Multiple Equilibria in an Optimal Control Model for Law Enforcement

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Abstract

In this paper, Becker's (1968) economic approach to crime and punishment is extended by including intertemporal aspects. We analyze a one-state control model to determine the optimal dynamic trade-off between damages caused by offenders and law enforcement expenditures. By using Pontryagin's maximum principle we obtain interesting insight into the dynamical structure of optimal law enforcement policies. It is found that inherently multiple steady states are generated which can be saddle-points, unstable nodes or focuses and boundary solutions. Moreover, thresholds (so-called Skiba points) between the basins of attraction are discussed. A bifurcation analysis is carried out to classify the various patterns of optimal law enforcement policies.

1 Introduction

In a seminal paper, Becker (1968) asked how many resources and how much punishment should be used to enforce different kinds of legislation. Becker posed this question equivalently, although more strangely, by asking how many offences *should* be permitted and how many offenders *should* go unpunished. The decision variables in this economic approach to crime and punishment are the expenditures on police and courts influencing the probability that an offence is convicted, and the type and size of punishment for those convicted. The goal is to find those expenditures and punishments that minimize the total social loss. This loss is the sum of damages from offences, costs of apprehension and conviction, and costs of carrying out the punishment imposed. According to Becker's own words, the main contribution of his essay is *"to demonstrate that optimal policies to combat illegal behaviour are part of an optimal allocation of resources"*.

Becker's economic approach of combating illegal behaviour is essentially *static*. The optimal amount of enforcement is determined once for all. In reality, however, the fight of the justice against crime is *intertemporal*.

The purpose of the present paper is to extend the economic framework of law enforcement by including the *dynamics* of offenders. To reach this goal we extend Becker's (1968) static supply function to a dynamic equation describing how the supply of offenders depends on the expected utility of an offence and the legal wage rate. The resulting dynamics of the state variable is replicator-like and has been used in illicit drug dynamics (cf. Caulkins, 1993). The criterion we want to optimize is the discounted stream of social costs resulting from the damages caused by the offenders and from the law enforcement expenditures.

The paper is organized as follows. In Section 2 we present our model. A special case is analyzed in Section 3. Finally, the results obtained are discussed in Section 4, in which also some extensions are presented.

2 The Model

In this section we present a simple (one-state), nonlinear optimal control model. The decision maker is a law enforcement agency, e.g. the drug police, and the offenders are, e.g., the dealers.

Let us denote by $O(t)$ the number of offences[1] at period t, and by $E(t)$ the rate of law enforcement at time t. To describe the social costs we have to specify the damage to society depending on the number of offences, $D(O)$.

[1] By assuming that each offender commits a fixed amount of offences per time unit, we may identify the number of offences with the number of offenders.

Like Becker (1968) we assume increasing marginal damage, i.e.

$$D(0) = 0, \quad D'(O) > 0, \quad D''(O) \geq 0\,.$$

The cost of apprehension and conviction, $C(E)$, is also assumed to be convex:

$$C(0) = 0, \quad C'(E) > 0, \quad C''(E) \geq 0\,.$$

In this paper we consider only one decision variable, namely the rate of law enforcement E. Moreover, we consider only one form of punishment, i.e. fines[2]. Thus, the authorities are faced with the problem to minimize the discounted total stream of loss,

$$\int_0^\infty e^{-rt}\left[D(O(t)) + C(E(t))\right] dt\,. \tag{1}$$

Here we assume a constant positive time preference rate r and an infinite planning horizon[3].

An important question at this stage is, how the control variable $E(t)$ influences the state variable $O(t)$. To fix the system dynamics we consider the individual utility expected from committing an offence (see also Becker, 1968, p. 177-178). Denote by y the income from the illegal activity and by f the fine. Here we assume both y and f as constant.

Moreover, let u be the utility function of the offender, and p the probability that an offence will be cleared by conviction. Then the expected utility of an offence is given by[4]

$$\mathbb{E}(u) = pu(y - f) + (1 - p)u(y)\,.$$

Assuming for simplicity a risk neutral offender, i.e. a linear individual utility, $u(y) = \pi y$, this results in

$$\mathbb{E}(u) = \pi y - \pi f p\,.$$

[2]This implies that the third term on Becker's social loss function, i.e. the punishment costs, can be neglected in the following.

[3]The case of a fixed finite planning horizon can also be easily dealt with. The case of a free terminal time claims also interest. Note that for finite terminal time T one has to specify a 'salvage value' for $O(T)$.

[4]As Leung (1991) remarks it would be more realistic to assume that at least a portion of the offender's gain is lost if he/she is arrested. However, this would not change the results of the following analysis. (For linear utility functions this can be immediately seen).

In what follows, we set for simplicity $\pi = f = 1$. This yields

$$\mathbb{E}(u) = y - p\,.$$

It is reasonable to assume that the conviction probability $p = p(e)$ depends on the law enforcement effort relative to the size of the population of offenders, $e = E/O$, not just enforcement effort. This mechanism has previously been described in the context of illicit drug consumption as *enforcement swamping* (Kleiman, 1993; Tragler et al., 2001). Moreover, we assume that the conviction probability is increasing in the law enforcement rate per offender, but with diminishing marginal efficiency. Formally,

$$p(0) = 0, \quad p'(e) > 0, \quad p''(e) < 0\,.$$

Such diminishing returns have been included in past models (e.g., Tragler et al., 2001) and reflect 'cream skimming'. The image is that some offenders are easier to convict than others are, and that the law enforcement system has some capacity and incentive to focus efforts on individuals who are more likely to be convicted. In other words, those who are convicted most easily, are convicted first.

Now we are able to specify the dependencies of the variable O as follows:

$$\dot{O}(t) = \kappa\,[y - p(e(t)) - \mathrm{w}] - \alpha p(e(t))O(t)\,. \tag{2}$$

In (2) w denotes the average rate of a legal activity[5]. In what follows, we set $\kappa = 1$, which can be achieved by a normalization of parameters. The term $\alpha p(e)O$ refers to that part of the convicted offenders which drop 'out of the business' per time unit because of arrest, restriction of movement or deterrence[6]. The nonnegative parameter α is assumed to be constant.

The meaning of equation (2) is as follows. The rate of change of offenders is proportional to the difference between the utility expected from committing a crime and their reservation wage. The reservation wage is simply what the offender could earn in alternative employment. Thus the model treats

[5]The assumption that the wage rate is constant among all offenders is, of course, highly stylized. It might be justified as crude approximation, in which the population of offenders is considered as a homogeneous aggregate.

[6]In case we would admit cost of punishment the term $\alpha p(e)O$ refers to the imprisoned offenders (see also footnote 2).

criminal activities as an economic entity where there is the potential for profit (from offences), the threat of being caught or risk (from enforcement), and an alternative form of profit. A potential criminal will be motivated to enter the pool of offenders only if his/her utility upon doing so exceeds that offered by an average legal activity. Caulkins (1993) has used a similar mechanism to model the dynamics of a drug dealer's market; see also Baveja et al. (1993) and Borisov et al. (2000). Compare also Clark (1985). A fairly general approach which is based on the same idea is the replicator dynamics playing a central role in evolutionary economics. For an application in a corruption game see Antoci and Sacco (1995).

Setting $y - \mathrm{w} = \beta$, we get the following ordinary differential equation:

$$\dot{O}(t) = \beta - [1 + \alpha O(t)]\, p(e(t))\,. \tag{3}$$

Clearly $\beta > 0$, since otherwise no risk neutral (or risk averted) agent would be criminal. Furthermore, it is reasonable to introduce a lower state constraint

$$O\,(t) \geq \underline{O} > 0\ , \quad \underline{O} \text{ small,} \tag{4}$$

in order to rule out singularities (which may occur at $O = 0$).

Summarizing, the authority tries to minimize the social loss (1) subject to the system dynamics (3) and the pure state constraint (4).

The initial number of offenders (or offences) is assumed to be given by

$$O(0) = O_0 \geq \underline{O}\,. \tag{5}$$

The minimization of the objective functional (1) subject to conditions (3), (4), and (5) defines an optimal control problem, where O is the state variable and E the control, which has clearly to be non-negative.

In the following section we will specify the functions $D(O(t))$, $C(E(t))$ and $p(e(t))$ in order to simplify the analysis. Time arguments t will be mostly omitted in the rest of the paper.

3 Analysis

3.1 Pontryagin's Maximum Principle

In what follows, we specify the functions for the social damage, $D(O)$, for the cost of apprehension and conviction, $C(E)$, and for the probability that an offence is cleared by conviction, $p(e)$, by

$$D(O) = O^\gamma \ , \quad \gamma \geq 1 \ ,$$

$$C(E) = E,$$

and

$$p(e) = \frac{e}{a+e} = \frac{E}{aO+E} = p\left(E, O\right) \ , \quad a > 0 \ ,$$

respectively, i.e., the probability of detection, p, is of Monod type.

Neglecting the pure state constraint (4) at the moment, our problem becomes

$$\max_{E(t)} \int_0^\infty e^{-rt} \left[-O(t)^\gamma - E(t)\right] dt \tag{6}$$

s.t.

$$\dot{O}(t) = \beta - \frac{E(t)\left[1 + \alpha O(t)\right]}{aO(t) + E(t)} \tag{7}$$

where $r > 0$, $\gamma \geq 1$, $\beta > 0$, $\alpha \geq 0$, and $a > 0$.

Applying the Maximum Principle (see, e.g., Feichtinger and Hartl, 1986) to (6) and (7) we first get the Hamiltonian

$$H = -O^\gamma - E + \lambda \dot{O} \ ,$$

where $\lambda = \lambda(t)$ denotes the current value costate variable. As follows from (10) below, λ is negative, which makes economic sense. The negativity of λ implies that the Hamiltonian is concave with respect to E (i.e., $H_{EE} \leq 0$), so that the necessary conditions of the Maximum Principle yield

$$H_E = \frac{\partial H}{\partial E} = 0 \tag{8}$$

(if we neglect boundary solutions at the moment) and

$$\dot{\lambda} = r\lambda - H_O. \tag{9}$$

Note that the standard sufficiency conditions of optimal control theory are not satisfied, since the maximized Hamiltonian is not concave with respect to the state variable.

Inserting (6) and (7) into (8) and (9), these two equations transform to

$$\lambda = -\frac{(aO+E)^2}{aO(1+\alpha O)} \tag{10}$$

and

$$\dot{\lambda} = \lambda\left[r + \frac{E(\alpha E - a)}{(aO+E)^2}\right] + \gamma O^{\gamma-1}. \tag{11}$$

Differentiating (10) with respect to time t we can equate the result with (11) in which we replace λ by the fraction given in (10). This procedure eliminates λ and results in one differential equation for E (instead of the two equations (10) and (11) for λ):

$$\begin{aligned}[2O(1+\alpha O)(aO+E)]\,\dot{E} = (E + 2\alpha EO - aO)\left[(\beta-1)E - \alpha EO + a\beta O\right] + \\ + O(1+\alpha O)\left[r(aO+E)^2 + E(\alpha E - a) - a\gamma O^{\gamma}(1+\alpha O)\right].\end{aligned} \tag{12}$$

In the following two subsections we will analyze the system of differential equations consisting of (7) and (12) first for ($\alpha = 0, \gamma = 1$ and 2), and then for ($\alpha = 1, \gamma = 1$). This specification of parameters allows to compute the steady states of the optimal solution to our model (6), (7). In most cases, it is also possible to characterize the stability properties of the steady states analytically.

3.2 Analysis for $\alpha = 0$

In this section we will see that for $\alpha = 0$, there is a unique steady state, which is unstable whenever feasible. This result holds both for linear and quadratic damage due to offences (i.e., $\gamma = 1$ and $\gamma = 2$, respectively).

3.2.1 The Special Case of Linear Damage ($\gamma = 1$)

We start with computing the equilibria of (7) and (12), which are simply the solutions of the system

$$\begin{aligned}\dot{O} &= 0, \\ \dot{E} &= 0.\end{aligned} \tag{13}$$

For $\alpha = 0$ and $\gamma = 1$ (i.e., linear damage), there is only one equilibrium,

$$\left(\hat{O}, \hat{E}\right) = \left(\frac{\beta(1-\beta+a\beta)}{r}, \frac{(1-\beta)(1-\beta+a\beta)}{ar}\right),$$

which either does not lie in the positive quadrant of the O-E-plane (if $1-\beta < 0$) or is unstable (if $1-\beta \geq 0$) as we have

$$\mu_1\mu_2 = \frac{ar^2}{2(1-\beta+a\beta)} \geq 0$$

and

$$\mu_1 + \mu_2 = r \geq 0$$

and thus

$$\mu_1 \geq 0, \mu_2 \geq 0$$

for the eigenvalues of the Jacobian of the system (7) and (12), μ_1 and μ_2.

3.2.2 The Special Case of Quadratic Damage ($\gamma = 2$)

If we choose $\alpha = 0$ and $\gamma = 2$ (i.e., quadratic damage), we get

$$\left(\hat{O}, \hat{E}\right) = \left(\frac{a\beta(1-\beta)}{ar - 2(1-\beta)^2}, \frac{a^2\beta^2}{ar - 2(1-\beta)^2}\right)$$

as (unique) steady state of our system. As we have

$$\mu_1\mu_2 = \frac{\left(ar - 2(1-\beta)^2\right)^2}{2a^2\beta} \geq 0$$

and

$$\mu_1 + \mu_2 = r \geq 0,$$

this equilibrium is also unstable.

3.3 Analysis for $(\alpha = 1, \gamma = 1)$

In this subsection we assume linear damage due to offences (i.e., $\gamma = 1$), but for the case of positive *alpha* (in particular, $\alpha = 1$, to get analytical results). We will see that we can have up to two interior (and feasible) steady states, which results in a very complex and interesting solution of our optimal control problem to be discussed in the next section.

Omitting equilibria with negative O and/or E we get two solutions $\left(\hat{O}_1, \hat{E}_1\right)$ and $\left(\hat{O}_2, \hat{E}_2\right)$ of (13) which may lie in the positive quadrant of the O-E-plane (depending on the specification of the values of the non-specified parameters a, β, and r):

$$\hat{O}_1 = \frac{2(1-\beta) - ar + SQRT}{2(ar-1)}, \tag{14}$$

$$\hat{E}_1 = \frac{a\beta\left[ar - 2(1-\beta) - SQRT\right]}{ar(2\beta-1) - SQRT}, \tag{15}$$

$$\hat{O}_2 = \frac{2(1-\beta) - ar - SQRT}{2(ar-1)}, \tag{16}$$

$$\hat{E}_2 = \frac{a\beta\left[ar - 2(1-\beta) + SQRT\right]}{ar(2\beta-1) + SQRT}, \tag{17}$$

where

$$SQRT = \sqrt{a\left\{-4\beta(1-\beta)[1 + r(1-a)] + ar^2\right\}}.$$

Proposition 1 *The signs of $\hat{O}_1$, $\hat{E}_1$, $\hat{O}_2$ and $\hat{E}_2$ with respect to the parameters a, β, and r are as given in Tables 1 and 2 (on the supposition that they exist, i.e. $SQRT$ is real).*

	$\beta \geq 1$					
	$ar \geq 1$			$ar \leq 1$		
	$a \leq 1$		$a \geq 1$	$a \leq 1$		$a \geq 1$
	$\beta \leq \frac{1}{1-a}$	$\beta \geq \frac{1}{1-a}$		$\beta \leq \frac{1}{1-a}$	$\beta \geq \frac{1}{1-a}$	
$\hat{O}_1$	$-$	$+$	$-$	$-$	$+$	$-$
$\hat{E}_1$	$+$	$-$	$+$	$+$	$-$	$+$
$\hat{O}_2$	$-$	$-$	$-$	$+$	$+$	$+$
$\hat{E}_2$	$+$	$+$	$+$	$+$	$+$	$+$

Table 1: Signs of the equilibria (14) - (17) with respect to different values of the parameters a, β and r for $\beta \geq 1$.

	$\beta < 1$					
	$ar \leq 2(1-\beta)$			$ar \geq 2(1-\beta)$		
	$ar \geq 1$	$ar \leq 1$		$ar \geq 1$	$ar \leq 1$	
		$\beta \geq \frac{1}{2}$	$\beta \leq \frac{1}{2}$		$\beta \geq \frac{1}{2}$	$\beta \leq \frac{1}{2}$
$\hat{O}_1$	+	−	−	+	+	+
$\hat{E}_1$	+	−	+	+	+	−
$\hat{O}_2$	−	−	−	−	+	+
$\hat{E}_2$	+	−	+	+	+	−

Table 2: Signs of the equilibria (14) - (17) with respect to different values of the parameters a, β and r for $\beta < 1$.

Proof: Rewrite (14) - (17) in the form

$$\hat{O}_{1,2} = \frac{2(1-\beta) - ar \pm \sqrt{ROOT}}{2(ar-1)}$$

and

$$\hat{E}_{1,2} = \frac{a\beta\left[ar - 2(1-\beta) \mp \sqrt{ROOT}\right]}{ar(2\beta-1) \mp \sqrt{ROOT}}$$

where

$$ROOT = a\left(-4\beta(1-\beta)[1 + r(1-a)] + ar^2\right).$$

Suppose now that

$$ROOT \geq 0$$

which is a necessary condition for the equilibria to exist. Further note that

$$2(1-\beta) - ar = \operatorname{sign}\left(2(1-\beta) - ar\right)\sqrt{[2(1-\beta) - ar]^2}$$

so that the numerators of $\hat{O}_{1,2}$ and $\hat{E}_{1,2}$ can be written as the sum of two roots affiliated with a positive or negative sign. To get the signs of the numerators of $\hat{O}_{1,2}$ and $\hat{E}_{1,2}$ we therefore look at the difference

$$ROOT - [2(1-\beta) - ar]^2 = 4(\beta-1)(1-\beta+a\beta)(1-ar)$$

of the radicands of these two roots to see which one 'dominates' the other (if they have opposite signs). Together with the signs of the according denominators (the denominator of $\hat{E}_{1,2}$ is derived analogously) it is then possible to determine the signs of the whole fractions. □

Let us now briefly discuss the results of Proposition 1 as they appear in Tables 1 and 2. As follows from Table 1, one has at most one equilibrium with positive signs both of the state O and the control E, if $\beta \geq 1$. In contrast to that, it is possible to have two equilibria in the positive quadrant, if one chooses $\beta < 1$ (see Table 2).

The following proposition fully characterizes the case $\beta \geq 1$.

Proposition 2 *For $\beta \geq 1$, the unique equilibrium in the positive quadrant $\left(\hat{O}_2, \hat{E}_2\right)$ (existing only if $ar \leq 1$) is a saddle-point.*

Proof: The eigenvalues $\mu_1^{(2)}$ and $\mu_2^{(2)}$ of the Jacobian of our system evaluated at the equilibrium point $\left(\hat{O}_2, \hat{E}_2\right)$ can be computed from

$$\mu_{1,2}^{(2)} = \frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - \frac{2(1-ar)SQRT\left[\theta\frac{SQRT}{a} - r(1-2\beta)\right]}{TERM + (1-2\beta)SQRT}}$$

where

$$TERM = 2\beta(1-\beta)\left\{1 + a[1 + r(1-a)]\right\} - ar$$

and

$$\theta = \operatorname{sign}\left(-4\beta(1-\beta)[1+r(1-a)] + ar^2\right)$$

($SQRT$ as defined above).

Suppose now that the following parameter restrictions hold:

$$\beta \geq 1, ar \leq 1.$$

Then we have (see above)

$$ROOT = a\left\{-4\beta(1-\beta)(1-ar+r)+ar^2\right\} \geq 0$$

which implies that

$$\theta = \text{sign}\left(\frac{ROOT}{a}\right) = +$$

and that $SQRT$ is real (and non-negative). Further,

$$TERM = 2\beta(1-\beta)[1+a(1-ar+r)] - ar \leq 0.$$

We thus have

$$\mu_{1,2}^{(2)} = \frac{1}{2}\left(r \pm \sqrt{r^2 + c_+}\right)$$

with a non-negative constant c_+ so that

$$\mu_1^{(2)} = \frac{1}{2}\left(r + \sqrt{r^2 + c_+}\right) \geq 0$$

and

$$\mu_2^{(2)} = \frac{1}{2}\left(r - \sqrt{r^2 + c_+}\right) \leq 0.$$

from which we conclude that in this case $\left(\hat{O}_2, \hat{E}_2\right)$ is a saddle. □

If $\beta < 1$, it is not possible to fully characterize the stability properties of the steady states analytically. However, if we impose some further restrictions on the remaining parameters (i.e., a, β, and r), we can derive another interesting result, which is summarized in the following proposition.

Proposition 3 *For $\beta \leq \frac{1}{2}$ and $ar \geq 1$, the unique equilibrium point in the positive quadrant $\left(\hat{O}_1, \hat{E}_1\right)$ is unstable.*

Proof: The eigenvalues of the Jacobian at the second equilibrium point, $\left(\hat{O}_1, \hat{E}_1\right)$, are given by

$$\mu_{1,2}^{(1)} = \frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - \frac{2(1-ar)SQRT\left[\theta\frac{SQRT}{a} + r(1-2\beta)\right]}{TERM - (1-2\beta)SQRT}}.$$

We have

$$\beta \leq \frac{1}{2}, ar \geq 1.$$

First, we again have (see above)

$$ROOT = a\left\{-4\beta(1-\beta)(1-ar) + r\left[ar - 4\beta(1-\beta)\right]\right\} \geq 0$$

implying that $\theta = +$ and that $SQRT$ is real (and non-negative). Suppose now that $TERM$ is positive which is equivalent to

$$2\beta(1-\beta)\left\{1 + a[1 + r(1-a)]\right\} \geq ar \geq 0.$$

Of course, $2\beta(1-\beta)\left\{1 + a[1 + r(1-a)]\right\}$ then reaches its maximum for $\beta = \frac{1}{2}$. But, setting $\beta = \frac{1}{2}$, we get

$$\begin{array}{ccl} & 2\beta(1-\beta)\left\{1 + a[1 + r(1-a)]\right\} - ar & = \\ = & \frac{1}{2}\left[1 + a(1 + r - ar)\right] - ar & = \\ = & \ldots & = \frac{1}{2}(1+a)(1-ar) \quad \leq 0 \end{array}$$

which is a contradiction to our assumption that $TERM$ is positive. For the eigenvalues we then have

$$\mu_{1,2}^{(1)} = \frac{1}{2}\left(r \pm \sqrt{r^2 - d_+}\right)$$

where d_+ is a non-negative constant. Therefore, at least the real parts of these eigenvalues are non-negative, i.e.

$$\mathrm{Re}\left(\mu_{1,2}^{(1)}\right) \geq 0.$$

□

This concludes the analytical results of our analysis. In the next section we first provide one particular bifurcation diagram to demonstrate the interesting behaviour which is exhibited by our optimal control model for law enforcement. We will then discuss the optimal solution first for the case of one unstable focus together with one saddle point, and then for the case of just one unstable focus. The numerical analysis was carried out with the help of the computer software package *Mathematica* (Wolfram, 1991).

4 Discussion

The bifurcation diagram depicted in Fig. 1 exhibits the parameter β at the abscissa and the equilibrium values for O at the ordinate.

It shows that for large values of $\beta = y - w$ (i.e., the difference of the income from the illegal activity and that of the alternative legal activity), there exist two long-run steady states, one (saddle-point) stable and one unstable equilibrium. While the former is depicted as thick line, the latter is thin.

If β decreases, the two equilibria gradually approach, and finally collide for a critical value of β_{crit} (= 0.854 for the given parameters). Below that value, i.e. for relatively small values of β, no interior equilibrium exists. Hence, here we observe a saddle-node bifurcation. Note that this behaviour makes economic sense.

Let us next briefly discuss the phase portrait of the state-control space as shown in Fig. 2, where we also take into account the lower state constraint (4). For this figure we used the same set of parameters as in Fig. 1, but with β fixed at the value 0.9. We observe two interior long-run steady states: one is an unstable focus in the south-west and the other is a saddle node in the north-east of the O-E-plane.[7]

We can prove numerically the existence of a 'Dechert-Nishimura-Skiba threshold' ('DNS-threshold', for short) O_{DNS} (also denoted as 'Skiba point', see Skiba, 1978) separating two basins of attraction.[8] As described in what follows, in this case different initial states may lead to different long-run equilibria. For a recent example of DNS-thresholds in economic applications see, e.g., Wirl and Feichtinger (2000).

In particular, for large and medium numbers of offenders the high steady state $\left(\hat{O}_2, \hat{E}_2\right)$ is approached along the branches of the stable manifold. For $O(0) > \hat{O}_2$, the optimal law enforcement rate is greater than its steady state level $\hat{E}_2$, but gradually decreases to this level, driving the number of offenders to its long-run steady state. Furthermore, for $O_{DNS} < O(0) < \hat{O}_2$,

[7]Note that the equilibria always lie on the isocline $\dot{O} = 0$, which in our case is a Monod function so that $E_1 \leq E_2$ whenever $O_1 \leq O_2$.

[8]The occurrence of multiple equilibria which are separated by thresholds is quite common in economic models. Skiba (1978) discussed such critical values, but his proof is incomplete. A first proof of existence for discrete time intertemporal optimization was given by Dechert and Nishimura (1983); see Dechert (1983) for a continuous time model.

the optimal enforcement is relatively low and gradually increases to $\hat{E}_2$, while $O(t)$ converges to $\hat{O}_2$.

Finally, there exists a third regime for sufficiently small initial numbers of offenders. Here the only candidate for optimal enforcement is the trajectory spiraling out of the lower steady state $\left(\hat{O}_1, \hat{E}_1\right)$ and ending in point $(\underline{O}, \underline{E})$ at the boundary $O = \underline{O}$. The optimality of this lower boundary equilibrium can be shown by using a Lagrangean function which includes the pure path constraint $O \geq 0$. Intuitively, the steady state is given by the $\dot{O} = 0$ isocline (in O-direction) and the complementary slackness condition resulting from the Lagrangean approach (in E-direction). Note that this lower boundary equilibrium is reached in finite time and the trajectory is vertical in this point (for a more detailed illustration see the left region of Fig. 3). For an illustration of this technique, compare also Feichtinger and Hartl (1986), pp. 218-219.

The economic interpretation of this solution is as follows. Let us start with the third regime. If a society is already 'rather clean', it pays to eradicate the offenders to a minimal level, $O = \underline{O}$. The corresponding enforcement feedback rule says that both $E(t)$ and $O(t)$ decrease monotonically to the relatively low levels $\underline{E}$ and $\underline{O}$, respectively, with the possible exception that law enforcement might increase initially.

For initial states above the threshold O_{DNS}, the optimal trade-off between damages created by offenders and the law enforcement expenditures results in an *upper steady state*, $\left(\hat{O}_2, \hat{E}_2\right)$. The optimal enforcement has to be relatively high (low) to reduce large (small) numbers of offenders.

It should be noted that the saddle-point paths provide the only candidates for an optimal solution satisfying the limiting transversality condition. Since the standard sufficiency conditions of optimal control theory are not satisfied, the path to the lower steady state $(\underline{O}, \underline{E})$ is also only an 'extremale'.

Fig. 3 illustrates the case of an unstable focus. As far as we know, such a case has not yet been discussed in any economic application of optimal control theory. By using Lagrangeans, it can be shown that the 'optimal' enforcement policy has the shape shown in Fig. 3. Note that in this case we also introduce an upper limit for the state,

$$O \leq \overline{O}.$$

We see that we again have a DNS-threshold, separating the basins of attraction of the two boundary equilibria, $(\underline{O}, \underline{E})$ and $\left(\overline{O}, \overline{E}\right)$. The economic interpretation is analogous to the case depicted in Fig. 2.

We conclude this paper by pointing to some possible extensions of our model. The dynamics (7) has been criticized by economists to be not founded by microeconomics. This possible 'lack of reality' has been taken into consideration in the work of Tragler et al. (2001), which has been empirically validated.

The idea of homogeneous agents is a fiction. In reality, agents behave *heterogeneously* in various aspects, e.g. in their offending intensity. While a complete inclusion of heterogeneity requires sophisticated methods, e.g. partial differential equations, a first approach will distinguish between *'light'* and *'heavy'* offenders; cf. Behrens et al. (2000).

Let us conclude with the remark that a full treatment of crime and punishment should use *game-theoretic methods*. Offenders interact intertemporally with law enforcement agencies in an interactive manner. One attempt along that avenue is made by Dawid and Feichtinger (1996).

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List of Captions

Figure 1 Bifurcation diagram showing the steady state values $\hat{O}_1$ (unstable focus; thin curve) and $\hat{O}_2$ (saddle point; thick curve) for $\beta \in [0, 2]$. We observe a saddle-node bifurcation at the critical value $\beta_{crit} = 0.854$. The other parameters where chosen as follows: $a = 1.5$, $\alpha = 1$, $\gamma = 1$, and $r = 0.5$.

Figure 2 Phase portrait in the O-E-space for the set of parameters $a = 1.5$, $\alpha = 1$, $\beta = 0.9$, $\gamma = 1$, and $r = 0.5$. The grey curves give the $\dot{O} = 0$ and the $\dot{E} = 0$ isoclines, while the vertical grey line at $\underline{O}$ (arbitrarily chosen at the level 0.001) indicates the lower state constraint. We observe two steady states: at $\left(\hat{O}_1, \hat{E}_1\right)$ there is an unstable focus while the equilibrium at $\left(\hat{O}_2, \hat{E}_2\right)$ is a saddle point. The vertical line at $O_{DNS} = 0.4$ indicates the DNS-threshold, separating the basins of attraction of the equilibria at the lower boundary $\underline{O}$ and the upper steady state value $\hat{O}_2$, respectively. The thick black curves with arrows hence are the optimal trajectories, providing a feedback rule for the optimal policy.

Figure 3 Phase portrait in the O-E-space for the set of parameters $a = 5$, $\alpha = 1$, $\beta = 0.4$, $\gamma = 1$, and $r = 0.5$. The grey curves give the $\dot{O} = 0$ and the $\dot{E} = 0$ isoclines, while the vertical grey lines at $\underline{O}$ (arbitrarily chosen at the level 0.05) and $\overline{O}$ (arbitrarily chosen at the level 1.3) indicate the lower and upper state constraints, respectively. We observe one steady state at $\left(\hat{O}_1, \hat{E}_1\right)$ which is an unstable focus. The vertical line at $O_{DNS} = 0.803$ indicates the DNS-threshold, separating the basins of attraction of the equilibria at the lower and upper boundaries, $\underline{O}$ and $\overline{O}$, respectively. The thick black curves with arrows hence are the optimal trajectories, providing a feedback rule for the optimal policy.

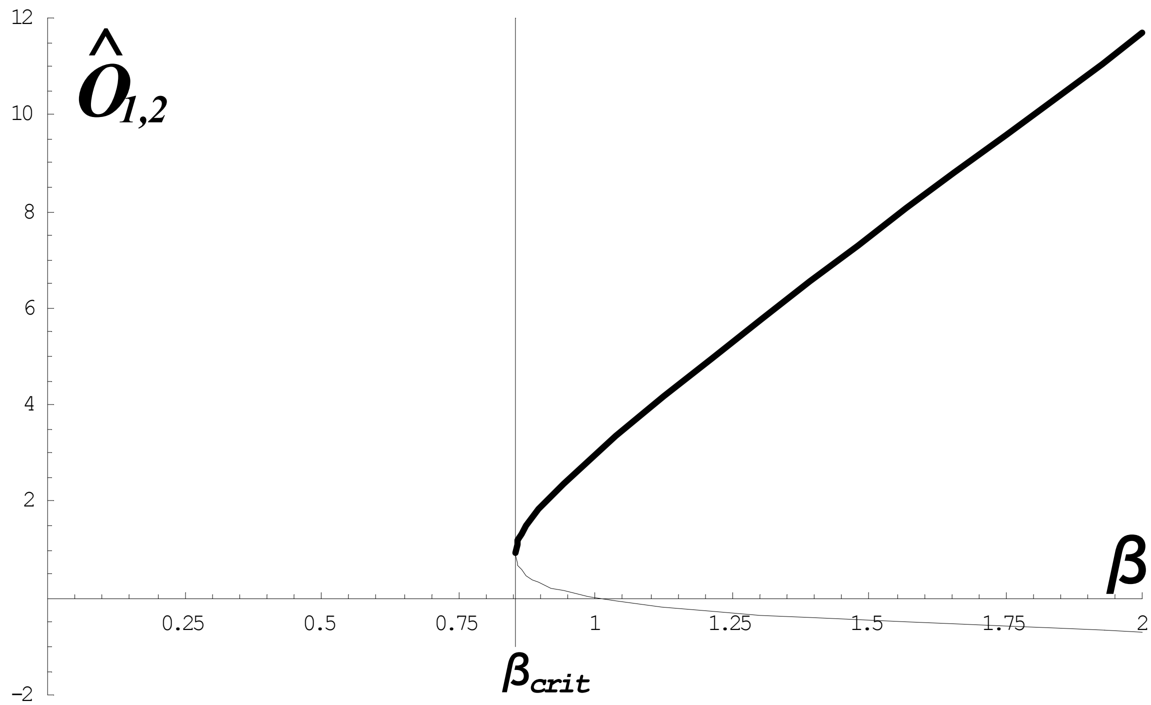
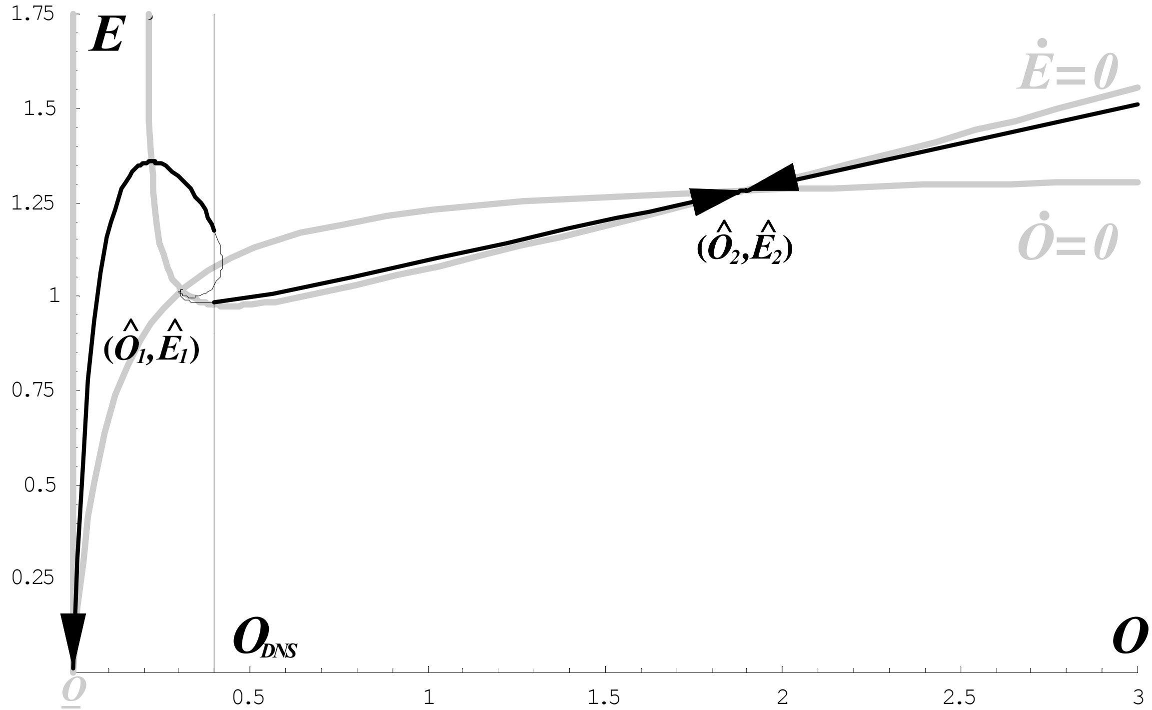


Figure 1



E
$\dot{E}=0$
$\dot{O}=0$
$(\hat{O}_1,\hat{E}_1)$
$(\hat{O}_2,\hat{E}_2)$
O_{DNS}
O
$\underline{O}$
1.75
1.5
1.25
1
0.75
0.5
0.25
0.5
1
1.5
2
2.5
3

Figure 2

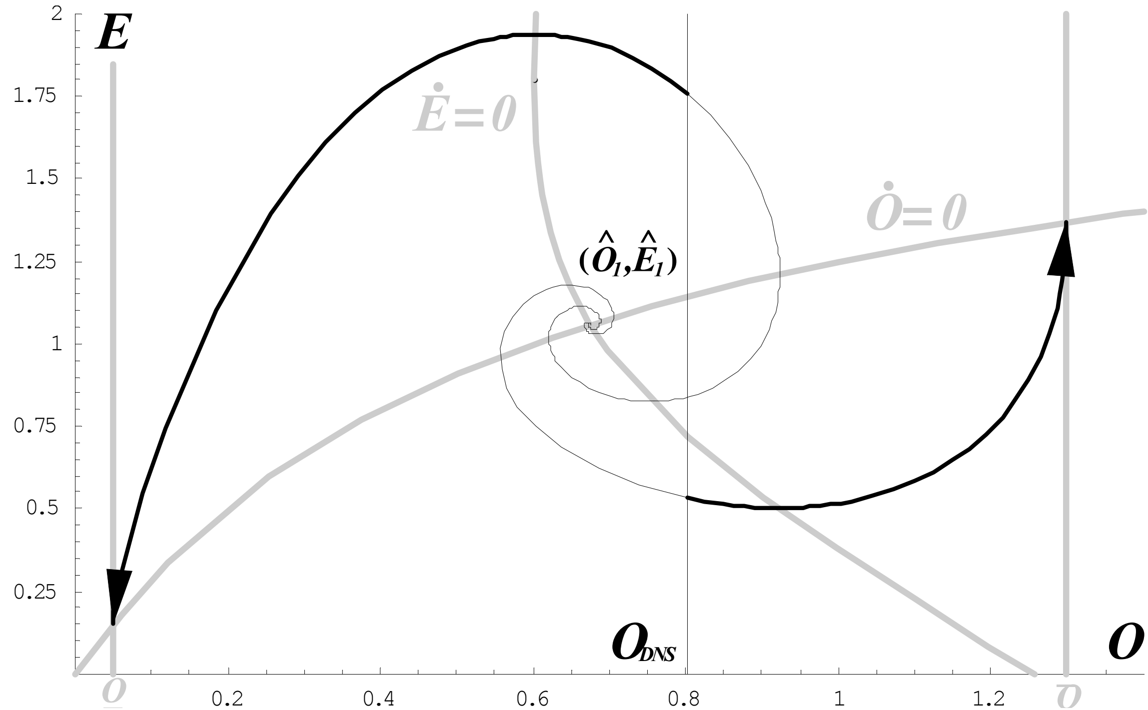


Figure 3