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Managing the Reputation of an Award to Motivate Performance

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Abstract Managers wish to motivate workers to exert effort. There is a large literature on the use of wages and monetary incentives for this purpose, but in practice the "honor" or "prestige" of an award can be a significant motivator as well, unless the award is given so often that its prestige is diluted. The model here focuses on management of the reputation of an award that may or may not have a fixed monetary component. The model is an optimal dynamic control model, so its solution suggests how to manage the award over time. The analysis is interesting because of a "false" steady state that is adjacent to but outside the admissible region and which otherwise has the qualitative properties of a steady state; there are (infinitely many) trajectories converging to it and (infinitely many) trajectories starting arbitrarily close to it. For all initial conditions there are infinitely many candidates for the optimal solution that cannot be evaluated in the standard way. We resolve that problem by proving a new proposition concerning the value of the

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utility functional when the limit of the Hamiltonian is non-zero. Managerial implications are derived.

1 Introduction

Economists have long been interested in the issue of motivating effort by employees through incentives. Usually the analysis focuses on monetary compensation, as in the efficiency wage literature (cf., [21]; [13]), but management practitioners recognize that there are other motivators, including awards and recognition. Napoleon was famous for using tin medals to induce soldiers to risk extreme hardship and death, quipping "With enough baubles and trinkets one can conquer the world."

Here we explore the question of how best to manage motivational awards whose value is at least in part reputational, not only monetary. For example, how often should they be given? Thirty years ago Hansen and Weisbrod observed ([10], pp. 422–423), somewhat facetiously but also somewhat seriously, "It is amazing that these questions have escaped analysis by economists. The general theory underlying awards is exceedingly simple. Awards are essentially free goods (apart from small computational costs), but at the same time, as the number of awards is expanded we should expect declining marginal social benefits. We urge that research be expanded into the virgin field of the optimal number and variety of awards."

This call has gone largely unheeded. As Baker et al. observe ([2], pp. 594–595): "Economists, while recognizing that nonmonetary rewards for performance can be important, tend to focus on monetary rewards because individuals are willing to substitute nonmonetary for monetary rewards and because money represents a generalized claim on resources and is therefore in general preferred over an equal dollar-value payment in kind." Labor economic models do certainly incorporate psychological perspectives and considerations. (Lazear, [15], gives a readable review.) But the models still almost exclusively focus on monetary bonuses and so ignore the possibility that giving awards too often can reduce their reputation for exclusivity and, hence, the value to the recipient. Reviews of outstanding questions still to be addressed in economic analysis of compensation (Baker et al. [2]; Lazear, [17]) do not mention award reputation. (Note: relational incentive contract models (e.g.,Bull, [3]) consider reputation effects, but those pertain to the reputation of the principal and agents, not the award.)

Thinking about reputational awards is important for at least three reasons. First, as Clark and Riis, [4], observe, sometimes firms are restricted to pay the same wage to all workers at a particular job level, e.g., because of collective bargaining agreements. In such circumstances, non-monetary awards, including honors, may be one of the few options available for rewarding exceptional performance.

Second, psychologists and behavioralists have long expressed concern that monetary incentives can undermine intrinsic motivations for working hard (e.g.,Deci, [5]; Hamner, [9]). Honors and awards may be less likely to do so.

Third, honorary awards are routinely used in practice, and prestigious awards can motivate effort all out of proportion to their cash value. There is a cottage consulting industry to help firms prepare for the Malcolm Baldrige National Quality Award, and the 40 or so state level equivalents, even though the Baldrige award includes no monetary component. Indeed, firms have to pay to compete for the Baldrige Award. That is not to say that winning honorary awards does not have real economic value. Hendricks and Singhal,[12], find that the stock market reacts positively to quality award announcements, particularly for awards from independent agencies such as the Baldrige Award, the Philip Crosby Medallion, etc. But that value is mediated through the award's reputation.

Our model is very simple in several respects. Workers are risk-neutral, so the results are not driven by differences in the extent of risk aversion as in many classic principal-agent models (Gibbons, [8]). Workers cannot "compete" by sabotaging each other's efforts (cf. Lazear, [14]). There are no career concerns (cf., Gibbons and Murphy, [7]). Also, we assume that the firm knows which employees are exerting extra effort (i.e., not "shirking" if one prefers negative labels). There are some parallels to the literature on tournaments (cf. Nalebuff and Stiglitz, [19]), but the emphasis here is not on ranking relative performance.

At the same time our model is more complicated than some standard models in certain respects. First, workers are heterogeneous with respect to their disutility of exerting additional effort. Second, and crucially, the model is dynamic so the firm seeks the optimal dynamic trajectory of incentives, not just a static solution. In part to keep this dynamic perspective from becoming unwieldly, but primarily for the sake of verisimilitude, we grant the firm perfect foresight and planning, but make the workers simply react to present condition. Workers cannot save or otherwise trade-off intertemporal utility. Nor can they coordinate their actions to act strategically.

The essence of the firm's decision problem is simple. Both the intrinsic honor and the signaling value of an award are reduced if it is given out too liberally. In the short run, the more rewards that are offered, the more workers are likely to be motivated, but the more people who receive an award the less prestigious it becomes. Of course if many people have received the award only because it has existed for many years, the award could still be prestigious. So even if an award's reputation is debased, there is some natural "recovery" of the award's reputation if few people are receiving it currently. (E.g., the Congressional Medal of Honor was given out very liberally during the Civil War, but is now perhaps the most prestigious award in the US.)

Hence, the central administrative question is how liberally to disperse the award over time, to all deserving workers or only a subset? Should prestigious awards be given more or less frequently than debased awards? Should the reputation of an award that is initially prestigious be preserved or exploited over time? If it is optimal to "invest" in building up the reputation of an award by giving it less frequently, should that investment be done aggressively or incrementally? Are the answers to any of these questions different if the monetary component of the award is small or large? Clearly the model introduced here is just a first step, but it does provide concrete advice concerning these managerial questions whose answers are not obvious a priori.

2 The Model

Suppose that workers decide to exert extra effort or not (binary choice). Management's objective is to maximize the discounted sum of the number of people exerting extra effort, less any costs of motivating that effort. Suppose that its control variable governs the proportion of people exerting extra effort who are rewarded. (E.g., all "deserving" workers are entered into a lottery (cf., Lazear, [16]), and a random sample are given the award at any given time.) Another way to think about this is as the rate at which deserving people are rewarded.

To analyze this problem one must first consider the workers' perspective. Suppose the workers are myopic. They make their current effort level decisions based only on the immediate payoff (sensible if there are many workers and they cannot coordinate). Suppose further that the disutility of exerting the extra effort is heterogeneous across individuals with cumulative distribution $\Phi(w)$. With N denoting the number of employees, $F(w) = N\Phi(w)$ is the number of employees for whom the cost of exerting extra effort is no greater than w. Finally, suppose that the award has both a simple financial component (worth v) and a psychic component that is inversely proportional to the reputation R for how common the award is. With u denoting the number of awards given per worker and unit time, the expected payoff if the worker exerts extra effort is hence $u\,(v + 1/R)$, and the number of workers who will exert extra effort is $F\,(u\,(v + 1/R))$. That is also the number of workers who are eligible for an award.[1]

With t as time argument, $R = R(t)$ and $u = u(t)$ are the state and control variables, respectively, in our optimal control model formulation. Note, however, that the time argument will mostly be omitted to enhance readability.

The dynamics of reputation R, where high values indicate a common or debased reputation, is given by

$$\dot{R}(t) = u(t)F\left(u(t)\left(v + \frac{1}{R(t)}\right)\right) - \delta R(t), \tag{1}$$

where δ is the rate of forgetting, i.e., the rate at which the commonness of the award's reputation decays. The first term in (1) is the rate at which

[1] One could generalize this by making the value of the reward an arbitrary, non-negative, decreasing function of the reputation stock, R.

eligible people receive the award u times the number of eligible people $F\left(u\left(v+1/R\right)\right)$, i.e., the overall rate at which people receive the award.

The employer's objective is to maximize the discounted stream of the revenue resulting from extra work minus the costs of the awards given. In mathematical terms,

$$\max_{u(t)} \int_0^{+\infty} e^{-rt} \left[\rho F\left(u(t)\left(v+\frac{1}{R(t)}\right)\right) - c\,u(t) F\left(u(t)\left(v+\frac{1}{R(t)}\right)\right)\right] dt,$$

where r is the discount rate, ρ is the incremental revenue accruing to the employer per employee who works hard, and c represents the costs to the employer per award given. Note that c includes the monetary value of the award, therefore $c \geq v$ must hold. Clearly, the control has to satisfy the constraint

$$u(t) \geq 0 \quad \text{for each} \quad t \geq 0.$$

3 Analysis

With

$$V(R) = v + 1/R, \tag{2}$$

our optimal control problem with one state (R) and one control (u) is given by

$$\max_{u} \int_0^{+\infty} e^{-rt} \left[\rho F\left(uV(R)\right) - cuF\left(uV(R)\right)\right] dt \tag{3}$$

s.t.

$$\dot{R} = uF\left(uV(R)\right) - \delta R. \tag{4}$$

For the number of people willing to work hard for expected reward w, $F(w)$, it would be reasonable to assume that it has the shape of a logistic function. In this case, however the analysis becomes rather complicated while the qualitative behavior of the model is very similar to that of the model with a linear function F. Hence, we prefer to analyze the model with

$$F(w) = dw, \tag{5}$$

because we do not lose information and the presentation wins transparency.

Before we go through the analysis of our model in full detail, we briefly summarize what will be shown in this section. Our problem is defined on the set $\Omega = \{(R,u) \in \mathbb{R}^2 | R > 0, u \geq 0\}$. Using Pontryagin's Maximum Principle (see, e.g., Feichtinger and Hartl, [6]; Leonard and Long, [18]) in Ω we get a phase diagram with one "true" steady state $(\widehat{R}, \hat{u})$ and one "false" steady state $(0,0)$. We use the expression "false" steady state, because $(0,0)$ is not a point of Ω, but it has the qualitative properties of a steady state: there are (infinitely many) trajectories converging to $(0,0)$, and there are (again infinitely many) trajectories starting arbitrarily close to $(0, 0)$. The steady state $(\widehat{R}, \hat{u})$ is a saddle point for each set of admissible parameters. Its stable manifold consists of a trajectory going from $(0, 0)$ to $(\widehat{R}, \hat{u})$ and a trajectory coming from infinity and going to $(\widehat{R}, \hat{u})$.

The set of candidates for the optimal solution consists of three classes of trajectories

1. The trajectories on the stable manifold of the saddle point $(\widehat{R}, \hat{u})$.
2. The set of trajectories going to $(0, 0)$, situated below the stable manifold of $(\widehat{R}, \hat{u})$ and above the horizontal axis.
3. The trajectory on the R-axis converging to the origin, for which $u = 0$.

More precisely, for each starting value R_0 of the reputation, there is a positive value $a(R_0)$ such that for each starting control $u_0 \in [0, a(R_0)]$, there is a trajectory starting at (R_0, u_0) that satisfies the necessary conditions of Pontryagin's Maximum Principle. $a(R_0)$ is the value of the control u on the stable manifold corresponding to R_0. For any initial value of the reputation, R_0, we therefore have infinitely many candidates for the optimal solution. A special candidate is $u = 0$, i.e. to do nothing.

By analytical arguments it follows that the trajectory starting at $(R_0, a(R_0))$ (which is a point on the stable manifold of $(\hat{R}, \hat{u})$) is the optimal solution. We proceed as follows: for an arbitrary $u_0 \in [0, a(R_0)]$, we compute the utility functional for the trajectory staring at (R_0, u_0) and compare it with the utility functional of the trajectory on the stable manifold starting at $(R_0, a(R_0))$. To compute the utility functional for the trajectory starting on the stable manifold we use a well-known proposition, which we cannot apply for the trajectory starting at (R_0, u_0) and going to $(0, 0)$, because the limit to infinity of the Hamiltonian along this trajectory is not equal to zero. We overcome this obstacle by proving a proposition that delivers the formula for the utility functional when the limit to infinity of the Hamiltonian along the trajectory is finite. The computation of this limit for the trajectories from the second class of candidates is analytical: from the ODE system we get an upper and a lower bound for the Hamiltonian, both of them having the same limit to infinity (see subsection 3.3).

3.1 The Maximum Principle

The current value Hamiltonian H for our reputation model (3), (4) with the functions $V(R)$ and $F(w)$ as specified in (2) and (5), respectively, and costate λ, has the form

$$H(R, u, \lambda) = (\rho - cu + \lambda u)duV(R) - \delta\lambda R.$$

Applying Pontryagin's Maximum Principle for (R, u) in the domain $\overset{\circ}{\Omega}= (0, +\infty) \times (0, +\infty)$ we obtain the canonical system

$$\frac{\partial H}{\partial u} = d\left(v + \frac{1}{R}\right)(\rho + 2(\lambda - c)u) = 0, \tag{6.a}$$

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial R} = (\rho - cu + \lambda u)\frac{du}{R^2} + (\delta + r)\lambda, \tag{6.b}$$

$$\dot{R} = \frac{\partial H}{\partial \lambda} = du^2\left(v + \frac{1}{R}\right) - \delta R. \tag{6.c}$$

From (6.a) we express λ as a function of u. Using this relation, from (6.b) and (6.c) we derive an ODE system in (R, u), which is given by

$$\dot{R} = R\left(\frac{du^2}{R^2} + \frac{dvu^2}{R} - \delta\right), \tag{7.a}$$

$$\dot{u} = u\left(\frac{du^2}{R^2} + \bar{c}u - (\delta + r)\right), \tag{7.b}$$

where $\bar{c} = \frac{2(\delta+r)}{\rho}c$. Note that the usual Mangasarian sufficiency conditions are not satisfied, since H is in general not concave wrt. R.

3.2 Interior Steady States: One Saddle Point

Our first aim is to find the equilibria of the ODE system (7.a-b) in the domain $\overset{\circ}{\Omega}= \{(R, u) \in \mathbb{R}^2 \mid R > 0, u > 0\}$.

Proposition 31 *The system (7.a-b) has only one steady state* $(\widehat{R}, \hat{u})$ *in* $\overset{\circ}{\Omega}$.

Proof. See Appendix A.

The first component of the steady state, $\widehat{R}$, is the unique positive solution of a cubic equation. The Cardano formulas give a cumbersome expression for $\widehat{R}$, which we do not mention here. The second component, $\hat{u}$, can be expressed as a function of $\widehat{R}$

$$\hat{u} = \sqrt{\frac{\delta}{d}}\frac{\widehat{R}}{\sqrt{1 + v\widehat{R}}}.$$

For all admissible parameters, the unique steady state in $\overset{\circ}{\Omega}$ is a saddle point. Close to it, the control on the stable manifolds increases when the state increases, and it decreases when the state decreases. In other words, the slope of the stable manifold expressed as a function of R is positive around this steady state. We obtain this result by analyzing the eigenvectors of the Jacobian evaluated at the saddle point.

In the following two propositions we summarize these first qualitative results on the steady state $(\widehat{R}, \hat{u})$.

Figure 1 about here

Proposition 32 *For each admissible set of parameters the steady state* $(\widehat{R}, \hat{u})$ *is a saddle point and its eigenvectors both point to North-East.*

Proof. See Appendix B.

In order to get further insights of the qualitative behavior of the canonical system (7.a), (7.b), in Figure 1 we give a phase diagram on the feasible region Ω.

The isoclines of the system (7.a-b) in the domain $\overset{\circ}{\Omega}$ are two smooth curves, which intersect exactly once in $\overset{\circ}{\Omega}$, at the steady state $(\widehat{R}, \hat{u})$. We express these isoclines, $\dot{R}/R = 0$ and $\dot{u}/u = 0$, as functions of R and denote them by I_R, $I_u : (0, +\infty) \to (0, +\infty)$, respectively. For all the admissible parameters the functions I_R and I_u are strictly increasing and we have

$$I_R(R) < I_u(R) \text{ for each } R \in (0, \widehat{R}), \tag{8}$$

$$I_R(R) > I_u(R) \text{ for each } R \in (\widehat{R}, +\infty). \tag{9}$$

Using the continuity of the ODE's in (7.a-b) on $\overset{\circ}{\Omega}$ we get the signs of $\dot{R}$ and $\dot{u}$ in the four isosectors determined by the isoclines I_R and I_u. This fact and Proposition 32 allow us to characterize the stable manifold in a small neighborhood of the steady state $(\widehat{R}, \hat{u})$ as follows.

Proposition 33 *There is a neighborhood of* $(\widehat{R}, \hat{u})$ *such that its stable manifold is situated between the graph of* I_u *and the line* $u = \hat{u}$.

This result can be extended to the whole stable manifold, as summarized in the following proposition.

Proposition 34 *1. The stable manifold of the steady state* $(\widehat{R}, \hat{u})$ *in* $\overset{\circ}{\Omega}$ *is the graph of a smooth function of* R, $u^s : (0, +\infty) \to I\!R$. *Furthermore,*
2. $\lim_{R \to 0} u^s(R) = 0$,
3.a. $u^s(R) \geq I_u(R)$ *for each* $R \in (0, \widehat{R})$,
3.b. $u^s(R) \leq I_u(R)$ *for each* $R \in (\widehat{R}, +\infty)$, *and*
4. u^s *is strictly increasing.*

Proof. See Appendix C.

Proposition 34 describes the stable manifold of $(\widehat{R}, \hat{u})$. It consists of two trajectories, one coming from the origin $(0, 0)$ and going to $(\widehat{R}, \hat{u})$, remaining all the time above the graph of I_u, and the other coming from infinity and going to $(\widehat{R}, \hat{u})$, remaining always below the graph of I_u. In particular, Proposition 34 states that for small values of R (i.e. cases, in which the reputation is very good; the award is of Nobel prize type) the optimal control is very small. Thus, if an award initially is very prestigious, it shoud be given sparingly in order not to debase the reputation too quickly.

3.3 The optimal solution

In a "standard" optimal control problem, we would now have sufficient information to state the optimal solution, i.e. the stable manifold of the saddle point equilibrium $(\widehat{R}, \hat{u})$. However, as we can see in the phase diagram in Fig. 1, the behavior of the flow close to the origin is truly "non-standard". Each sufficiently small neighborhood of $(0,0)$ in Ω can be divided into three sectors. The sector between the R-axis and the instable manifold of $(\widehat{R}, \hat{u})$ is a parabolic sector (for so-called nonhyperbolic critical points in $\mathbb{R}^2$, see, e.g., Andronov et al., [1], or Perko, [20]). Between the instable and the stable manifold of $(\widehat{R}, \hat{u})$ there is an elliptic sector, and between the stable manifold of $(\widehat{R}, \hat{u})$ and the u-axis there is another parabolic sector. We can say that the flow here has the shape of a quarter of a nonhyperbolic steady state with two elliptic and two parabolic sections.[2]

Hence, for each initial reputation $R_0 \in (0, +\infty)$ there are infinitely many candidates for the optimal control. On the one hand, each trajectory starting at (R_0, u_0) with $0 \leq u_0 < u^s(R_0)$ is an infinite time horizon trajectory going to $(0,0)$ and therefore a possible solution. On the other hand, the trajectory starting at $(R_0, u^s(R_0))$ going to the steady state $(\widehat{R}, \hat{u})$ along its stable manifold is another possible solution.

To determine the optimal solution, we compute the utility functional for each candidate and then compare the results. For this aim, we recall the definition of the maximized Hamiltonian, $H^0(R,u)$

$$H^0(R,u) = \max_{\lambda} H(R,u,\lambda),$$

and we introduce the notation

$$U(R,u) = \rho F\left(uV(R)\right) - cuF\left(uV(R)\right).$$

In order to give a formula for the utility functional for a trajectory with an infinite time horizon, which does not satisfy the classical condition

$$\lim_{t\to+\infty} e^{-rt} H^0(R(t), u(t)) = 0,$$

we need the following proposition.

Proposition 35 *For each trajectory (R,u) starting at $(R_0, u_0) \in \overset{\circ}{\Omega}$ fulfilling the necessary conditions of Pontryagin's Maximum Principle and the condition that*

$$\lim_{t\to+\infty} e^{-rt} H^0(R(t), u(t)) \text{ is finite,}$$

the utility functional is given by

$$\int_0^{+\infty} e^{-rt} U(R(t), u(t)) dt = \frac{1}{r}\left(H^0(R_0, u_0) - \lim_{t\to+\infty} e^{-rt} H^0(R(t), u(t))\right).$$

[2] A classical example of such a steady state is the origin of the system $\dot{x} = x^2 + xy$, $\dot{y} = \frac{1}{2}y^2 + xy$.

Proof. See Appendix D.

To apply Proposition 35 to the trajectories that satisfy the necessary conditions of Pontryagin's Maximum Principle we hence have to compute $\lim_{t\to+\infty} e^{-rt} H^0(R(t), u(t))$.

3.3.1 The limits. In this subsection we want to compute

$$\lim_{t\to+\infty} e^{-rt} H^0(R(t), u(t)) \tag{10}$$

for those trajectories $(R(t), u(t))$, which satisfy the necessary conditions of Pontryagin's Maximum Principle and approach $(0,0)$ as $t \to +\infty$.

The current value Hamiltonian is

$$H(R,u,\lambda) = (\rho - cu + \lambda u)du\left(v + \frac{1}{R}\right) - \delta\lambda R.$$

The maximizing condition for interior solutions, $\frac{\partial H}{\partial u} = 0$, implies $2(\lambda - c)u + \rho = 0$ and therefore $\lambda - c \le 0$ in Ω. Note that since $\lambda - c \le 0$, the Hamiltonian H is concave in u, because

$$\frac{\partial^2 H}{\partial u^2} = 2(\lambda - c)d\left(v + \frac{1}{R}\right).$$

Using this information we get the maximized Hamiltonian

$$H^0(R,\lambda) = -\frac{\rho^2\, d}{4(\lambda - c)}\left(v + \frac{1}{R}\right) - \delta\lambda R.$$

An equivalent form of H^0 can be derived analogously by substituting λ instead of u from the Hamiltonian maximizing condition, yielding

$$H^0(R,u) = \frac{\rho}{2}\left(\frac{du}{R} + \frac{\delta R}{u} + dvu\right) - \delta c R.$$

Using Lemma E1 from the Appendix E, we can compute the limit (10), which we need in order to get the utility functional for the trajectories going to $(0,0)$ from Proposition 35.

Proposition 36 *If* (R,u) *starting at* $(R_0,u_0) \in \overset{\circ}{\Omega}$ *satisfies the necessary conditions of Pontryagin's Maximum Principle and*

$$\lim_{t\to+\infty} (R(t), u(t)) = (0,0),$$

then

$$\lim_{t\to+\infty} e^{-rt} H^0(R(t), u(t)) = \frac{\rho\delta}{2}\frac{R_0}{u_0}.$$

Proof. See Appendix E.

Note that for the special case $c = 0$ it is possible to obtain explicitly the expressions for the solution (R,u) starting at (R_0,u_0) and to compute directly the limit.

3.3.2 The comparison between the stable manifolds of the steady state and any trajectory in $\overset{\circ}{\Omega}$ going to $(0,0)$. Let $R_0 > 0$ be an initial reputation. We denote by (R^*, u^*) the trajectory starting on the stable manifold of $(\widehat{R}, \hat{u})$, at $(R_0, u^s(R_0))$. Let $(R_0, u_0) \in \overset{\circ}{\Omega}$ such that there is a trajectory (R, u) starting at (R_0, u_0) and going to $(0,0)$. Then

$$0 < u_0 < u^s(R_0). \tag{11}$$

We want to compare the values of the utility functionals for these two trajectories.

For (R^*, u^*), we have $\lim_{t\to+\infty} H^0(R^*(t), u^*(t)) = 0$, so the utility functional is (by a weak form of Proposition 35, cf. Feichtinger and Hartl, [6]) equal to

$$\frac{1}{r} H^0(R^*(0), u^*(0)) = \frac{1}{r}\left[\frac{\rho}{2}\left(\frac{du^s(R_0)}{R_0} + \frac{\delta R_0}{u^s(R_0)} + dvu^s(R_0)\right) - c\delta R_0\right]. \tag{12}$$

For the trajectory (R, u) starting at (R_0, u_0), according to Proposition 35 and 36 the utility function is equal to

$$\begin{aligned} \frac{1}{r} H^0(R_0, u_0) - \frac{1}{r}\lim_{t\to+\infty} e^{-rt} H^0(R(t), u(t)) &= \\ = \frac{1}{r}\left[\frac{\rho}{2}\left(\frac{du_0}{R_0} + dvu_0\right) - c\delta R_0\right]. \end{aligned} \tag{13}$$

Computing the difference between the utility functional values (12) and (13), we get

$$\frac{\rho}{2r}\left[(u^s(R_0) - u_0)\, d\left(v + \frac{1}{R_0}\right) + \frac{\delta R_0}{u^s(R_0)}\right].$$

From (11), $u^s(R_0) - u_0 > 0$, so this difference is positive. Hence for any initial reputation R_0 the solution with the initial control $u^s(R_0)$ is better than the solution with the initial control u_0. In other words, we can exclude the interior trajectories converging to $(0,0)$ as optimal solutions. What remains to do is to compare the stable manifold of the steady state with the border trajectory on the R-axis.

3.3.3 The comparison between the stable manifolds of the steady state and the trajectory on the R-axis. Let $R_0 > 0$ be an initial reputation. Then the trajectory starting at $(R_0, 0)$ goes along the R-axis to $(0,0)$. This trajectory on the R-axis given by $(R_0 e^{-\delta t}, 0)$ is not in $\overset{\circ}{\Omega}$ but it could of course be a solution of our maximization problem. We will now show that the utility functional for the trajectory (R^*, u^*) starting at $(R_0, u^s(R_0))$ on the stable manifold of $(\widehat{R}, \hat{u})$ is positive, while the utility functional for the trajectory starting at $(R_0, 0)$ is zero.

For $u \equiv 0$ we have

$$\int_0^{+\infty} e^{-rt}(\rho - cu)duV(R)dt = 0,$$

i.e. the utility function for the trajectory on the R-axis is in fact zero. For (R^*, u^*) starting at $(R_0, u^s(R_0))$, the utility functional is given by (12). To determine whether the last expression in (12) is positive we define

$$R_c := \frac{d\rho^2 v + \sqrt{d^2\rho^4 v^2 + 4d\rho^2 c^2 \delta}}{2c^2\delta}$$

and two functions c_1, $c_2 : [R_c, +\infty) \to \mathbb{R}_+$ given by

$$c_{1,2}(R) := \frac{c\delta R^2 \mp R\sqrt{c^2\delta^2 R^2 - d\delta\rho^2 - d\delta\rho^2 vR}}{d\rho(1 + vR)}.$$

For each $(R, u) \in \Omega$ which is not situated between the graphs of $u = c_1(R)$ and $u = c_2(R)$, the Hamiltonian $H^0(R, u)$ is positive. Obviously, $c_1(R) < c_2(R)$ for each $R \in [R_c, +\infty)$. Furthermore, we have the following lemma.

Lemma 37 *1. The function c_1 is strictly decreasing, and I_u is strictly increasing.*

2. $\lim_{R\to+\infty} c_1(R) = \lim_{R\to+\infty} I_u(R) = \frac{\rho}{2c}$.

Proof. To prove the first part of this lemma it suffices to compute the derivatives of c_1 and I_u and to show that they are negative and positive, respectively. The second part is also straightforward. Q.E.D.

It remains to show is that the value of the utility functional for the solution starting at $(R_0, u^s(R_0))$ on the stable manifold is positive. If $R_0 \geq R_c$, then by the monotonicity of u^s, the relation 3.b from Proposition 34, the second and the first part of Lemma 37 we have

$$u^s(R_0) \leq \lim_{R\to+\infty} u^s(R) \leq \lim_{R\to+\infty} I_u(R) = \lim_{R\to+\infty} c_1(R) < c_1(R_0).$$

Therefore the initial point $(R_0, u^s(R_0))$ is in the domain where the maximized Hamiltonian is positive, i.e. $H^0(R_0, u^s(R_0)) > 0$. If $R_0 < R_c$ then $H^0(R_0, u^s(R_0)) > 0$ by the definition of R_c.

4 Interpretation

We have proved that despite the flow's complex behavior around the origin $(0, 0)$, the stable manifolds of the unique interior saddle point equilibrium are always the optimal trajectories. The next task is to interpret these solutions to address the managerial questions raised in the introduction.

4.1 Basic Properties of All Solutions

Since there is just a single equilibrium it follows that it is optimal for any award, regardless of its initial reputation, to end up with a given final reputation. There is no threshold separating initial reputations such that high-prestige rewards remain prestigious and those below that threshold never become so.

Furthermore, there is a monotonic relationship between the current reputation of the award and the frequency with which it should be given at that moment. The more prestigious the award currently is, the less often it should be given. If it is optimal to adjust the reputation from its initial value, by altering the frequency with which it is given, that adjustment should be made smoothly. As long as the parameter values are held constant, it is never optimal to first increase and then decrease the frequency with which the award is given, or vice versa.

4.2 Special Case of a Pure Honor (Monetary Component $v = 0$)

When there is no monetary component to the award ($v = 0$) the steady state has an analytical expression

$$(\widehat{R}, \hat{u}) = \left(\sqrt{\frac{d}{\delta}} \frac{\rho r}{2c(r+\delta)}, \frac{\rho r}{2c(r+\delta)} \right).$$

In this case, the long-run policy prescription is simple. The frequency with which the award is given in steady state is the product of (1) the value of having a worker exert extra effort relative to the (administrative) cost of issuing an award and (2) the discount rate divided by the discount rate plus the rate at which the dilution of the award's reputation decays over time δ. The first term is obvious; the second is interesting. When managers are far-sighted and workers forgetful, the award will be given sparingly so that its reputation is high. When managers are myopic and past award frequencies heavily color the current reputation, the award will be given liberally and thereby be debased.

Intriguingly, when $v = 0$ the steady state reputation $1/\widehat{R}$ is exactly $\sqrt{\delta/d}$ divided by the steady state award frequency. It is obvious that high rates δ of forgetting past awarding should increase the equilibrium reputation relative to the award frequency. It is less obvious a priori but eminently reasonable in retrospect that the same should be true for greater disutility of work (smaller d). Greater resistance requires greater reward as a motivator. It was completely non-obvious a priori that the relationship should vary as the square root of the ratio of these two quantities.

Figure 2 about here

4.3 Case of an Award with Both Monetary and Honorific Dimensions

Unfortunately analytical expressions for the equilibrium are not available for the case $v > 0$. All we know is that

$$\hat{u} = \sqrt{\frac{\delta}{d}} \frac{\widehat{R}}{\sqrt{1 + v\widehat{R}}}.$$

We do not know whether the equilibrium reputation will be better or worse when v is increased from zero, but we do know that whatever the equilibrium reputation, the corresponding frequency with which the award is given is less than when that same equilibrium reputation is achieved for $v = 0$. That makes sense, since the award is more expensive to give when $v > 0$.

We can learn more, however, by looking at numerical examples. Clearly a crucial term in the model is the total value of earning one reward, $V(R) = v + 1/R$, which we find both in the objective functional and the state equation. Obviously, what matters in our model is not the value of the parameter v itself, but only the sum of v and the reward's reputation, $1/R$. In other words, a low value of the monetary value of the reward, v, can be compensated by a good reputation, i.e. a high value of $1/R$, and the same is true for a bad reputation together with a high monetary component.

What makes things a little bit complicated here is the fact that while v (a system parameter) is a constant, R a dynamic variable that changes over time. This means that the ratio between v and $1/R$ changes continuously. For our sensitivity analysis we therefore put v into relation with the steady state value of the reputation, $1/\widehat{R}$, which is of course a constant.

We consider three cases with respect to v: (1) pure honor ($v = 0$), (2) in steady state the honor and monetary compensation are equally valuable ($v = 2.78$), and (7.a-b) in steady state the honor is only one-tenth as valuable as the monetary component ($v = 7.95$). In particular, Figure 2 plots the stable and unstable manifolds for these three cases when $r = 0.1, c = 50, d = 0.000125, \delta = 0.3$ and $\rho = 2000$.

The figure shows that increasing the (constant) monetary component of the award "spreads" the stable and unstable manifolds apart, making the stable manifold more concave and pushing their intersection (the saddle point equilibrium) up and out, so the steady state reputation declines (i.e., $\widehat{R}$ increases). That makes sense. To the extent that the reward's value lies in its reputation, that reputation must be preserved. The more prominent the financial component, the less dependent management is on the reputation of the award, freeing it to give the award more liberally.

Figure 3 about here

4.4 Varying the Value of Increased Effort for a Fixed Monetary Award

Next consider how the solution varies if the value of inducing one person to work harder ρ increases for a fixed monetary award value v. For different values of ρ, the evolution of the optimal control u^* in time is qualitatively similar, but quantitatively different (see Figure 3). Starting with a strong initial reputation ($R_0 = 0.5$), the award frequency increases initially for all values of ρ, but the larger ρ is, the sharper the increase and the higher the final value of the control. That makes perfect sense (in Figure 3, $\rho_1 < \rho_2 < \rho_3$). All other things equal, the more valuable it is to get a worker to exert extra effort, the more aggressive management should be in the distribution of awards.

4.5 Connecting to Case of a Purely Monetary Award

Note the value of the award is $v + 1/R$. Thus the coefficient on the award's psychic value in the worker's objective function is fixed at unity. However, we can simulate a change in that coefficient by changing other, related parameters. The value of exerting extra effort is ρ, and the cost of exerting that effort is governed by d (small d corresponds to large cost). Hence, in particular, multiplying v, ρ, and c by a given constant and dividing d by that same constant is equivalent to dividing the coefficient on reputation by that same constant. Doing this shows that when the reputational effects are unimportant (not because R is large but rather because its leading coefficient is less than one), the solution becomes very simple. As soon as R is much above zero, the optimal policy is to place u at essentially its steady state value. I.e., if the reputational effects are omitted, this reduces down to a simple, static problem of incentivizing via monetary bonuses.

5 Extensions

There are many natural extensions, including allowing the monetary component of the award to vary over time, considering different distributions for the disutility of work, making the psychic value be a different (but presumably still decreasing) function of the state value R, etc. We elaborate here on an extension that is more subtle.

Just as awards can have both a monetary and a reputational component, where the power of the reputational dimension is diluted with frequent application, so to can sanctions have both a direct component and stigma, and the power of stigmatization can be diluted if the sanction is given too frequently. Nathaniel Hawthorne's Scarlet Letter was almost akin to a death sentence in Colonial America; it would not be so today. In more contemporary terms, criminologists have discussed “stigma swamping”, the idea that criminal justice sanctions, e.g., arrest for drug possession, can undermine their own effectiveness when used too often. That does not mean that the direct disutility of fines and incarceration are reduced if they are shared with many peers. However, post-release, it may be easier to reestablish employment, social networks, and self-esteem if one’s conviction was perceived to be routine or expected, rather than highly deviant within one’s peer group.

This situation is very similar to the one above, except now the work is criminal work so extra effort brings additional benefits to the “worker”, not extra cost from exertion, and the “award” becomes a sanction. The policy maker selects p, the probability an offense is sanctioned, not u. Part of the cost of being sanctioned is the associated stigma, and the more people who have been sanctioned recently, the lower the stigma. So the state variable R would be the inverse of how stigmatizing it is to be caught and punished for the offense in question, and as before this would decay via some “forgetting rate”. The cost of being sanctioned might be $(\nu + 1/R)$, where ν represents the stigma-independent component of the sanction (fine, prison term, etc.). The policy maker would decide how to vary p over time in order to minimize the amount of extra (criminal) effort plus the cost of administering sanctions. If p is too low, then there is no deterrent to crime, not even from fines and prison. If p is too high then R becomes large and the only deterrent is fines and prison; there is little stigma associated with getting arrested.

6 Conclusions

This paper makes three key contributions. First, it proposes and solves a model that extends the literature on incentivizing workers by considering the case of an award with a reputational component, not just a monetary value. The solution yields answers to practical managerial problems, some of which would have been hard to predict a priori.

Second, the model illustrates a rather strange and interesting structure with a point that looks for all intents and purposes like an equilibrium except that it is not within the admissible region. In particular, there are (infinitely many) trajectories converging to it and (infinitely many) trajectories starting arbitrarily close to it. We label this point a “false equilibrium”.

Third, we develop a proposition that yields the value of the utility functional when the limit of the Hamiltonian is non-zero. This proposition can be used to solve problems with a "false equilibrium" such as the one encountered here.

Hence, considering this simple model of reputational awards led to both practical and methodological contributions. It is our sense that further inquiries into other models of reputational awards will be similarly productive.

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Appendix

A Proof of Proposition 31

From system (7.a-b) we obtain the equation

$$\sqrt{\frac{\delta}{d}}\frac{R}{\sqrt{1+Rv}} = \frac{-\bar{c}R^2 + R\sqrt{\bar{c}^2R^2 + 4d(r+\delta)}}{2d}. \tag{14}$$

The cubic equation that we get by squaring twice the equation (14) has the following canonical form

$$e(R) := -\frac{\delta\bar{c}^2 v}{d}R^3 + \left[(\delta+r)^2v^2 - \frac{\delta\bar{c}^2}{d}\right]R^2 + 2rv(\delta+r)R^1 + r^2R^0 = 0.$$

We prove that the equation $e(R) = 0$ has exactly one positive solution. The equation has at least one positive root because $e(0)e(+\infty) < 0$. Suppose that it has another positive root. Then it must have three real roots. The coefficient of R^3 is negative and the coefficient of R^0 is positive, hence the roots of the cubic equation are either all positive, or one is positive and two are negative. On the other hand, the coefficient of R^1 is positive, therefore the first case (all roots positive) does not occur. Q.E.D.

B Proof of Proposition 32

The Jacobian at any $(R, u) \in \Omega$ is given by

$$\frac{\partial(\dot{R}, \dot{u})}{\partial(R, u)}(R, u) = \begin{bmatrix} \frac{3du^2}{R^2} + 2\bar{c}u - (\delta+r) & -\frac{2du^3}{R^3} \\ 2du(v + \frac{1}{R}) & -\frac{du^2}{R^2} - \delta \end{bmatrix}.$$

The steady state $(\widehat{R}, \hat{u})$ satisfies the system

$$\frac{\dot{R}}{R} = \frac{du^2}{R^2} + \frac{dvu^2}{R} - \delta \tag{15.a}$$

$$\frac{\dot{u}}{u} = \frac{du^2}{R^2} + \bar{c}u - (\delta+r). \tag{15.b}$$

Hence the Jacobian at $(\widehat{R}, \hat{u})$ has the form

$$\frac{\partial(\dot{R}, \dot{u})}{\partial(R, u)}(\widehat{R}, \hat{u}) = \begin{bmatrix} \frac{du^2}{R^2} + (\delta+r) & -\frac{2du^3}{R^3} \\ \frac{2\delta R}{u} & -\frac{du^2}{R^2} - \delta \end{bmatrix}. \tag{16}$$

The eigenvalues of (16) are the roots of the quadratic equation

$$E(\mu) := \mu^2 - r\mu - \left[\left(\frac{du^2}{R^2} - \delta\right)^2 + r\left(\frac{du^2}{R^2} + \delta\right)\right] = 0,$$

which has real roots of negative product. We therefore can conclude that the steady state $(\widehat{R}, \hat{u})$ is a saddle point.

Let μ_1 be the positive eigenvalue of (16) and let $(v_1, 1)$ be the corresponding eigenvector. In a similar way, let μ_2 be the negative eigenvalue of (16) and let $(v_1, 1)$ be the corresponding eigenvector. Denoting by $[a_{ij}]_{i,j=1,2}$ the Jacobian matrix (16) for the steady state $(\widehat{R}, \hat{u})$ we have

$$\text{sign}(v_2) = \text{sign}\left(-\frac{a_{12}}{a_{11} - \mu_2}\right) = \text{sign}\left(\frac{\frac{2du_0^3}{R_0^3}}{\frac{du_0^2}{\widehat{R}^2} + (\delta + r) - \mu_2}\right) > 0,$$

so v_2 is positive. The sign of the product $v_1 v_2$ is given by

$$\begin{aligned} \text{sign}(v_1 v_2) &= \text{sign}\left((-\tfrac{a_{12}}{a_{11}-\mu_1})(-\tfrac{a_{12}}{a_{11}-\mu_2})\right) = \\ &= \text{sign}(a_{11}^2 - a_{11}(\mu_1 + \mu_2) + \mu_1\mu_2) = \text{sign}(E(a_{11})). \end{aligned}$$

Since $E(a_{11}) = E\left(\frac{du^2}{R^2} + (\delta + r)\right) = 4\delta d\frac{\hat{u}^2}{\widehat{R}^2} > 0$, v_1 and v_2 have the same sign and therefore they are both positive. Q.E.D.

C Proof of Proposition 34

Let (R_0, u_0) be a point in a neighborhood of $(\widehat{R}, \hat{u})$ as defined in Proposition 33, situated on the stable manifold to $(\widehat{R}, \hat{u})$ such that $R_0 < \widehat{R}$, and let $\omega^L = (R^L, u^L)$ be the trajectory passing through (R_0, u_0) and going to $(\widehat{R}, \hat{u})$. Using ω^L we will define later the function u^s on $(0, \widehat{R})$.

We denote by J the maximal time interval of existence for the solution ω^L of the system (7.a-b) defined on $\overline{\Omega}$. Then there is t_- such that $J = (t_-, +\infty) \subseteq (-\infty, +\infty)$.

First we prove that R^L is strictly increasing on J. Let $J' = (a_-, +\infty)$ be the maximal interval on which R^L is strictly increasing. As a consequence of Proposition 33, R^L is strictly increasing on $[0, +\infty)$, therefore $J' \neq \emptyset$.

Suppose $a_- > t_-$. Then $\dot{R}^L(a_-) = 0$. For $t = 0$, from Proposition 33 we get

$$u^L(t) - I_u(R^L(t)) > 0.$$

On the other hand, since $R^L(a_-) < \widehat{R}$, we have

$$u^L(a_-) - I_u(R^L(a_-)) = I_R(R^L(a_-)) - I_u(R^L(a_-)) < 0. \tag{17}$$

Then there is $b \in (a_-, 0)$ such that $u^L(b) = I_u(R^L(b))$, i.e. $\dot{u}^L(b) = 0$.

Let $b_- \in (a_-, 0)$ be the smallest real number such that $\dot{u}^L(b_-) = 0$. By (17) the trajectory (R^L, u^L) on (a_-, b_-) remains under the graph of I_u, where $\dot{u} \leq 0$, so $u^L(a_-) \geq u^L(b_-)$. Using the monotonicity of I_R and that of R^L on $(a_-, +\infty)$ we get

$$u^L(a_-) = I_R(R^L(a_-)) < I_R(R^L(b_-)) < I_u(R^L(b_-)) = u^L(b_-),$$

contradiction. Therefore $a_- = t_-$, and R^L is strictly increasing on J.

Next, we prove that the range $\omega^L(J)$ of the trajectory ω^L is bounded. We analyze the two components R^L and u^L of ω^L separately. As we have just proved, R^L is strictly increasing on J. Hence $0 < R^L(t) < \lim_{t\to+\infty} R^L(t) = \widehat{R}$, for each $t \in J$.

Concerning u, let us assume that $u^L(J)$ is unbounded. Let $\bar{u} > 2\hat{u}$ be a real number such that

$$\frac{du^2}{\widehat{R}^2} + \bar{c}u - (\delta + r) > 0 \text{ for each } u > \bar{u}.$$

Since $u^L(J)$ is unbounded, there is $a \in (t_-, +\infty)$ such that $u^L(a) > \bar{u}$. We define

$$b_+ = \sup\{b > a \,|\, u^L(t) > \bar{u} \text{ for each } t \in [a, b)\}.$$

By the continuity of u^L, $b_+ > a$. For each $t \in [a, b_+)$ we have

$$\begin{aligned} \dot{u}^L(t) = u^L(t)\left(\tfrac{d(u^L(t))^2}{(R^L(t))^2} + \bar{c}u^L(t) - (\delta + r)\right) > \\ > \quad \bar{u}\left(\tfrac{d(u^L(t))^2}{\widehat{R}^2} + \bar{c}u^L(t) - (\delta + r)\right) \quad > 0, \end{aligned}$$

therefore u^L is strictly increasing on $[a, b_+)$. Then for each $n \in \mathbb{N}$ and $b_n \in [a, b_+)$ we have

$$2\hat{u} < \bar{u} < u^L(a) \leq u^L(b_n). \tag{18}$$

Suppose $b_+ < +\infty$. Then $b_n \to b_+$ implies $\bar{u} < u^L(b_+)$. Since u^L is continuous, there is ε such that for each $t \in (b_+ - \varepsilon, b_+ + \varepsilon)$ we have $u^L(t) > \bar{u}$, which contradicts the definition of b_+. Therefore $b_+ = +\infty$. For $b_n \to b_+$ from (18) we obtain a contradiction,

$$0 < 2\hat{u} < \bar{u} < u^L(a) \leq \hat{u},$$

so we finally conclude that $u^L(J)$ cannot be unbounded. In addition we have $\omega^L(J) \subset \Omega_0$, where $\Omega_0 := (0, \widehat{R}) \times (0, \bar{u} + 1)$.

Now we prove that

$$\lim_{t \searrow t_-} \omega^L(t) = (0, 0).$$

This means that the control goes to zero and no award will be given when the reputation is so good that it goes to infinity i.e. $R \to 0$. We suppose that there is a sequence $(t_n)_{n\in\mathbb{N}} \searrow t_-$ such that $\omega^L(t_n)$ does not converge to $(0, 0)$. The set $\overline{\Omega}_0$ is compact and $\omega^L(t_n) \in \Omega_0$ for each $n \in \mathbb{N}$ therefore there is a subsequence of $\omega^L(t_n)$ converging to some $(R_1, u_1) \in \overline{\Omega}_0$. Without loss of generality we take $\omega^L(t_n) \to (R_1, u_1)$ for $n \to +\infty$. A consequence of the Bendixson-Poincaré theorem (see Hartmann [11]) asserts that on an open simple connected plane set, containing no steady states each solution on its maximal interval of existence I does not remain in any compact subset, for t going to an element of ∂I. Applying this corollary to $\Omega_m = [\frac{1}{m}, \hat{R} - \frac{1}{m}] \times [\frac{1}{m}, \bar{u} + \frac{m-1}{m}]$, for each positive integer m, we obtain

$$(R_1, u_1) \in \cap_{m=1}^{+\infty} \overline{\Omega_0 \setminus \Omega_m} = \partial\, \Omega_0 =$$

$$= [0, \widehat{R}] \times \{0\} \cup \{0\} \times [0, \bar{u} + 1] \cup \{\widehat{R}\} \times [0, \bar{u} + 1] \cup [0, \widehat{R}] \times \{\bar{u} + 1\}.$$

But $(R_1, u_1) \notin \{\widehat{R}\} \times [0, \bar{u} + 1]$ and $(R_1, u_1) \notin [0, \widehat{R}] \times \{\bar{u} + 1\}$, because $R(t)$ is strictly increasing on $(t_-, +\infty)$ therefore $\lim_{n\to+\infty} R(t_n) = R_1 < \widehat{R}$ respectively $u(t) \le \bar{u}$ for each $t \in (t_-, +\infty)$ therefore $\lim_{n\to+\infty} u(t_n) < \hat{u}+1$.

It remains to show that $(R_1, u_1) = (0, 0)$ for the following two cases

- $(R_1, u_1) \in [0, \widehat{R}] \times \{0\}$
- $(R_1, u_1) \in \{0\} \times [0, \bar{u} + 1]$.

In the first case, suppose that $R_1 \neq 0$. The system (7.a-b) can be C^1-extended on a neighborhood U of $(R_1, 0)$ in $(0, +\infty) \times \mathbb{R}$, therefore the trajectory ω^L can be prolonged to a solution that passes through $(R_1, 0)$. On the other hand

$$t \mapsto (R_1 e^{-\delta t}, 0), \text{ for } t \in \mathbb{R}$$

is another solution of (7.a-b) in $(0, +\infty) \times \mathbb{R}$, which passes through $(R_1, 0)$. This contradicts the uniqueness of solution in U for the ODE system (7.a-b).

In the second case, suppose that $u_1 \neq 0$. From the system (15.a-b) we get

$$2\frac{\dot{u}}{u} - \frac{\dot{R}}{R} = \frac{du^2}{R^2}\left(1 - \frac{v}{R}\right) + \bar{c}u - 2\delta - r.$$

Since $\lim_{t\searrow t_-} \frac{u^2}{R} = +\infty$, it follows that for each t close enough to t_- we have $\left(\ln\left(\frac{u^2}{R}\right)(t) - t\right)' > 0$. Then, since $t_- < 0$, we obtain the following contradiction

$$+\infty = \lim_{t\searrow t_-} \frac{u^2}{R}(t) \le \lim_{t\searrow t_-} \frac{u_0^2}{R_0}\, e^t < \frac{u_0^2}{R_0}.$$

Therefore our assumption $u_1 \neq 0$ is false and $\lim_{t\searrow t_-} \omega^L = (0, 0)$.

We define u^s on $(0, \widehat{R})$ as follows: for each $R \in (0, \widehat{R})$ there is a unique $t \in (t_-, +\infty)$ such that $R^L(t) = R$. Then $u^s(R) := u^L(t)$. In a similar way we can define u^s on $(\widehat{R}, +\infty)$. In other words we have proved part 1 and part 2 of the proposition.

Now we prove that $u^L(t) \ge I_u(R^L(t))$ for each $t \in (t_-, +\infty)$. Suppose the contrary: there is $t_0 \in (t_-, +\infty)$ such that $u^L(t_0) < I_u(R^L(t_0))$. By the Proposition 33 there is $t_1 > t_0$ such that $u^L(t_1) > I_u(R^L(t_1))$. Then there is $b \in (t_0, t_1)$ and $\varepsilon > 0$ such that $u^L(b) = I_u(R^L(b))$ and

$$u^L(t) < I_u(R^L(t)) \text{ for each } t \in (b - \varepsilon, b).$$

It follows that $\dot{u}^L(t) < 0$ for each $t \in (b-\varepsilon, b)$ therefore $u^L(b-\varepsilon) > u^L(b)$. On the other hand I_u is strictly increasing, so

$$u^L(b-\varepsilon) < I_u(R^L(b-\varepsilon)) < I_u(R^L(b)) = u^L(b).$$

Contradiction, therefore $u^s(R^L(t)) = u^L(t) \geq I_u(R^L(t))$ for each $t \in (t_-, +\infty)$ and since $R^L(J) = (0, \widehat{R})$ we have proved the part 3.a of the proposition. The part 3.b is similar to 3.a and the part 4 is a simple consequence of part 3.a-b. Q.E.D.

Remark C1 *If we consider* $(0,0)$ *a "false" steady state then we can say that* $(0,0)$ *is an "unstable" equilibrium because there is trajectory going to* $(0,0)$ *and a trajectory going from* $(0,0)$*. A trajectory going from* $(0,0)$ *is* ω^L *as we have seen in the previous proposition. A trajectory going to* $(0,0)$ *is every trajectory in* Ω *closed to* $t \mapsto (R_0 e^{-\delta t}, 0)$*, with* $R_0 \in I\!R_+$ *which is a solution of (7.a-b) in the extended domain* $(0,+\infty) \times I\!R$*.*

D Proof of Proposition 35

Since model is autonomous, we have

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial R}\dot{R} + \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial \lambda}\dot{\lambda} = \\ &= 0 + (r\lambda - \dot{\lambda})\dot{R} + \dot{R}\dot{\lambda} = \\ &= r\lambda\dot{R}. \end{aligned} \tag{19}$$

Now

$$\begin{aligned} \lim_{t\to+\infty} e^{-rt} H(R(t), u(t), \lambda(t)) - H(R_0, u_0, \lambda_0) &= \\ = \int_0^{+\infty} \frac{d}{dt}[e^{-rt} H(R(t), u(t), \lambda(t))] dt &= \\ = \int_0^{+\infty} e^{-rt} \left[\frac{dH}{dt}(R(t), u(t), \lambda(t)) - rH(R(t), u(t), \lambda(t)) \right] dt &\overset{(19)}{=} \\ = \int_0^{+\infty} e^{-rt} [r\lambda(t)\dot{R}(t) - rH(R(t), u(t), \lambda(t))] dt &= \\ = -r \int_0^{+\infty} e^{-rt} U(R(t), u(t)) dt & \end{aligned}$$

Along each trajectory that satisfies the Pontryagin's Maximum Principle the value of H equals the value of H^0, hence

$$\int_0^{+\infty} e^{-rt} U(R(t), u(t)) dt = \frac{1}{r} \left[H^0(R_0, u_0) - \lim_{t\to+\infty} e^{-rt} H^0(R(t), u(t)) \right].$$

Q.E.D.

E Proof of Proposition 36.

First we prove the following lemma, which proof is based on successive approximations similar to that in the proof of the Gronwall's inequality

Lemma E1 *Let* (R, u) *be a trajectory starting at* $(R_0, u_0) \in \overset{\circ}{\Omega}$ *and satisfying the necessary conditions of Pontryagin's Maximum Principle in* $\overset{\circ}{\Omega}$ *. If*

$$\lim_{t\to+\infty}(R(t), u(t)) = (0, 0),$$

then

(E1.1) $\lim_{t\to+\infty} \frac{u}{R}(t) = 0$*, and*
(E1.2) $\lim_{t\to+\infty} e^{-rt}\frac{R}{u}(t) = \frac{R_0}{u_0}$.

Proof. (E1.1) The trajectory (R, u) is a solution of system (7.a-b) in $\overset{\circ}{\Omega}$, therefore it satisfies system (15.a-b). We have

$$\frac{\dot{u}}{u} - \frac{\dot{R}}{R} = \bar{c}u - r - \frac{dvu^2}{R} \leq \bar{c}u - r.$$

Since $\lim_{t\to+\infty} u(t) = 0$, for t big enough we get $\frac{\dot{u}}{u} - \frac{\dot{R}}{R} \leq -\frac{r}{2}$. Now $\left(\ln\left(\frac{u}{R}\right)(t) + \frac{r}{2}t\right)' \leq 0$ and therefore

$$\frac{u}{R}(t) \leq e^{-\frac{r}{2}t}\frac{u_0}{R_0}. \tag{20}$$

For $t \to +\infty$, we have

$$0 \leq \lim_{t\to+\infty}\frac{u}{R}(t) \leq \lim_{t\to+\infty} e^{-\frac{r}{2}t}\frac{u_0}{R_0} = 0.$$

(E1.2) From system (15.a-b) we obtain

$$\frac{\dot{R}}{R} - \frac{\dot{u}}{u} = r + \frac{dvu^2}{R} - \bar{c}u = r + u\left(\frac{dvu}{R} - \bar{c}\right) \leq r$$

for t big enough, because by (E1.b) we have $\lim_{t\to+\infty}\frac{dvu}{R} = 0$. Then $\left(\ln\left(\frac{R}{u}\right)(t) - rt\right)' \leq 0$ implies

$$\frac{R}{u}(t) \leq e^{rt}\frac{R_0}{u_0}. \tag{21}$$

From (20) and (21) we have

$$e^{\frac{r}{2}t}\frac{R_0}{u_0} \leq \frac{R}{u}(t) \leq e^{rt}\frac{R_0}{u_0}$$

therefore $\lim_{t\to+\infty} e^{-rt}\frac{R}{u}(t) = \frac{R_0}{u_0}$. Q.E.D.

It remains to prove the Proposition 36. Using the formula for the maximized Hamiltonian

$$\lim_{t\to+\infty} e^{-rt} H^0(R(t), u(t)) = \lim_{t\to+\infty} e^{-rt} \left[\frac{\rho}{2}\left(\frac{du}{R} + \frac{\delta R}{u}\right) + dvu - \delta cR\right] =$$

$$= \frac{\rho d}{2} \lim_{t\to+\infty} e^{-rt} \lim_{t\to+\infty} \frac{u}{R}(t) + \frac{\rho\delta}{2} \lim_{t\to+\infty} e^{-rt} \frac{R}{u}(t) + \lim_{t\to+\infty} e^{-rt}(dvu - \delta cR) =$$

$$= 0 + \frac{\rho\delta}{2}\frac{R_0}{u_0} + 0 = \frac{\rho\delta}{2}\frac{R_0}{u_0}.$$

Q.E.D.

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Fig. 1 The general phase diagram in the state-control plane.

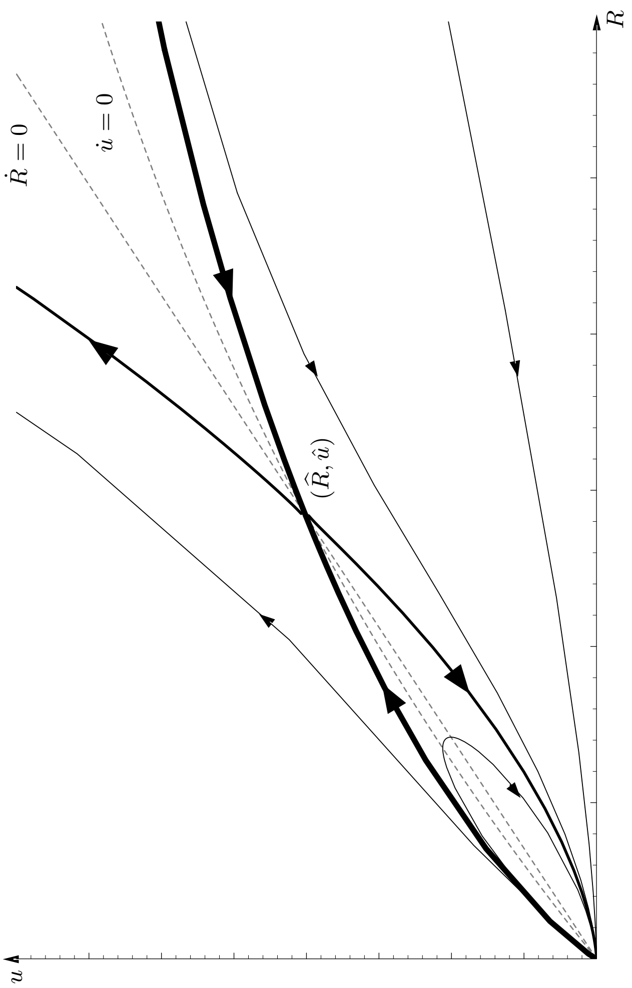

Fig. 2 The optimal solution for different values of v $(0, 2.78$ and $7.98)$ such that $R_0 v$ equals $0, 1$ and 10.

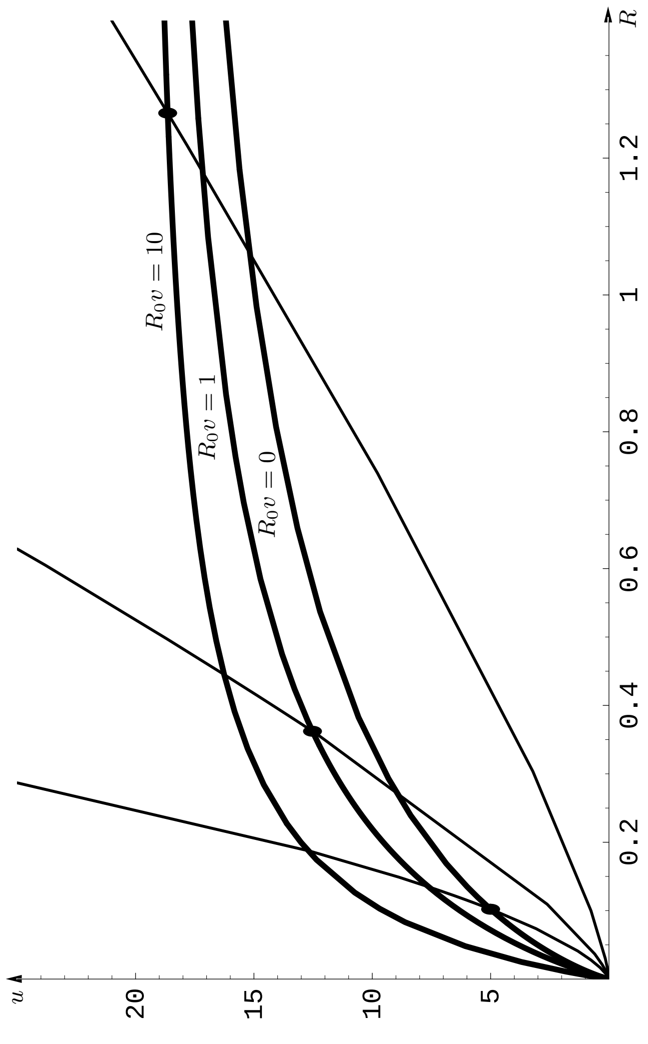

Fig. 3 The evolution of the control in time with starting reputation $R_0 = 0.5$ for $\rho_1 = 1500, \rho_2 = 2000, \rho_3 = 2500$. The other parameters are $r = 0.1, c = 50, d = 0.0125, v = 1, \delta = 0.3$.