^2r^2\right) &> 0 \Longleftrightarrow \\ k\,(r-2)^2\,(r+2)^2 + 8r^2 &> 0. \end{aligned}$$

Thus, equilibrium $x = 0$ always has real eigenvalues. Furthermore we have

$$1 + k - \sqrt{1 + 2k + k^2 r^2} > 0$$

because of

$$\begin{aligned} (1+k)^2 - \left(1 + 2k + k^2 r^2\right) &> 0, \\ -k^2 (r-1)(r+1) &> 0, \end{aligned}$$

which is equivalent to $r < 1$ since only positive discount rates make sense. Thus,

$$\begin{aligned} \frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{1 + k + \sqrt{1 + 2k + k^2 r^2}}{k}} &> 0, \\ \frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{1 + k - \sqrt{1 + 2k + k^2 r^2}}{k}} &> 0, \\ \frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{1 + k + \sqrt{1 + 2k + k^2 r^2}}{k}} &< 0, \\ \frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{1 + k - \sqrt{1 + 2k + k^2 r^2}}{k}} &< 0. \end{aligned}$$

■

We note that by introducing the control v into the dynamical system has not changed the stability property of the equilibrium $x = y = 0$. It remained a saddle point, now in the four dimensional space.

4.1.2 Equilibria $x = -1$ and $x = 1$

The eigenvalues of (21) which simplifies to

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & \frac{1}{k} \\ 0 & 0 & r & -2 \\ 0 & 2 & 1 & r \end{bmatrix} \tag{24}$$

are

$$\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{1 - 2k \pm \sqrt{1 - 4k - 2k^2 r^2}}{k}}, \tag{25}$$

and we get:

Proposition 2. *In the equilibria $x = \pm 1$ two cases can occur:*
a) for small r and k, i.e. for $k < \frac{1}{4}$ *and* $r < \frac{\sqrt{(2-8k)}}{2k}$ *we have two positive and two negative eigenvalues,*
b) otherwise all eigenvalues are conjugate complex (with two positive and two negative real parts)

Proof. Here, $1-4k-2k^2r^2>0$ for $0<k<\frac{-1+\sqrt{1+\frac{1}{2}r^2}}{r^2}$ or $0<r<\frac{\sqrt{(2-8k)}}{2k}$. Otherwise (i.e. for large r and/or k) all eigenvalues are complex; see also Figure 2.

Insert Figure 2 here

If $1-4k-2k^2r^2>0$, then also the last two eigenvalues are real:

$$\begin{aligned} \frac{r^2}{4}+\frac{1-2k-\sqrt{1-4k-2k^2r^2}}{k} &> 0 \Longleftrightarrow \\ kr^2+4-8k-4\sqrt{1-4k-2k^2r^2} &> 0, \end{aligned}$$

which can only be true for $kr^2+4-8k>0$, i.e. for $k<\frac{4}{8-r^2}$. Then we can take squares and get

$$\begin{aligned} \left(kr^2+4-8k\right)^2 &> 16\left(1-4k-2k^2r^2\right), \\ r^4k^2+8r^2k+16k^2r^2+64k^2 &> 0, \end{aligned}$$

which is always satisfied. We note that for all reasonable values of r, condition $kr^2+4-8k>0$ follows from $1-4k-2k^2r^2>0$.

Let us now obtain the signs of the eigenvalues (if they are real). For this we show

$$1-2k-\sqrt{(1-4k-2k^2r^2)}>0.$$

For real eigenvalues, we always have $k<2$ (see figure above) and we can take squares:

$$\begin{aligned} (1-2k)^2-\left(1-4k-2k^2r^2\right) &> 0 \Longleftrightarrow \\ 4k^2+2k^2r^2 &> 0, \end{aligned}$$

which is always true.

Thus, for $0<k<\frac{-1+\sqrt{1+\frac{1}{2}r^2}}{r^2}$ or $0<r<\frac{\sqrt{(2-8k)}}{2k}$ we have obtained the signs:

$$\begin{aligned} \frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{1-2k+\sqrt{1-4k-2k^2r^2}}{k}} &> 0, \\ \frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{1-2k-\sqrt{1-4k-2k^2r^2}}{k}} &> 0, \\ \frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{1-2k+\sqrt{1-4k-2k^2r^2}}{k}} &< 0, \\ \frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{1-2k-\sqrt{1-4k-2k^2r^2}}{k}} &< 0. \end{aligned}$$

■

Thus, introducing the control v into the dynamical system has changed the stability property of the equilibria $x = \pm 1$, $y = 0$. They are now saddle points in the four dimensional space (if r and k are not too large; otherwise they are foci).

4.1.3 Numerical Examples

All saddle points in ± 1 For $r = 0.15$ and $k = 0.2$ the inner root in (25) is positive, $1 - 4k - 2k^2 r^2 = 0.1982 > 0$ and all eigenvalues are real:

The eigenvalues in 0 are -3.3781, -0.22105, 0.37105 and 3.5281, while the eigenvalues in ± 1 are -2.2123, -0.80797, 0.95797 and 2.3623.

All complex eigenvalues in ± 1 For $r = 0.15$ and $k = 0.3$ the inner root in (25) is negative, $1 - 4k - 2k^2 r^2 = -0.20405 < 0$ and all eigenvalues are complex:

The eigenvalues in 0 are -2.8504, -0.2713, 0.4213 and 3.0004, while the eigenvalues in ± 1 are $-1.22 \pm 0.58137i$ and $1.37 \pm 0.58137i$.

4.2 The Two New Equilibria

Clearly, it suffices to analyze $x = \sqrt{\frac{1-r^2}{3}}$, since the problem is symmetric around $x = 0$ and the stability of $x = -\sqrt{\frac{1-r^2}{3}}$ is the same. The Jacobian evaluated at $x = \sqrt{\frac{1-r^2}{3}}$ and $(x+1)\,x\,(x-1) + \frac{\mu}{k} = 0$ is

$$i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 3x^2 - 1 & 0 & 0 & \frac{1}{k} \\ -6\mu x & 0 & r & 1 - 3x^2 \\ 0 & 2 & 1 & r \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -r^2 & 0 & 0 & \frac{1}{k} \\ -\frac{2}{3}k\left(1 - r^2\right)\left(2 + r^2\right) & 0 & r & r^2 \\ 0 & 2 & 1 & r \end{bmatrix}$$

with eigenvalues:

$$\frac{r}{2} \pm \frac{\sqrt{45k^2r^2 + 36k \pm 12k\sqrt{3k^2(r^4 - 2r^2 + 4) + 18kr^2 + 9}}}{6k}$$
$$= \frac{r}{2} \pm \sqrt{\frac{15kr^2 + 12 \pm 4\sqrt{3k^2(r^4 - 2r^2 + 4) + 18kr^2 + 9}}{12k}}.$$

We derive:

Proposition 3. *For reasonably small discount rates ($r < 0.89725$) two cases can occur in the new equilibria $x = \pm\sqrt{\frac{1-r^2}{3}}$:*
a) for very small k, i.e. for $k < \frac{24r^2}{64 - 32r^2 - 59r^4}$ *we have all real eigenvalues one being negative and 3 being positive.*
b) otherwise two eigenvalues are real and two are conjugate complex.

Proof. We note, that $r^4 - 2r^2 + 4 > 0$ and that the inner root is always real. Now,

$$\begin{aligned} 15kr^2+12-4\sqrt{3k^2\left(r^4-2r^2+4\right)+18kr^2+9} &> 0 \Longleftrightarrow \\ \left(15kr^2+12\right)^2-\left(4\sqrt{3k^2\left(r^4-2r^2+4\right)+18kr^2+9}\right)^2 &> 0 \Longleftrightarrow \\ 3k\left(k\left(59r^4+32r^2-64\right)+24r^2\right) &> 0 \end{aligned} \quad (26)$$

is always satisfied for large r, i.e. for $59r^4+32r^2-64>0$, which can be written as $r>\frac{2}{59}\sqrt{354\sqrt{7}-236}=0.897\,25$.

For smaller r, relation (26) is equivalent to

$$k<\frac{24r^2}{64-32r^2-59r^4},$$

i.e., for reasonably small discount rates r, the parameter k must be very small in order to allow for all real eigenvalues; See Figure 3.

Insert Figure3 here

In order to obtain the sign of the eigenvalues, provided that the are real, we observe from (26) that $\sqrt{\frac{15kr^2+12-4\sqrt{3k^2(r^4-2r^2+4)+18kr^2+9}}{12k}}$ is less than $\frac{r}{2}$ for $r<1$:

$$\sqrt{\frac{15kr^2+12-4\sqrt{3k^2\left(r^4-2r^2+4\right)+18kr^2+9}}{12k}}<\frac{r}{2}$$

is equivalent to (by taking squares)

$$\begin{aligned} 3kr^2+3 &< \sqrt{3k^2\left(r^4-2r^2+4\right)+18kr^2+9} \Longleftrightarrow \\ 9k^2r^4+18kr^2+9 &< 3k^2r^4-6k^2r^2+12k^2+18kr^2+9 \Longleftrightarrow \\ r^2\left(r^2+1\right) &< 2. \end{aligned}$$

It remains to show that $\sqrt{\frac{15kr^2+12+4\sqrt{3k^2(r^4-2r^2+4)+18kr^2+9}}{12k}}$ is bigger than $\frac{r}{2}$, which is rather straightforward (by taking squares):

$$\begin{aligned} \sqrt{\frac{15kr^2+12+4\sqrt{3k^2\left(r^4-2r^2+4\right)+18kr^2+9}}{12k}} &> \frac{r}{2} \Longleftrightarrow \\ 3kr^2+3+\sqrt{3k^2\left(r^4-2r^2+4\right)+18kr^2+9} &> 0. \end{aligned}$$

■

4.2.1 Numerical Examples

Two complex, two real eigenvalues For $r=0.15$ and $k=0.1$, Case a) occurs in Proposition 3: $k>\frac{24r^2}{64-32r^2-59r^4}=0.0085375$. The two new equilibria are $x=\pm\sqrt{\frac{1-r^2}{3}}=\pm 0.570\,82$ and the eigenvalues are: -4.4101, 4.5601

and $0.075 \pm 0.24477i$, which corresponds to the findings of Proposition 3. The eigenvector of the Jacobian

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -0.0225 & 0 & 0 & 10.0 \\ -0.1318 & 0 & 0.15 & 0.0225 \\ 0 & 2 & 1 & 0.15 \end{bmatrix}$$

w.r.t. the negative eigenvalue -4.4101 is $[0.28875,\ 1.2734,\ 0.011113,\ -0.56093]$. Thus, the "Skiba curves" (see the next section) are upward sloping as expected, since its slope is positive: $1.2734/0.28875 = 4.41 > 0$.

However we do not really know whether this unstable equilibrium is a point on the Skiba-curve, since - as in the one-dimensional case - there may be an overlap since the model is non-concave.

All real eigenvalues For $r = 0.5$ and $k = 0.1$, Case b) occurs in Proposition 3: $k < \frac{24r^2}{64-32r^2-59r^4} = 0.1147$. The two new equilibria are $x = \pm\sqrt{\frac{1-r^2}{3}} = \pm 0.5$ and the eigenvalues are: -4.2906, 0.16186, 0.33814 and 4.7906. The eigenvector of the Jacobian

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -0.25 & 0 & 0 & 10.0 \\ -0.1125 & 0 & 0.5 & 0.25 \\ 0 & 2 & 1 & 0.5 \end{bmatrix}$$

w.r.t. the negative eigenvalue -4.2906 is $[0.19836,\ 0.85108,\ 0.023456,\ -0.36021]$. Thus, the "Skiba curves" (see the next section) are upward sloping as expected, since its slope is positive: $0.85108/0.19836 = 4.2906 > 0$.

However we do not really know whether this unstable equilibrium is a point on the Skiba-curve, since - as in the one-dimensional case - there may be an overlap since the model is non-concave. This fact is checked numerically in the next section.

5 The DNS Curves

Let us first make a few introductory statements on comment on DNS curces or Skiba curves. In one-dimensional control problems, Skiba (1978, Ref. 4) has observed that multiple equilibria can occur, and that the long run optimal equilibrium is history-dependent in this case. This fact was more precicely analyzed by Dechert and Nishimura (1983, Ref. 5) which is why we call these curves DNS (Dechert - Nishimura - Skiba) curves (we note that in part of the litarature they were called Skiba curves). While most previous literature has assumed that Skiba points arise because of nonconcavities of the Hamiltonian and that an unstable equilibrium is always a focus, it has recently been shown that neither of these standard beliefs are true; see Feichtingeret al. (2000, Ref. 6) and Hartl et al. (2000, Ref. 7).

In control problems with two-dimensional state space, similar effects will occur in case of multiple equilibria, where the threshold between the basins of attraction of the different equilibria are now one-dimensional Skiba curves in the two-dimensional state space. Contrived Examples for such Skiba curves were obtained some fifteen years ago by juxtaposing and perturbing two one-dimensional systems with DNS points (Dechert, personal communication, May 27th, 2000). But only recently have two-dimensional DNS curves emerged in "natural" problems (Haunschmied et al., 2000, Ref. 8), and previously published examples have had a much simpler structure.

We now compute the DNS- curves for our model. We select parameter values, for which all equilibria have real eigenvalues and numerically obtain the desired threshold curves as the projection of the one-dimensional stable manifold corresponding to the two new "threshold"-equilibria into the state space. This is done by solving the canonical system backwards in time starting near the new equilibria in the direction of the eigenvectors w.r.t. the (single) negative eigenvalue. The resulting curves are depicted in Figure 4.

Insert Figure 4 here

It may be interesting to check the behavior of these Skiba-curves for larger values of x and y. It turns out, that these look like spirals but they never intersect. This is depicted in Figure 5.

Insert Figure 5 here

We have also investigated numerically whether there can be overlaps of the stable manifolds w.r.t. the equilibria $x = -1$, $x = 0$, and $x = +1$. Computing the canonical system backwards in time starting from various points on the stable manifold in the neighborhood of the three saddle point equilibria (using the linearized system), it turned out that these trajectories remained all within the areas of attraction plotted in Figure 5. Thus, these curves are indeed threshold curves and could be called "Skiba"-curves.

6 Conclusions

We have proposed a simple model of the evolution of a politician's positions in a two-dimensional issue space. The solution (embodied in Figures 4 and 5) has three equilibria which represent being moderate overall ($y = 0$) but taking a position for, against, or in between (x = -1, +1, or 0) on some "litmus test" issue. Regardless of where the politician starts in this issue space, he or she always converges to one of those three positions. However, the optimal behavior is not to simply and directly approach the equilibrium that is closest to the politician's original position.

On the contrary, the basins of attraction to the three equilibria are separated by so-called DNS curves that spiral outward. A ray extending from the origin passes through alternating regions for which it is optimal to approach different equilibria. For example, a ray extending from the origin (most neutral position) into the second quadrant (original positions tending to be liberal both in general and with respect to the litmus test issue) pass through successive regions where it is optimal to converge to the equilibria that are neutral, liberal, neutral, conservative, neutral, liberal, neutral, conservative, etc.

This solution is highly consistent with the notion that "politics makes strange bedfellows." Politicians are drawn to any given final position (e.g., being conservative on the litmus test issue) from all four quadrants of the original issue space. Furthermore, the solution indicates that one cannot take for granted the final support of someone who initially seems to be closely aligned with one's views. If that other person lies on the other side of the DNS curve - which can be an arbitrarily small distance away - it is optimal for them to adopt a completely different position.

The solution is also striking mathematically. We have demonstrated a simple way of computing the threshold curves between the basins of attraction of multiple equilibria, and have in fact found two such two-dimensional DNS curves separating three equilibria.

Acknowledgments

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List of Figure Captions:

Figure 1. Typical trajectories of the uncontrolled system, $v = 0$. The separatrices (stable and unstable manifolds) are connected and are shown as a dotted line.

Figure 2. Below this curve we have four real eigenvalues, above all eigenvalues are complex in the equilibria $x = \pm 1$.

Figure 3. Below this curve we have four real eigenvalues, above two eigenvalues are complex and two are real in $x = \pm\sqrt{1-r^2}/\sqrt{3}$.

Figure 4. The Skiba-Curves for the parameters $r = 0.5$ and $k = 0.1$.

Figure 5. The Skiba-curves for larger values of x and y. The solid line separates the regions of attraction of the equilibria $x = 0$ and $x = +1$, while the dotted line separates the regions of attraction of the equilibria $x = 0$ and $x = -1$. The narrow area is thus the region of attraction of the equilibrium $x = 0$.

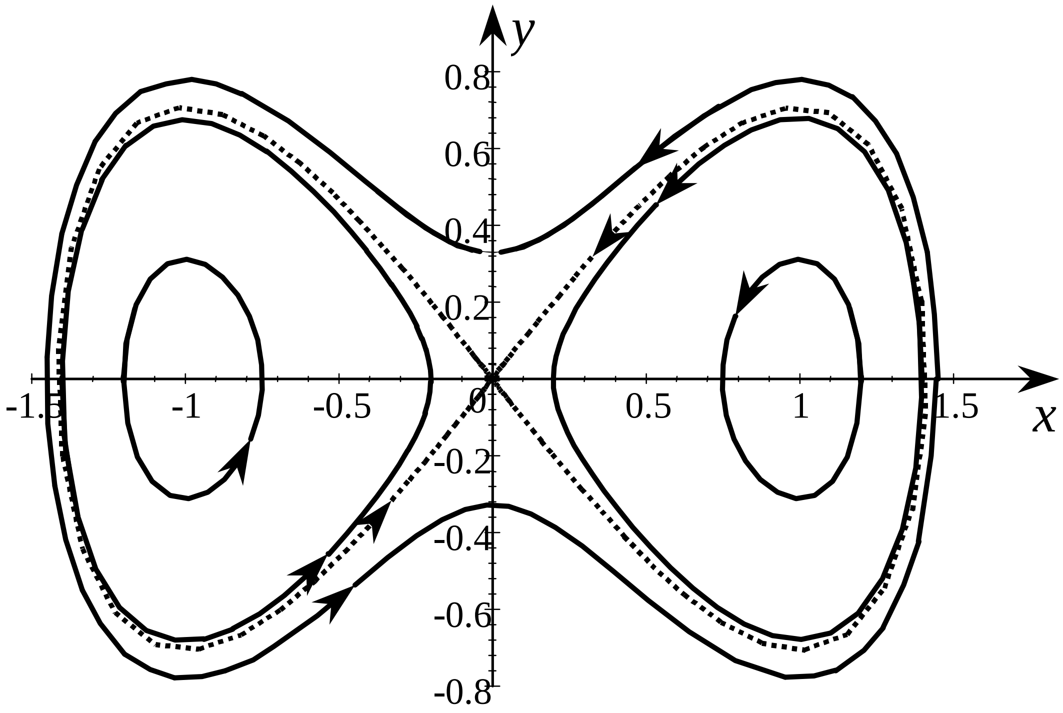
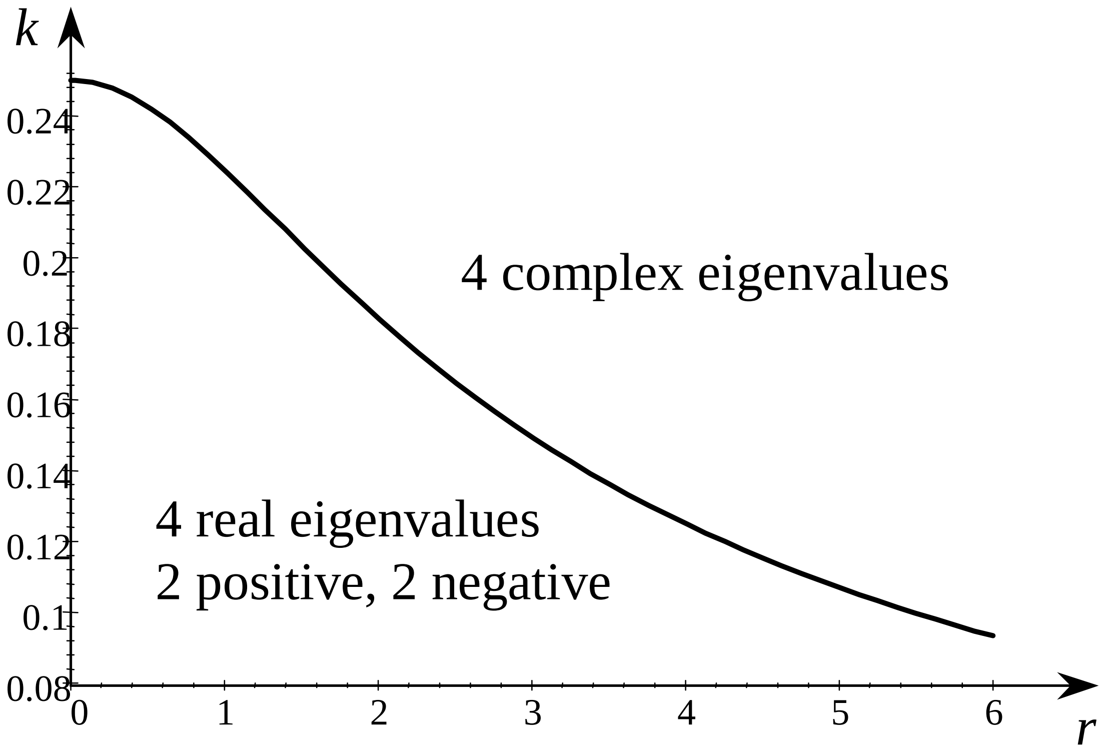


Figure 1



Why Politics Makes Strange Bedfellows: A Dynamic Model with DNS Curves

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Figure 2

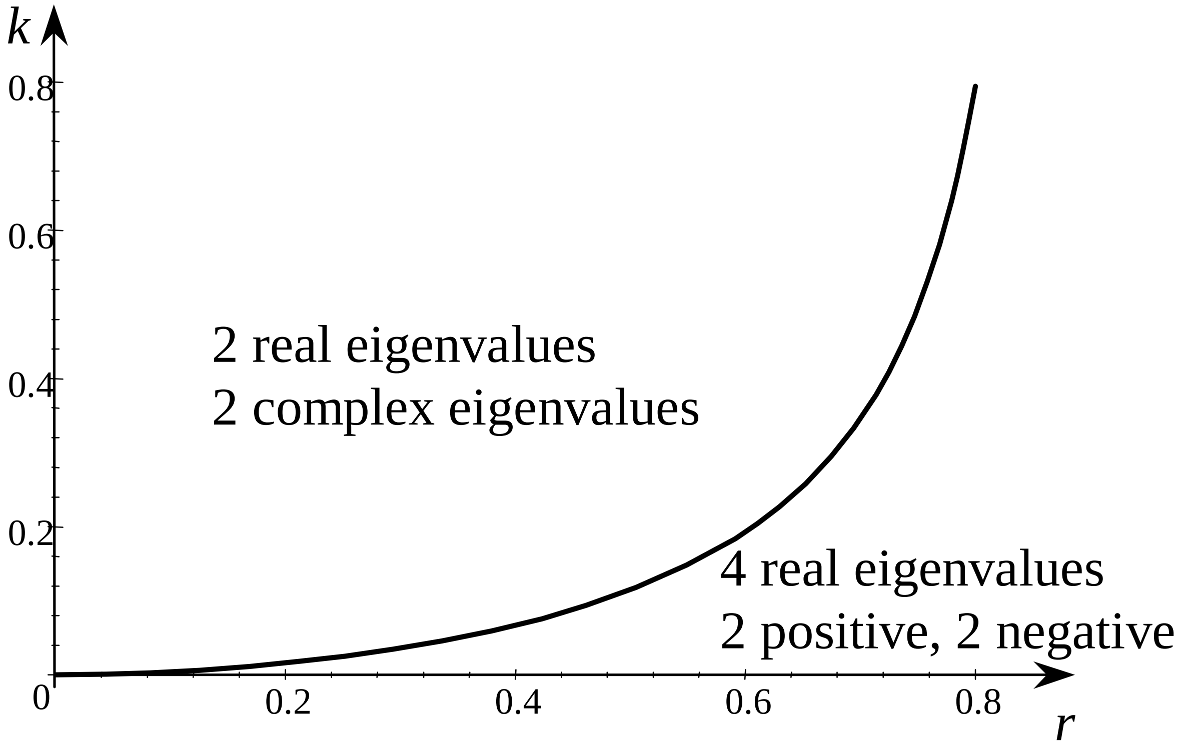
k
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
r
2 real eigenvalues
2 complex eigenvalues
4 real eigenvalues
2 positive, 2 negative

Figure 3

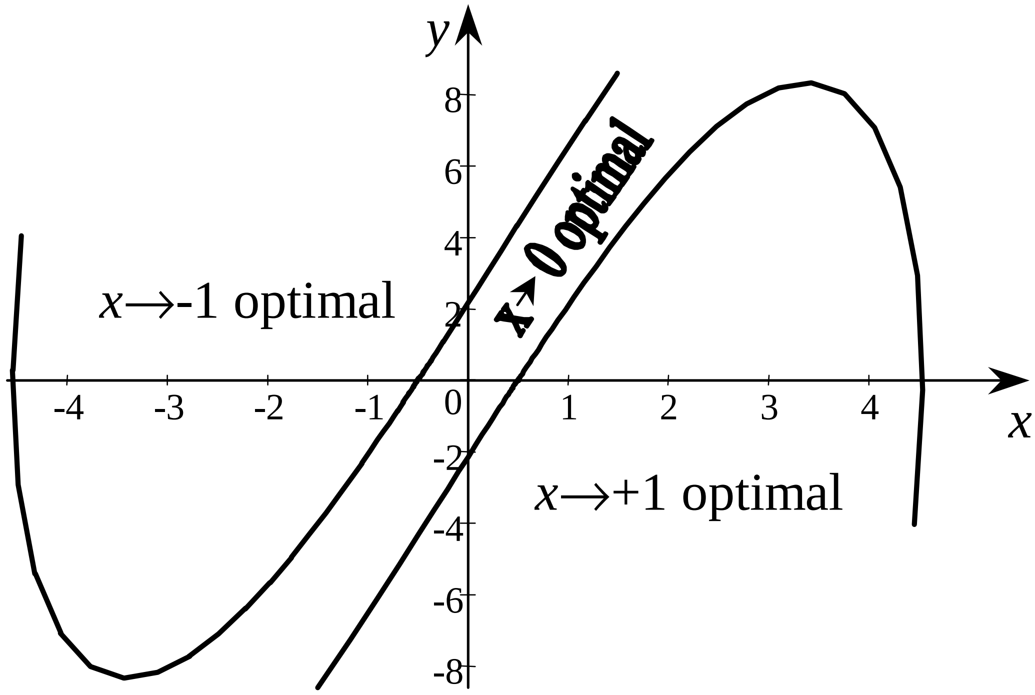

Figure 4

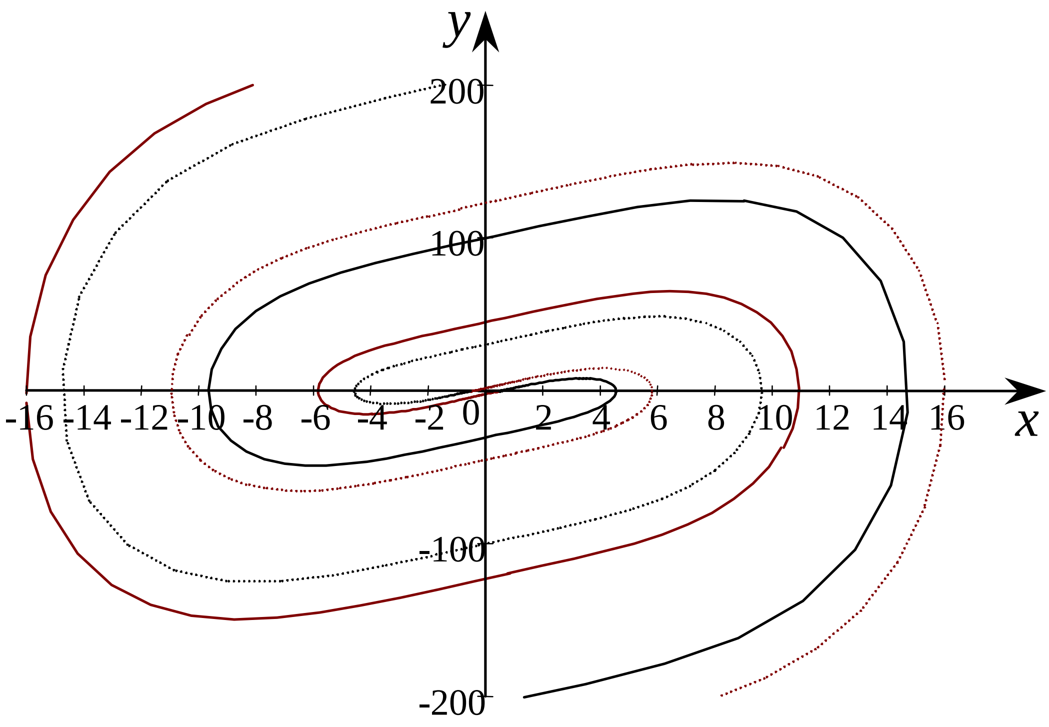

Figure 5