Financially constrained capital investments: the effects of disembodied and embodied technological progress*

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Abstract

Empirical studies stress the significance of financing constraints in business investment. Especially high tech investment is likely to be affected by capital market imperfections. The reason is that their returns are highly uncertain so that it is difficult to get outside finance for this kind of investment. This paper studies the combined effect of technological progress and the capital market being imperfect on the firm's investment behavior. We show that it is crucial to make a distinction between embodied and disembodied technological progress. Embodied technological progress affects only the capital goods built after the technological breakthrough while disembodied technological progress influences the productivity of all capital goods installed. It is shown that where disembodied technological progress leads to a positive anticipation phase before

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the technological breakthrough occurs, a negative anticipation phase will occur before an embodied technological breakthrough. During this negative anticipation phase the firm builds up a stock of liquid financial assets in order to be able to finance increased investments in the improved capital goods from after the technological breakthrough.

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1 Introduction

The aim of this paper is to study the simultaneous effect of technological progress and an imperfect capital market on the firm's investment behavior. With respect to technological progress we make a distinction between embodied and disembodied technological progress. Disembodied technological progress affects all capital goods including those installed in the past. This implies that under disembodied technological progress the whole capital stock is affected independently of its age distribution. This can for instance occur when technological progress positively affects the infrastructure or the schooling of workers.

Embodied technological progress at time $\bar{t}$ only affects capital goods built after time $\bar{t}$, because only those capital goods incorporate the latest technological advances (Boucekkine and Pommeret (2004)). Therefore, to investigate the effects of embodied technological progress on the firm's investment behavior, it is necessary to distinguish capital stocks of different building years, which is typically done in a vintage capital stock model. Embodied technological progress is the main driver of economic growth. Greenwood et al. (1997) states that 60% of labor productivity in post war US is investment specific. Yorokoglu (1998) points at the very high pace of improvement of information technology capital: in less than a decade (1980s) personal computers are 20 times faster, while the price of new computers did not change over the years.

Chari and Hopenhayn (1991) posed three questions regarding innovation dynamics. The first question was: why are new technologies adopted so slowly? In general this has to do with learning. As an example we can refer to the LCD industry, where a considerable ramp up time (time needed to start the production line) coexists with a yield (amount of good products relative to the total amount of products) that strongly increases over time. The second question was: why do people often invest in old technologies while at the same time apparently superior technologies are available? To analyze this observation, a vintage capital stock model is needed that allows investment in older machines, thus in non-frontier vintages[1]. Such a model was developed in Feichtinger et al. (2006), in which one possible explanation was found: just before a technological breakthrough firms do not want to buy expensive capital goods, because after this technological breakthrough a more productive new generation of capital goods will become available. Instead, firms prefer to invest in cheaper older machines with a shorter lifetime that can be scrapped around the timing of the technological breakthrough.

Chari and Hopenhayn's third question was: how are decisions to adopt new technologies affected by the prospect that even better technologies will arrive in the future? This clearly relates to technological progress of the embodied type. In Feichtinger et al. (2004) it was found that the decision to adopt new technologies is not affected by future technological progress. The reason is that in that model the firm exerts no market power. Hence, increasing its own production does not have an effect on output prices, so that the NPVs of capital stocks of different age do not influence each other. In Feichtinger et al. (2006) market power was introduced and negative anticipation effects were found in the sense that during a period of rapid technological progress the firm delays investments in new machines in order to undertake the investments after the technological progress has taken place. As just mentioned, the reduced acquisition of new machines is compensated by investing more in older machines so that the production level can be kept at approximately the same level. A similar negative anticipation effect was found in Pakko (2002). That paper adopts a general equilibrium framework in which new vintages draw labor away from the old ones. Consequently, when technological progress make new vintages more attractive, more capital goods of these vintages will be attracted. Hence, the new vintages will draw more labor away in the future from the current vintage and therefore reduces its marginal product and thus its profitability. This causes the reduction of investments in current vintages and thus the negative anticipation effect.

Besides market power or a redistribution of labor among the vintages, there can be another reason to delay current investments, and that is when money is scarce, i.e. the firm must earn the money first before it is able to invest. Purpose of this paper is to analyze the effects of technological progress on the capital accumulation process of a firm that operates on an imperfect capital market. This implies that while establishing its investment plan the firm has to take into account financial constraints (for instance, raising funds to invest in a new generation in the LCD industry causes problems for Taiwan's major players, see, e.g., http://www.ebnonline.com/computer/news/story/OEG20020815S0022). Such a financial constraint can be expected to exert offsetting effects on the desirability of current investments. Whereas investing now reduces the financial means to invest in the immediate

[1]For this reason the work by, e.g., Solow et al. (1966), Malcomson (1975), Benhabib and Rustichini (1991), and Boucekkine et al. (1997, 1998, 1999, 2001) cannot be used, because there it is only possible to invest in the newest generation of capital goods. The common denominator of the just mentioned contributions is the application of delayed differential equations. Contrastringly, by applying a partial differential equation approach we (and also Feichtinger et al. (2004, 2005, 2006)) are able to introduce the possibility of investing in older capital goods.

future, it also raises production and thus future revenues, which provides financial means for investments later on.

Early applied research on investment (see Meyer and Kuh, 1957), as well as a later study by Fazzari et al. (1988), stressed the significance of financing constraints in business investment. Eisner (1978) found that the timing of investment in small firms is more sensitive to profits than it is in large firms. According to Carpenter and Peterson (2002), a large number of recent empirical studies report evidence that firm's investments depend on its financial condition. Especially high tech investment is likely to be affected by capital market imperfections. The reason is that their returns are highly uncertain so that it is difficult to get outside finance for this kind of investment.

We study the problem of a firm dealing with disembodied technological progress by setting up a non-autonomous optimal control model, where investments are constrained by limited financial means. A state variable representing the firm's financial assets is introduced. These financial assets are liquid so that their value can be used at any time for spendings on investments in capital stock and dividends. The stock of financial assets can be increased by revenue on selling products and interest obtained when the financial assets are positive. To analyze the problem of embodied technological progress, we have to take into account that the productivity of capital goods of different vintages may differ. For this reason we intend to extend the dynamic theory of the firm by developing a vintage capital goods model in which the productivity of different vintages differ, and financial assets similar as in the "disembodied model" are introduced.

As in Feichtinger et al. (2006), in the model with embodied technological progress and an imperfect capital market we also find a negative anticipation effect. However, while in that paper this was caused by market power, here the financial structure leads to this effect: during a time interval before a technological breakthrough occurs, the firm saves money in order to increase investments in machines embodying the new technology as soon as they are available. Saving money implies that in such a time interval the firm cuts down on investments and paying dividends, and increases the stock of financial assets instead. The extra financial assets are used to finance extra investments in the better technology from after the technological breakthrough. Here a substitution effect takes place in that the firm cuts down on investments in the worse technologies from before the breakthrough in order to increase purchasing the better technologies from after the breakthrough. So the arrival of good news in the form of a technological breakthrough coming up leads to a (temporary) negative effect in the form of reduced firm growth taking place before the technology shock.

Compared to Feichtinger et al. (2006) a difference can be detected with respect to investments in older machines before the technological breakthrough: where in Feichtinger et al. (2006) the reduced investments in new machines were accompanied by increased investments in older machines, in the present paper we see no such effect. The reason is that the firm prefers to acquire financial assets instead in order to be able to finance more investments in the improved capital stock from after the technological breakthrough.

Another remarkable finding is that if we consider disembodied technological progress rather than embodied technological progress, the negative anticipation effect changes into a positive anticipation effect. The reason is that now the technological progress also raises the productivity of machines already installed. Hence, it becomes more attractive to purchase current machines already at a time interval that starts before the technological breakthrough occurs, which will lead to an investment boom there. The implication is that the increase in investments takes place gradually in the disembodied case, while it leads to non-monotonic behavior in the form of a period of reduced investments followed by an investment jump in the embodied case. In other words, investment is continuous over time in the disembodied case, while investment in new machines jump up just after the technological breakthrough has occurred.

The continuity properties are opposite when we consider production rather than investment. In the disembodied case production jumps up at the time of the technological breakthrough (since productivity of all machines in use jumps up), while the production level is continuous in the embodied case. There production first decreases (caused by the negative anticipation effect), but then increases gradually over time during a time period that begins right at the time of the technology shock. Of course, after the technological breakthrough production will eventually reach a higher level than it had before the technology shock. The negative anticipation effect also causes a production decrease in Pakko (2002), but in Feichtinger et al. (2006) production remains constant before the technology shock since less investments in new machines are compensated by increased investments in older ones.

The paper is organized as follows. Section 2 presents the analysis of a benchmark model in which there is no technological progress. Section 3 studies the implications for optimal investment behavior from disembodied technological progress, while Section 4 does the same but then for embodied technological progress. The proofs from Sections 2, 3, and 4 are presented in Appendices 1, 2, and 3, respectively.

2 No Technological Progress

This section presents the benchmark model without technological progress. Consider a model of a firm, producing a single good by means of a homogeneous capital stock (K). The capital stock depreciates by a fixed rate δ and is used to produce goods. The number of goods produced by one machine in one year is given by the constant f. The goods are sold on the output market against a constant unit price p (we thus assume that the firm is small, in the sense that it can not influence the unit price of output), so that revenue equals pfK. The main results of our paper also hold in a qualitative sense when the firm has market power, i.e. $p'(fK) < 0$, but we choose not to include this so that we do not need to disentangle results coming from an imperfect capital market (Feichtinger et al., 2006, shows that market power leads to a negative anticipation phase in case of embodied technological progress, and we will show later on that this will also hold in the case of an imperfect capital market).

Capital stock can be increased by investing (I). In order to invest the firm has to incur acquisition and adjustment costs. Unit costs of acquisition are normalized to one, while adjustment costs are denoted by $\frac{c}{2}I^2$. The adjustment costs, being for example, adaptation costs, search costs or installation costs, are thus assumed to be quadratic. This is not essential, but leads to a simple expression for the optimal purchases.

The firm operates on an imperfect financial capital market. Therefore, according to, e.g., Hubbard (1998) it is important to consider investment and financial policy jointly: firms may, for example, accumulate liquidity to have more financial means available for investing in the future. This can especially be beneficial in case capital goods become better over time due to technological progress. In our model the firm can build up a stock of financial, or liquid, assets (B). Financial assets are sufficiently liquid in the sense that their value can be immediately spend on investments in case the firm wishes to do so. The return on these financial assets is assumed to be fixed and denoted by the interest rate i. The stock of financial assets increases with the difference between the sum of revenues on interest (iB) and selling products (pfK), and the sum of investment expenses ($I + \frac{c}{2}I^2$) and dividends (D):

$$\dot{B} = iB + pfK - I - \frac{c}{2}I^2 - D, \quad B(0) = B_0 \geq 0.$$

In this model it is assumed that the firm cannot borrow. This is a considerable simplification but note that our main aim is to study investment behavior under the joint occurrence of technological progress and an imperfect capital market. The latter characteristic implies that money is scarce, and for this feature to be preserved it is not essential whether borrowing is allowed or not. Not being able to borrow implies that

$$B(t) \geq 0.$$

The firm chooses to buy or sell machines at different points of time in order to maximize the total discounted dividend stream. Denoting the discount rate by r, the resulting dynamic model of the firm is given by

$$\max_{I,D} \int_0^\infty e^{-rt} D(t)\, \mathrm{d}t \tag{1}$$

subject to

$$\dot{K} = I - \delta K, \quad K(0) = K_0 > 0, \tag{2}$$

$$\dot{B} = iB + pfK - I - \frac{c}{2}I^2 - D, \quad B(0) = B_0 \geq 0, \tag{3}$$

$$B \geq 0, \tag{4}$$

$$D \geq 0. \tag{5}$$

It is reasonable to assume that marginal revenue exceeds the user cost of capital, i.e.,

$$fp > r + \delta. \tag{6}$$

In order to focus on the firm's investment behavior where we want to compare policies like (i) immediate dividend payouts, (ii) investing first to obtain more revenue later and (iii) increasing the stock of financial assets first to invest in more productive capital goods later (this concerns the case of technological progress), we assume that

$$i < r. \tag{7}$$

Note that imposing $i \geq r$ will imply that increasing B will always dominate paying dividends. Under $i < r$ it holds that, if $B > 0$, holding financial assets now in order to invest more in capital stock later comes at a cost because the return on paying dividends is higher.

In case it is not possible to invest in financial assets, i.e. $B(t) = 0$, the model is similar to Kort (1988) or Van Hilten et al. (1993, Section 5.4.2). The following lemma proves that an optimal solution exists and that the optimal investment rate is continuous over time. It is formulated for a non-constant productivity $f(t)$ so that it can also be used in the next section where technological progress is considered.

Lemma 1 *Assume that r, b, c, δ, and K_0 are strictly positive, that $f(s) \geq f > 0$ is piecewise continuous and bounded, and that (6) and (7) hold. If the problem (1)–(5) has a solution, then the optimal $I(t)$ is a continuous function. Moreover, the problem has a solution if the constraint $D \leq \bar{D}$ is added, with any positive number $\bar{D}$.*

Remark 1 This lemma applies also to the case of technological progress, $f(t)$. The additional constraint $D \leq \bar{D}$ is essential, since for $B(0) > 0$ it could be "optimal" to pay an impulsive dividend at time zero (see the last paragraph of this section). However, the subsequent analysis will show that if $B(0) = 0$, then the constraint $D \leq \bar{D}$ will not be binding for a sufficiently large $\bar{D}$. Therefore, in this case the bound $\bar{D}$ is not needed for existence.

The Hamiltonian and Lagrangian functions are (see e.g. Feichtinger and Hartl, 1986)

$$\begin{aligned} H &= D + \lambda_1 (I - \delta K) + \lambda_2 \left(iB + pfK - I - \frac{c}{2} I^2 - D \right), \\ L &= D + \lambda_1 (I - \delta K) + \lambda_2 \left(iB + pfK - I - \frac{c}{2} I^2 - D \right) + D\mu + B\eta. \end{aligned}$$

The costates must satisfy the adjoint equations

$$\dot{\lambda}_1 = r\lambda_1 - L_K = (r + \delta)\, \lambda_1 - \lambda_2 pf, \tag{8}$$

$$\dot{\lambda}_2 = r\lambda_2 - L_B = (r - i)\, \lambda_2 - \eta, \tag{9}$$

while the first order conditions for the controls are

$$L_I = \lambda_1 - \lambda_2 (1 + cI) = 0, \tag{10}$$
$$L_D = 1 - \lambda_2 + \mu = 0. \tag{11}$$

In general it holds that in the presence of the state constraint (4), the corresponding adjoint variable λ_2 could jump. However, Lemma 1 in connection with (10) implies that this jump cannot occur in our model.

We solve the problem by path coupling. The following paths can be distinguished. The proofs of the last two rows of this table are contained in Appendix 1.

	Path 1	Path 2	Path 3	Path 4
η	0	+	+	0
μ	+	0	+	0
feasible?	yes	yes	yes	no
final path?	no	yes	no	no

We proceed by describing the main characteristics of the feasible paths. A complete mathematical description can be found in Appendix 1.

Path 1: holding financial assets:

Here we have $\eta = 0$, $\mu > 0$, so that $D = 0$ and $B \geq 0$. This implies that

$$\dot{B} = iB + pfK - I - \frac{c}{2} I^2.$$

Path 2: no financial assets - pay dividends:

Here we have $\eta > 0$, $\mu = 0$, which implies $D \geq 0$, while $B = 0$. This means that

$$\dot{B} = pfK - I - \frac{c}{2} I^2 - D = 0.$$

If the path is in steady state, then this steady state is given by

$$1 + c\hat{I} = \frac{pf}{(r + \delta)}, \tag{12}$$

meaning that marginal investment expenses equal the discounted revenue stream, and

$$\hat{K} = \frac{\hat{I}}{\delta} = \frac{fp - (r + \delta)}{\delta c (r + \delta)}.$$

If Path 2 occurs as a final path, capital stock converges to this steady state $\hat{K}$ following the well known flexible accelerator rule (e.g. Lucas (1967)):

$$\dot{K} = \delta \left(\hat{K} - K \right).$$

Path 3: maximal investment:

Here we have $\eta > 0$, $\mu > 0$ which implies $D = 0$ while $B = 0$. So, it holds that

$$\dot{B} = pfK - I - \frac{c}{2} I^2 = 0.$$

The following proposition contains the main result of this section. It shows that, if the firms starts out with a positive stock of financial assets, it starts to use these financial assets to finance investments (Path 1). This is followed by a phase where the firm holds zero financial assets while growing at its maximum. This takes place as long as investment is that low that the discounted revenue stream exceeds marginal investment expenses (Path 2). As soon as they become equal (cf. (12)), the firm fixes investment at $\hat{I}$ and approaches the long run steady state value $\hat{K}$ according to the flexible accelerator rule (Path 3). Hence, whenever the stock of financial assets is positive, it is decreasing over time. This is because in a scenario without technological progress there is no incentive to accumulate liquidity in order to increase investments later at the expense of reducing investments now. This will change in the sections to come, where technological progress is explicitly taken into account.

Proposition 1 *Define*

$$K^* = \frac{\hat{I} + \frac{c}{2} \hat{I}^2}{pf}. \tag{13}$$

The optimal solution of the problem with constant productivity f depends on the initial values of the state variables:

Scenario 1: $B(0) > 0$ and $K(0) < K^*$. Path 1 $\rightarrow$ Path 3 $\rightarrow$ Path 2,

Scenario 2: $B(0) = 0$ and $K(0) < K^*$. Path 3 $\rightarrow$ Path 2,

Scenario 3: $B(0) = 0$ and $K(0) \geq K^*$. Path 2.

Figure 1 illustrates the optimal solution.

[Figure 1 about here.]

The above proposition does not provide a solution in case $K(0) > K^*$ and $B(0) > 0$. In this case the firm spends its financial assets impulsively on dividends at time zero, as depicted in Figure 2. In general it holds that at the initial point of time an impulsive dividend is paid out when $B(0)$ is too large compared to the corresponding $K(0)$. For sufficiently small values of $K(0)$ the impulse dividend takes place in such a way that, just after the impulse dividend occurred, on Path 1 $B(t)$ is gradually reduced to zero before the capital stock reaches the level K^*.[2]

[Figure 2 about here.]

3 Disembodied Technological Progress

Disembodied technological progress means that at the moment it appears technological progress increases the productivity of the whole capital stock, so not only of the new capital goods. This could be caused by an improved infrastructure, including schooling of workers.

The model is similar to the one in the previous section with the exception that now $f = f(t)$:

$$\max_{I,D} \int_0^\infty e^{-rt} D(t)\,\mathrm{d}t \tag{14}$$

subject to

$$\dot{K} = I - \delta K, \quad K(0) = K_0 > 0, \tag{15}$$

$$\dot{B} = iB + pf(t)K - I - \frac{c}{2}I^2 - D, \quad B(0) = B_0 \geq 0, \tag{16}$$

$$B \geq 0,$$

$$D \geq 0. \tag{17}$$

We consider a scenario where technological progress takes place by a productivity jump at time $\bar{t}$:

$$f(t) = \begin{cases} f & \text{for } t \leq \bar{t}, \\ F & \text{for } t > \bar{t}, \end{cases} \tag{18}$$

in which f and F are positive constants such that

$$f < F.$$

[2]In Appendix 1 it is shown that the sequence Path 1 $\rightarrow$ Path 2 is not possible, implying that Path 1 cannot contain the point $(0, K^*)$ or any point $(0, K)$ for $K > K^*$. So there must exist a point $\tilde{K} \in (0, K^*)$ representing the maximal value of K for which Path 1 can occur. To prevent occurrence of Path 1 for $K > \tilde{K}$ a policy of impulsive dividends can be applied to get rid of superfluous financial assets. However, note that occurrence of impulsive dividends is less relevant for the technological progress problem. This is because financial assets will not be collected now in order to pay impulsive dividends later. The reason is that such a policy is dominated by paying more dividends immediately (note that $r > i$).

This means that we have process innovation, meaning that more goods are produced using a capital good after the timing of the technological breakthrough. Although we thus consider the case of process innovation rather than product innovation, it is good to note that our results carry over to the case of product innovation[3]. To see this, consider a situation where a technological breakthrough increases the product's quality. This implies that the firm can ask a higher price, i.e.

$$p(t) = \begin{cases} p & \text{for } t \leq \bar{t}, \\ P & \text{for } t > \bar{t}, \end{cases} \tag{19}$$

with $P > p$. Then it is clear that here process and product innovation have the same effect: both under (18) and (19) unit revenue increases due to the technology shock.

For $t > \bar{t}$ Proposition 1 applies with f replaced by F. What remains to be solved is the problem for $t < \bar{t}$. Analogous to the analysis of the problem without technological progress in the previous section, three paths can occur in each of the technology regions. The Paths 1, 2, and 3 for the non-technological progress problem relate to Paths 1f, 2f, and 3f in the time interval before the jump, while these paths are denoted by 1F, 2F, and 3F, respectively, for the time interval after the jump (note that Path 4 is irrelevant because of its infeasibility).

We consider the scenario where at the initial point of time the firm has reached Path 2f and that it is in its optimal steady state associated with technology level f, i.e.

$$K(0) = \hat{K}_f = \frac{fp - (r+\delta)}{\delta c (r+\delta)}.$$

The aim is to compare the optimal behavior of two firms: one, the anticipating firm, having perfect information about the technology shock at $\bar{t}$ from the very beginning, and the second, non-anticipating, firm that does not expect the shock and only learns about it at the moment it happens. Consequently, before time $\bar{t}$ the latter firm's investment behavior is such as if this breakthrough will not take place and the investment behavior of the non-anticipating firm satisfies

$$I(t) = \hat{I}^f = \frac{fp - (r+\delta)}{c(r+\delta)} \text{ for } t \leq \bar{t}. \tag{20}$$

In this section the main result is the following.

Proposition 2 *Consider the investment behavior of a non-anticipating and of an anticipating firm, for which at $t = 0$ Path 2f applies with $I(0) = \hat{I}^f$ (that is, both firms are at their optimal steady state). There exists an $\varepsilon > 0$ such that on the time interval $[\bar{t} - \varepsilon, \bar{t}]$ the anticipating firm invests more than the non-anticipating firm. This means that for the investment behavior of the anticipating firm it holds that*

$$I(t) > \hat{I}^f \textit{ for } t \in [\bar{t} - \varepsilon, \bar{t}]. \tag{21}$$

[3]We thank Omar Licandro for making this observation.

In Proposition 1 we determined the complete trajectories for the problem without technological progress. Doing the same for the model of this section turned out to be an enormous task. However, fortunately Proposition 2 shows that without fulfilling this task we are still able to determine the result we are after, which is to establish the behavior of the investing firm facing a future technological breakthrough. We found that the technological breakthrough at time $\bar{t}$ induces a positive anticipation phase that takes place before this time. The intuition of this result is straightforward. Since disembodied technological progress also enhances the productivity of existing machines, it becomes more attractive to invest in them just before a technological breakthrough is realized.

To have an idea about the length of the time interval at which this positive anticipation phase occurs, we consider a scenario where the firm does not own any financial assets, i.e. $B(t) = 0$. Then a straightforward application of the maximum principle shows that the investment behavior of the anticipating firm before the technology jump is given by

$$I(t) = \hat{I}^f + \frac{p}{c[r+\delta]} e^{-[r+\delta][\bar{t}-t]} [F - f] \text{ for } t \leq \bar{t}. \tag{22}$$

Comparing (20) and (22), we find that the anticipating firm invests more than the other one from the very first time it learns about the future technological jump, $t = 0$, implying that the ε from Proposition 2 equals $\bar{t}$.

The next proposition is about the stock of financial assets. It shows that the period of accumulation of financial capital (if any) for the anticipating firm finishes strictly before the technological breakthrough takes place. This implies that the firm starts using the financial capital to finance the positive anticipation phase before $\bar{t}$.

Proposition 3 *If for the anticipating firm $B(\bar{t}) > 0$, then $\dot{B}(t) < 0$ on some interval $[\bar{t} - \varepsilon, \bar{t}]$.*

4 Embodied Technological Progress

Since embodied technological progress causes differences in the productivity of capital stocks of different ages, we need to develop a vintage capital stock model to analyze its implications. To do so we consider a model of a firm producing a single good by means of a continuum of vintage capital technologies, with finite life time. At every period the firm has access to machines in the primary and secondary market and faces adjustment costs in both markets. Productivity of these machines is influenced by technological progress. Like in the previous sections the firm faces an imperfect capital market and has the possibility to build up a stock of financial assets. As a consequence of the financing constraint and the adjustment costs, it will turn out that the firm adopts the frontier technology slowly.

To set up the vintage model, we have to distinguish between different generations of machines. To do so we explicitly introduce the age of the machine, which is denoted by a. Each machine has a fixed maximal lifetime ω, so that $a \in [0, \omega]$. However, the investment can be negative and the firm has the possibility to sell the machines before the end of the lifetime.

The machines are used to produce goods. The number of goods produced in year t by a machine of age a and vintage $t-a$ is given by $f(t-a)$. When we have embodied technological progress, the output produced by a machine of some fixed age a increases with the vintage $t-a$. The idea is that more modern machines are more productive because they embody superior technology. In that case f is an increasing function and may jump upwards at some particular point(s) of time.

The stock of capital goods of age a at time t is denoted by $K(t,a)$. Then the total output produced at time t is defined as

$$Q(t) = \int_0^\omega f(t-a)\, K(t,a)\, \mathrm{d}a.$$

We again assume that the firm is small, in the sense that it cannot influence the unit price of output p.

Capital stock can be increased by investing. Following, e.g., Barucci and Gozzi (1998), and Xepapadeas and De Zeeuw (1999), it is assumed that markets exist for machines of any age from 0 to ω. Let $b(a)$ be the cost of buying a machine of age a. Thus the acquisition price does not depend on calendar time: at the same time that machines get more productive their acquisition price remains the same. As pointed out by Yorokoglu (1998), this for instance holds for computing technology, that, within one decade, improved on the order of twenty times in terms of both speed and memory capacities, without increasing the cost.

Investments in machines of age a in year t are denoted by $I(t,a)$. The total cost (or revenue) to the firm from transactions in the machine market is $b(a)I(t,a) + \frac{c}{2}\left[I(t,a)\right]^2$, with the second term reflecting the adjustment costs resulting from buying or selling machines. We separately consider investment in the frontier vintage: let $I_0(t)$ be the investment in new machines at time t. Then the cost of purchase is $b_0 I_0(t) + \frac{c}{2}(I_0(t))^2$, where b_0 is the acquisition price and c corresponds to the adjustment cost. This separation of I_0 and $I(t,a)$ allows to consider only investments in new machines as a separate framework. Clearly, $b(a)$ is decreasing, $b(0) \leq b_0$ (older machines cannot be more expensive than newer machines) and $b(\omega) = 0$ (a machine at the maximum age is not worth anything).

It holds that due to embodied technological progress, leading to an increase of $f(t-a)$, the relative price of equipment in terms of output declines. This is in accordance with Greenwood et al. (1997), who mentions the examples of new and more powerful computers, faster and more efficient means of telecommunication, robotization of assembly lines, and so on.

The evolution of capital is described by

$$K_t(t,a) + K_a(t,a) = I(t,a) - \delta K(t,a), \quad K(0,a) = K_0(a), \;\; K(t,0) = I_0(t),$$

where the subscripts denote partial differentiation and, as before, δ is the depreciation rate[4]. This specification is based on the fact that time and age move together: machines aged $a + \mathrm{d}t$ at time $t + \mathrm{d}t$, are those aged a at time t, minus their depreciation, plus (minus) machines of this vintage bought (sold) at time t.

The firm aims to maximize the total discounted dividend stream. It does so by determining its optimal investment strategy, which consists of buying or selling machines of different ages. In this way the firm chooses the optimal age distribution of machines at each point in time.

We impose that an investment needs to be financed before it is undertaken. To do so, as in the previous section we introduce $B(t)$ – the financial assets that can be instantaneously converted into investments and dividends. On the other hand, the firm can increase the financial assets by revenue pQ and interest payments iB. This leads to the following dynamic equation:

$$\dot{B}(t) = iB(t) + pQ(t) - D(t) - b_0 I_0(t) - \frac{c}{2}[I_0(t)]^2 - \int_0^\omega \Big[b(a)I(t,a) + \frac{c}{2}[I(t,a)]^2\Big]\,\mathrm{d}a.$$

With the same objective function as in the disembodied case, the whole model becomes:

$$\max_{I(t,a),I_0(t),D(t)} \int_0^\infty e^{-rt} D(t)\,\mathrm{d}t, \tag{23}$$

$$\begin{aligned}
K_t(t,a) + K_a(t,a) &= I(t,a) - \delta K(t,a), \quad K(0,a) = K_0(a), \;\; K(t,0) = I_0(t), && (24)\\
\dot{B}(t) &= iB(t) + pQ(t) - D(t) - W(t), \quad B(0) = B_0, && (25)\\
Q(t) &= \int_0^\omega f(t-a)K(t,a)\,\mathrm{d}a, && (26)\\
W(t) &= b_0 I_0(t) + \frac{c}{2}[I_0(t)]^2 + \int_0^\omega \Big[b(a)I(t,a) + \frac{c}{2}[I(t,a)]^2\Big]\,\mathrm{d}a && (27)\\
I_0(t) &\geq 0, && (28)\\
B(t) &\geq 0, && (29)\\
D(t) &\geq 0, && (30)\\
K(t,a) &\geq 0. && (31)
\end{aligned}$$

[4]To simplify notation we consider δ and c to be constant rather than age-dependent. Furthermore, we do not include changes in productivity due to deterioration or learning. Qualitatively similar results can be obtained by having the parameters age dependent and/or allowing for deterioration or learning effects in productivity (cf. Feichtinger et al. 2004, 2006).

In the embodied model we have to explicitly formulate the state constraint (31) for K, since it need not be automatically satisfied for *all* ages (see Remark 3 below). Also selling new machines is not possible, since they were not available before, which explains (28).

Embodied technological progress is introduced in Feichtinger et al. (2004), where the output price is fixed and the capital market is perfect. Due to these features, different predictions of future technological progress have no influence on current investments, since these do not influence the NPVs of later investments. Feichtinger et al. (2006) considers the same framework with the difference that the firm has market power so that output price depends on production. Then current investments increase production which reduces output price and thus the NPVs of future investments. In this case a technological breakthrough is preceded by a negative anticipation phase in which the firm cuts down on investments. The result is an increased output price so that the firm can more profitably invest in the more productive capital goods from after the technological brakthrough. The present model reintroduces the constant output price, but assumes an imperfect capital market instead. In this framework investment decisions are intertemporally related, since the firm can save to increase financial assets now, so that it has more financial means to invest later.

Our aim is to study the behavior of the firm around a technological breakthrough (shock), and compare the results with the above mentioned contributions and with the case of disembodied technological progress presented in the previous section. For this purpose we take the same scenario (18) for the unit production function $f(\cdot)$, and consider again an anticipating and a nonanticipating firm. Like in the previous section, also here it holds that results do not change if we take (19) instead of (18), implying that our analysis covers both process and product innovation.

To start our analysis, again we restrict ourselves to a scenario where the user cost of capital is positive (cf. (6)) This translates into

$$pf > b_0 g(0), \quad pf > b(a)g(a) \;\; \forall a \in [0,\omega), \tag{32}$$

where

$$g(a) = \frac{r+\delta}{1-e^{-(r+\delta)(\omega-a)}}$$

The following lemma shows that the problem has a unique "classical" solution, provided that initial impulse dividend payouts (cf. Figure 2) are excluded. Clearly, initial impulse dividends do not occur when either $B(0) = 0$, or when an additional constraint $D \leq \bar{D}$ is imposed.

Lemma 2 *Assume that $b(a) \geq 0$ and $f(t) \geq f > 0$ are piece-wise continuous and bounded, $\delta \geq 0$, $c > 0$, $i \geq 0$, $i < r$, and (32) holds. Then problem (23)–(24) with the additional constraint $D \leq \bar{D}$ (with any positive number $\bar{D}$) has a unique solution.*

Remark 2 Similarly as in Remark 1, the additional constraint $D \leq \bar{D}$ is not needed if $B(0) = 0$. However, we do not explicitly prove this here.

The remaining analysis is conducted under the assumptions in Lemma 2.

Lemma 3 *For the optimal solution of problem (23)–(31) the equality $D(t) = 0$ holds in every interval where $B(t) > 0$.*

This lemma implies that the optimal solution has a similar structure as in the disembodied case in the sense that in some intervals we have $B(t) = 0$ so that "dividend = revenue - investment", while in the remaining planning period it will hold that $D(t) = 0$ (all earnings are retained). Clearly, on the final path the contribution to the objective should be positive. This implies that dividends are positive so that $B(t) = 0$ there.

The next lemma describes the benchmark solution in a scenario without technological progress and zero financial assets throughout. The latter implies that all revenue is immediately spent on investments and dividends. The lemma claims that for a given age investment is constant over time and is determined such that marginal investment expenses equal the discounted revenue stream. This in fact also describes the optimal investment behavior of the nonanticipating firm during the initial time period that lasts until the time of the occurrence of the technological breakthrough. As in Proposition 1 of Feichtinger et al. (2005), the lemma shows that for $t \geq \omega$ capital stock and investments are time invariant, implying that the steady state with respect to calendar time is reached.

Lemma 4 *Let $f(t) = f$ be constant on $[0, \infty)$. Consider the investment functions*

$$\hat{I}_0(t) = \hat{I}_0 = \frac{1}{c}\left[\frac{pf}{g(0)} - b_0\right], \quad \hat{I}(t,a) = \hat{I}(a) = \frac{1}{c}\left[\frac{pf}{g(a)} - b(a)\right] \tag{33}$$

together with the solution $\hat{K}(a)$ of the equation

$$K_a(a) = -\delta K(a) + \hat{I}(a), \quad K(0) = \hat{I}_0. \tag{34}$$

Then $(\hat{I}_0, \hat{I})$ is the only optimal control in problem (23)–(27) with $B(t)$ forced to stay equal to zero (notice that the constraints (30) and (31) are disregarded here). The corresponding trajectory $K(t, \cdot)$ coincides with $\hat{K}(a)$ for $t \geq \omega$. Moreover, if $K_0(\cdot)$ is greater than or sufficiently close to $\hat{K}(\cdot)$ (uniformly), the constraints (30)–(31) are satisfied, so that $(\hat{I}_0, \hat{I})$ is the optimal solution of (23)–(31) with the additional constraint $B(t) = 0$.

Remark 3 In the case of disembodied technological progress (cf. Section 3) we found that on every optimal path starting from a positive capital stock, we have $I(t) > 0$ and, consequently, $K(t) > 0$. This is not the case for embodied technological progress. Although from Lemma 4 it can be derived that $I_0 > 0$ and $I > 0$ along the final path, in the case of a technological shock it could happen that $I(t, a) < 0$ for some intermediate t and a. It could even happen that $K(t, a) < 0$, if this is not prevented by the constraint (31). That is to say that the state constraint (31) can be active for some t and a, which significantly complicates the problem.

The following proposition points at the implications of the technology shock for optimal firm behavior. First, in order to benefit from the increased productivity due to technological progress, investments in new machines go up at the moment that the shock occurs. Second, in case of a large enough shock it is optimal for the firm to invest some earnings in financial assets before the shock occurs, so that larger funds are available for investments in the more productive machines of the vintages from after the technology shock. This implies that under embodied technological progress it is optimal to hold a positive amount of financial assets, although the return of paying dividends ($r > i$) and of investing (cf. (32)) is higher.

Proposition 4 *Let I_0 be the optimal investment in new machines of the anticipating firm and let it be strictly positive shortly after the technological shock at $\bar{t}$.*

Then
(i) I_0 jumps up[5] *at $\bar{t}$;*
(ii) if the magnitude of the technological shock, $(F - f)/f$, is sufficiently large, then $B(t) > 0$ for some $t > \bar{t}$ and arbitrarily close to $\bar{t}$.

A similar discontinuity can also be proved for investments in old machines: over time we consider machines with the same age, thus a is constant. While we let t increase starting from $t < \bar{t} + a$, an upward jump can be observed as soon as t reaches the level $\bar{t} + a$. At this time the investments in capital stock of age a belong to the first vintage that benefits from the technology jump.

[5] More precisely, $I_0(\bar{t}-) < I_0(\bar{t}+)$, where $I_0(\bar{t}-)$ and $I_0(\bar{t}+)$ are the limit values of I_0 left and right from $\bar{t}$:

$$I_0(\bar{t}-) = \liminf_{h \to 0+} \frac{1}{h} \int_{\bar{t}-h}^{\bar{t}} I(t)\,\mathrm{d}t, \quad I_0(\bar{t}+) = \liminf_{h \to 0+} \frac{1}{h} \int_{\bar{t}}^{\bar{t}+h} I(t)\,\mathrm{d}t.$$

Proposition 4 (i) formally implies that $I_0(\bar{t}+) > 0$. As shown in the proof, if it also holds that $I_0(\bar{t}-) > 0$, then the size of the jump is estimated by

$$I_0(\bar{t}+) - I_0(\bar{t}-) \geq \frac{F - f}{f} \left(\frac{b_0}{c} + I_0(\bar{t}-) \right),$$

The following lemma says that production is continuous even at the moment that the technology jump occurs. This is because production of the existing capital stock from before the jump does not change (this is the crucial difference with the disembodied case), while the stock of capital whose productivity benefits from the technology jump, first needs to be built up.

Lemma 5 *The production $Q(t)$ is (Lipschitz) continuous along the optimal path.*

The next proposition presents an anticipation effect that is just opposite to the one in the disembodied case. To simplify the exposition and to make a clear parallel with the disembodied case we consider investments in new machines only. The result is that just before the technology shock occurs, the anticipating firm spends less money on investment than the nonanticipating firm, i.e. $I(t) < \hat{I}_0$ on $[\bar{t} - \varepsilon, \bar{t})$. Instead, this money is used to increase the stock of financial assets, so that after the technological breakthrough this stock can be employed to finance more investments in new machines with higher productivity. That is, the anticipating firm saves until time $\bar{t}$, and starts using the accumulated financial assets for investments just after the technological shock has occurred. Even more, we prove that the investments in new machines shrink to zero before the technology shock if the latter is large enough. We conclude that a kind of substitution effect arises: reducing investment in the less efficient technologies from before the breakthrough is optimal in order to purchase the more efficient technologies from after the breakthrough.

Of course, the downside of an investment decrease is that immediate production and thus immediate revenue is reduced, which negatively affects financial means. However, investment outlays are earned back during a longer time period, so that, provided that this investment reduction takes place sufficiently close to the time of the technology shock, the reduced cash inflow in the form of immediate revenue reduction is more than compensated by the reduced cash outflow due to less investment expenses.

Proposition 5 *Let the technological shock, $F - f$, be sufficiently large, and let $I_0(t)$, $I(t, a)$ be the optimal investments of the anticipating firm. Then it holds that*

i) $\lim_{t \to \bar{t}-} I_0(t) = 0$. Consequently, the anticipating firm invests less than the non-anticipating one during some interval $[\bar{t} - \varepsilon, \bar{t}]$, provided that the optimal investment of the latter is positive there.

ii) If only investments in new machines are allowed (i.e. $I(t, a) = 0$) and if for the anticipating firm $B(\bar{t}) > 0$, then $B(t)$ is strictly increasing on some interval $[\bar{t} - \varepsilon, \bar{t}]$.

As we already mentioned in the Introduction, a negative anticipation effect also occurs in Feichtinger et al. (2006). Since in that model price decreases with output, the negative anticipation phase concerning investment in new machines causes a reduction in output and thus an increase in the price of output. This triggers an investment boom in older machines, which were cheaper and had a shorter lifetime, so that they could be scrapped around the moment of the technological breakthrough. In the present paper the investment behavior in older machines during the negative anticipation phase is quite different: here no investment boom occurs, since (i) the output price is constant in this model and (ii) above investing in older machines the firm prefers to acquire financial assets that can be used later on to finance the new efficient machines from after the technological breakthrough.

5 Conclusions

This paper investigates how a firm should invest to anticipate future technology shocks, while the capital market is imperfect. In fact, the resulting investment behavior depends heavily on whether technological progress is disembodied or embodied. Disembodied technological progress means that, independent from their building year, all machines benefit from a technological breakthrough. Consequently, a disembodied technology shock raises the NPV of all capital goods in use at the time of the breakthrough. This results in a positive anticipation phase of investments taking place before the technological breakthrough actually occurs. During this positive anticipation phase the firm cuts down on financial assets in order to raise the amount of investments.

Embodied technological progress implies that only the capital goods with building year from after the time of the technological breakthrough are affected. Anticipating an embodied technological breakthrough, the firm builds up a stock of financial assets in order to have more financial means available to acquire the better machines from after the breakthrough. Building up this stock of financial assets before the technology shock occurs, requires that the firm should reduce investments at that time. This results in a negative anticipation phase with respect to investment, which is opposite to what takes place before a disembodied technology shock.

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Appendix 1. Proofs for the case of no technological progress

Proof of Lemma 1. The second claim of the lemma is similar (and simpler) than that of Lemma 2 in Section 4.

In the case of a continuous function f the proof of the continuity of I can be done by examining the proof of Corollary 6.2 in Feichtinger and Hartl (1985). Although it cannot be used directly, since strong convexity with respect to D is lacking, its proof implies continuity of I.

If f has a jump point (which is then an isolated jump point since f is piecewise continuous) then the proof involves a continuous approximation f_n of f, for which the above arguments are applicable. Since f_n appears only in the drift term of the differential equations for B, it does not affect the control-dependent part of the Hamiltonian. The corresponding solutions of the state/co-state equations are uniformly bounded and equi-continuous. This allows to take the limit in the space of continuous functions and obtain continuity of the co-state variables corresponding to f, hence continuity of I. Q.E.D.

Proof of Proposition 1. Before we start out the path coupling procedure, we first analyze the paths separately.

Path 1: holding financial assets:

Here we have $\eta = 0$, $\mu > 0$, which implies $D = 0$, while $B \geq 0$. This means that

$$\dot{B} = iB + pfK - I - \frac{c}{2}I^2, \tag{35}$$

$$\begin{aligned} \dot{\lambda}_1 &= (r+\delta)\,\lambda_1 - \lambda_2 pf, & (36)\\ \dot{\lambda}_2 &= (r-i)\,\lambda_2, & (37) \end{aligned}$$

$$\begin{aligned} L_I &= \lambda_1 - \lambda_2\,(1+cI) = 0, & (38)\\ L_D &= 1 - \lambda_2 + \mu = 0. & (39) \end{aligned}$$

Differentiate (38) with respect to time and obtain that

$$\begin{aligned} \dot{\lambda}_1 - \dot{\lambda}_2\,(1+cI) - \lambda_2 c\dot{I} &= 0,\\ (r+\delta)\,\lambda_2\,(1+cI) - \lambda_2 pf - (r-i)\,\lambda_2\,(1+cI) &= \lambda_2 c\dot{I}. \end{aligned}$$

Due to (39) we know that $\lambda_2 = 1 + \mu > 0$, so that

$$c\dot{I} = (i+\delta)\,(1+cI) - pf, \tag{40}$$

$$\dot{I}\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0 \Leftrightarrow I \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \frac{fp - (i+\delta)}{c\,(i+\delta)}. \tag{41}$$

From $\lambda_2 = 1 + \mu > 0$ and (37) we also know that this path cannot be a final path (otherwise λ_2 grows to infinity due to the fact that $r > i$).

Path 2: no financial assets - pay dividends:

Here we have $\eta > 0$, $\mu = 0$, which implies $D \geq 0$, while $B = 0$. This leads to

$$\dot{B} = pfK - I - \frac{c}{2}I^2 - D = 0, \tag{42}$$

$$\begin{aligned} \dot{\lambda}_1 &= (r+\delta)\lambda_1 - \lambda_2 pf, & (43)\\ \dot{\lambda}_2 &= (r-i)\lambda_2 - \eta, & (44) \end{aligned}$$

$$\begin{aligned} L_I &= \lambda_1 - \lambda_2(1+cI) = 0, & (45)\\ L_D &= 1 - \lambda_2 = 0 \quad \Rightarrow \quad \lambda_2 = 1. & (46) \end{aligned}$$

From (44) and (46) we obtain that

$$\eta = r - i > 0,$$

while from (45) we derive that

$$\lambda_1 = 1 + cI.$$

Differentiating this w.r.t. time and using (43) gives

$$\begin{aligned} c\dot{I} &= \dot{\lambda}_1 = (r+\delta)\lambda_1 - pf \qquad (47)\\ &= (r+\delta)(1+cI) - pf, \end{aligned}$$

$$\dot{I}\begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \Leftrightarrow I \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \frac{fp - (r+\delta)}{c(r+\delta)}.$$

Since $D > 0$, this path gives a positive contribution to the objective. Therefore, this is a candidate for a final path. For this reason we now determine the steady state on this path. From (47) we obtain that

$$\begin{aligned} \dot{I} &= 0 \Longrightarrow (r+\delta)\left(1 + c\hat{I}\right) = pf, \\ \hat{I} &= \frac{fp - (r+\delta)}{c(r+\delta)}, & (48)\\ \hat{K} &= \frac{\hat{I}}{\delta} = \frac{fp - (r+\delta)}{\delta c(r+\delta)}. & (49) \end{aligned}$$

The following lemma proves that revenue is sufficient to sustain the steady state.

Lemma 6

$$pf\hat{K} - \hat{I} - \frac{c}{2}\hat{I}^2 > 0 \tag{50}$$

Proof. Applying (48) and (49) relation (50) is easily shown to be equivalent to $fp > \frac{(r+\delta)\delta}{2r+\delta}$, which is implied by assumption (6). Q.E.D.

Due to (47) we can obtain an analytical solution for I:

$$I(t) = \left(I_{\text{start}} - \hat{I}\right) e^{(r+\delta)t} + \hat{I}, \tag{51}$$

which on a final path diverges to $\pm\infty$ at rate $r+\delta$ for $I_o \neq \hat{I}$. By (46) and (45) we have $\lambda_1 = 1 + cI$ so that also λ_1 diverges to $\pm\infty$ at rate $r+\delta$. For this reason it is clear that on a final path, $I(t) = \hat{I}$, so that we have the well known flexible accelerator rule

$$\dot{K} = \delta\left(\hat{K} - K\right).$$

This implies that on a final Path 2 capital stock converges to its steady state $\hat{K}$.

Path 3: maximal investment:

Here we have $\eta > 0$, $\mu > 0$, which implies that $D = 0$, while $B = 0$. Then we have:

$$\dot{B} = pfK - I - \frac{c}{2}I^2 = 0, \tag{52}$$

$$\begin{aligned} \dot{\lambda}_1 &= (r+\delta)\lambda_1 - \lambda_2 pf, & (53)\\ \dot{\lambda}_2 &= (r-i)\lambda_2 - \eta, & (54) \end{aligned}$$

$$\begin{aligned} L_I &= \lambda_1 - \lambda_2(1+cI) = 0, & (55)\\ L_D &= 1 - \lambda_2 + \mu = 0 \quad \Longrightarrow \quad \lambda_2 = 1 + \mu > 0. & (56) \end{aligned}$$

Differentiating (52) w.r.t. time and using this equation gives:

$$\left(pf - \delta\left(1 + \frac{c}{2}I\right)\right) I = \dot{I}(1+cI). \tag{57}$$

Differentiating (56) w.r.t. time leads to:

$$\dot{\mu} = \dot{\lambda}_2 = (r - i)\,\lambda_2 - \eta = (r - i)\,(1 + \mu) - \eta.$$

Note that the contribution to the objective of this path is zero. Therefore, Path 3 does not qualify as a final path. This is because an alternative policy of setting $I = 0$ implies that $D = pfK$ which can be maintained forever and gives a positive contribution to the objective.

Path 4: positive financial assets and dividend - infeasible:

Here we have $\eta = 0$, $\mu = 0$, which implies $D \geq 0$, while $B \geq 0$. The above simplifies into

$$\dot{\lambda}_1 = (r + \delta)\,\lambda_1 - \lambda_2 pf, \tag{58}$$
$$\dot{\lambda}_2 = (r - i)\,\lambda_2, \tag{59}$$

$$L_I = \lambda_1 - \lambda_2\,(1 + cI) = 0, \tag{60}$$
$$L_D = 1 - \lambda_2 = 0. \tag{61}$$

From (61) we get $\lambda_2 = 1$ which is a contradiction to (59) . So this path is infeasible.

Path coupling procedure

We have just one candidate for a final path. We proceed with investigating trajectories with Path 2 as a final path. First we check Path 1 $\rightarrow$ Path 2.

Path 1 $\rightarrow$ Path 2

We know that on Path 1 $\dot{\mu} > 0$. From (39) and (46) it follows that λ_2 jumps downwards at this coupling point t_{12}. The coupling point is an entry point to the boundary arc of the state constraint $B \geq 0$. Therefore, $\dot{B}\left(t_{12}^{-}\right) \leq 0$, where $\dot{B}\left(t_{12}^{-}\right)$ is the l.h.s. limit of $\dot{B}$ as t approaches the coupling point t_{12}.

From Feichtinger and Hartl (1986, Corollary 6.3) we know that a downward jump in λ_2 is only possible if the entry to the boundary arc is tangential and at least one of the controls is discontinuous.

From (38) and (45) we obtain that a downward jump of λ_2 should go along with an upward jump of I. However, from Lemma 6 it follows that I is continuous over time, so that I cannot jump upwards at t_{12}.

Therefore, the coupling Path 1 $\rightarrow$ Path 2 is not possible.

Path 3 $\rightarrow$ Path 2

To check feasibility of this coupling we have to determine whether the state and the costate variables can be continuous at the coupling point t_{32}. Continuity of the costate variables implies that also investment is continuous, by (45) and (55).

We proceed by presenting the financial assets equation just before and just after the coupling point.

At t_{32}^{+} we have $\dot{B} = pfK - \hat{I} - \frac{c}{2}\hat{I}^2 - D = 0$, while at t_{32}^{-} it holds that

$$\dot{B} = pfK - \hat{I} - \frac{c}{2}\hat{I}^2 = 0. \tag{62}$$

Since K is continuous, we know that $K(t_{32}) = K^*$ and also D must be continuous at t_{32} implying that $D\left(t_{32}^{+}\right) = 0$. From Lemma 6 we now obtain that

$$K(t_{32}) = K^* < \hat{K},$$

simply by comparing $pfK - \hat{I} - \frac{c}{2}\hat{I}^2 = 0$ and $pf\hat{K} - \hat{I} - \frac{c}{2}\hat{I}^2 > 0$.

We conclude that in this combination, on Path 2 capital stock *increases* towards its steady state value.

We now establish the development of investment and capital stock on Path 3. We start out with

$$\dot{\mu} = (r - i)(1 + \mu) - \eta.$$

Continuity of λ_2 requires that $\mu\left(t_{32}^{-}\right) = 0$ and $\dot{\mu}\left(t_{32}^{-}\right) \leq 0$, i.e., $\eta\left(t_{32}^{-}\right) \geq r - i$. At t_{32}^{-} we can also insert the formula for $\hat{I}$ in (57) to eventually obtain that

$$\dot{I}\left(t_{32}^{-}\right) = \left(\frac{2rpf + \delta\left(pf - (r + \delta)\right)}{2(r + \delta)}\right)\frac{fp - (r + \delta)}{cfp} > 0.$$

Combining this result with (57) it is obtained that

$$\dot{I} > 0 \text{ throughout Path 3.} \tag{63}$$

By (52) this implies that also $\dot{K} > 0$ on this path.

Path 1 → Path 3 → Path 2

At the end of Path 3 it holds that $I = \hat{I}$. This implies via (63) that $I < \hat{I}$ holds at the beginning of Path 3 and at the end of Path 1 (since I is continuous). Since $\frac{fp-(i+\delta)}{c(i+\delta)} > \hat{I}$, we obtain from (41) that

$$\dot{I} < 0 \text{ along Path 1.} \tag{64}$$

On the sequence Path 1 → Path 3 → Path 2 investment I admits its smallest value at t_{13}. Since this is the start of Path 3, we know that K is increasing there. Since throughout Path 1 it holds that $I > I\left(t_{13}\right)$, it follows automatically that K is increasing on Path 1.

Next we show that B is decreasing throughout Path 1. From (35) it follows that

$$\ddot{B} = i\dot{B} + pf\dot{K} - \dot{I} - cI\dot{I}.$$

Now assume that B is *not* decreasing throughout Path 1. This would imply that at one point in time $\tau < t_{13}$ we have $\dot{B}\left(\tau\right) = 0$ while $\ddot{B} \leq 0$. This contradicts $\ddot{B} = pf\dot{K} - \dot{I} - cI\dot{I} > 0$ there.

Path 3 →Path 1 → Path 3 → Path 2 and Path 2 →Path 1 → Path 3 → Path 2

In the previous investigation of Path 1 → Path 3 → Path 2 we have shown that $B > 0$ and B is decreasing throughout Path 1. Hence, before Path 1 there cannot be a Path 3 or a Path 2 with $B = 0$.

Path 2 → Path 3 → Path 2

Since $I = \hat{I}$ on the final Path 2 and $\dot{I} > 0$ on Path 3 (cf. (63)) we know that $I\left(t_{23}\right) < \hat{I}$. From (51) we obtain that $\dot{I} < 0$ throughout the first Path 2.

Differentiate (42) w.r.t. time to obtain that

$$pf\dot{K} - \dot{I} - cI\dot{I} - \dot{D} = 0. \tag{65}$$

From the negativity of $\dot{I}$ and $\dot{D}$ at t_{23}^{-} (note that continuity of D is implied by (42) and continuity of K and I), it follows that $\dot{K}\left(t_{23}^{-}\right) < 0$. On the other hand, we know that $\dot{K}\left(t_{23}^{+}\right) \geq 0$. This implies that I is discontinuous at t_{23} which is a contradiction to (45) and $\left(55\right)$. Hence, Path 2 → Path 3 → Path 2 is not possible. Q.E.D.

Appendix 2. Proofs for the case of disembodied technological progress

Proof of Proposition 2. We already concluded in Section 3 that for $t > \bar{t}$ Proposition 1 applies with f replaced by F. This implies that for $t > \bar{t}$ the following sequences can occur:

$$\begin{aligned} \text{I: } & \text{Path 2F,} \\ \text{II: } & \text{Path 3F} \rightarrow \text{Path 2F,} \\ \text{III: } & \text{Path 1F} \rightarrow \text{Path 3F} \rightarrow \text{Path 2F.} \end{aligned}$$

The procedure of the proof is to address all possibilities on $[\bar{t} - \varepsilon, \bar{t}]$. This implies that we must consider all possible predecessors of the sequences I-III, which gives the following list:

Trajectory 1: Path 1f $\rightarrow$ Path 2F
Trajectory 2: Path 2f $\rightarrow$ Path 2F
Trajectory 3: Path 3f $\rightarrow$ Path 2F
Trajectory 4: Path 1f $\rightarrow$ Path 3F $\rightarrow$ Path 2F
Trajectory 5: Path 2f $\rightarrow$ Path 3F $\rightarrow$ Path 2F
Trajectory 6: Path 3f $\rightarrow$ Path 3F $\rightarrow$ Path 2F
Trajectory 7: Path 1f $\rightarrow$ Path 1F $\rightarrow$ Path 3F $\rightarrow$ Path 2F
Trajectory 8: Path 2f $\rightarrow$ Path 1F $\rightarrow$ Path 3F $\rightarrow$ Path 2F
Trajectory 9: Path 3f $\rightarrow$ Path 1F $\rightarrow$ Path 3F $\rightarrow$ Path 2F

The first path mentioned in each of the Trajectories 1-9 is the path that applies on $[\bar{t} - \varepsilon, \bar{t}]$. It is important to realize that this need not be the path that applies at the initial point of time, also because it is required that the complete trajectory starts out with Path 2f. To determine the complete trajectories is an enormous task, but fortunately for the purpose of proving the proposition it is not necessary to accomplish this.

Next we check whether inequality (21) holds on each of the nine trajectories.

Trajectory 1:

Suppose that inequality (21) does not hold here, i.e.

$$I(t) \leq \hat{I}^f = \frac{fp - [r + \delta]}{c[r + \delta]} \text{ for } t \in [\bar{t} - \varepsilon, \bar{t}]. \tag{66}$$

From (7) and (40) it is then obtained that

$$\dot{I}(t) < 0 \text{ for } t \in [\bar{t} - \varepsilon, \bar{t}].$$

This implies that the behavior on Path 1f is qualitatively similar as on Path 1F, which means that on this path it holds that

$$\dot{I} < 0, \dot{K} > 0, \dot{B} < 0.$$

Since the state variable B must be continuous over time, this implies that no other path can be coupled before this trajectory. Hence, the complete trajectory cannot have Path 2f as an initial path, so that we do not need to consider this sequence any further, while (66) applies.

Trajectories 2-3:

On Path 2F it holds that

$$I = \hat{I}^F = \frac{Fp - [r+\delta]}{c[r+\delta]} > \frac{fp - [r+\delta]}{c[r+\delta]} = \hat{I}^f.$$

This implies that

$$I(\bar{t}) = \hat{I}^F > \hat{I}^f.$$

Together with I being continuous over time (cf. Lemma 1), it follows that a number $\varepsilon > 0$ exists such that (21) holds.

Trajectory 4:

The proof is analogous to Trajectory 1.

Trajectory 5:

Suppose that inequality (21) does not hold here, i.e.

$$I(t) \le \hat{I}^f = \frac{fp - [r+\delta]}{c[r+\delta]} \text{ for } t \in [\bar{t} - \varepsilon, \bar{t}]. \tag{67}$$

Combining (67) with (47) gives that either

$$\dot{I} = 0 \tag{68}$$

holds, or

$$\dot{I} < 0. \tag{69}$$

First consider (68). This implies that $I(t) = \hat{I}_f = \frac{fp-[r+\delta]}{c[r+\delta]}$. Now, from (48)-(50) and (42) we conclude that $D > 0$ on Path 2f while $D = 0$ on Path 3F. This means that D jumps downwards at $\bar{t}$, which is impossible, because D is continuous according to Lemma 1.

Next, consider (69). Continuity of the dividend rate requires that

$$\dot{D}(\bar{t}^-) \le 0.$$

On Path 2f it can be obtained from (42) that at time $\bar{t}^-$ we have

$$pf\dot{K} - [1 + cI]\,\dot{I} = \dot{D} \leq 0,$$

which via (69) implies that $\dot{K}\left(\bar{t}^-\right) < 0$. However, on Path 3F it holds that $\dot{K} > 0$, so that a discontinuity of I arises on $\bar{t}$, which is not feasible. This implies that this trajectory cannot occur under (67).

Trajectory 6:

Since $\dot{K} > 0$ holds on Path 3F, this implies via the continuity of K and I that also on Path 3f it holds that $\dot{K} > 0$ and $\dot{I} > 0$. Path 3f can be preceded either by Path 1f or Path 2f. In the case of Path 1f and when (21) does not hold on Path 3f, the fact that $\dot{I} > 0$ on Path 3f implies that $I < \hat{I}^f$ at the end of Path 1f. Analogous to the proof of Trajectory 1 we conclude that no path can precede Path 1f, so that we do not need to consider this possibility any further.

In case Path 2f precedes Path 3f, continuity of I implies that $\dot{K} \geq 0$ on Path 2f. This in turn implies that $I \geq \hat{I}^f$ on Path 2f, which via $\dot{I} > 0$ on Path 3f leads to the conclusion that (21) holds.

Trajectory 7:

The proof is analogous to Trajectory 1.

Trajectory 8-9:

At the beginning of Path 1F it holds that $B > 0$, while $B = 0$ at both Path 2f and Path 3f. Hence, B is discontinuous, which makes these trajectories in feasible. Q.E.D.

Proof of Proposition 3. Since $D(t) = 0$ on intervals where $B(t) > 0$, we have for $t > \bar{t}$

$$\dot{B}(t) = iB(t) + pFK(t) - I - \frac{c}{2}I^2 \leq 0.$$

Then due to the continuity of B, K and I and $f < F$ we obtain

$$iB(t) + pfK(t) - I - \frac{c}{2}I^2 < 0$$

on some interval $[\bar{t} - \varepsilon, \bar{t}]$, which implies the claim of the proposition. Q.E.D.

Appendix 3. Proofs for the case of embodied technological progress

Proof of Lemma 2. Due to the strict convexity with respect to I_0 and I, it could be proved that for every admissible maximizing sequence the corresponding controls $I_0(t)$ and $I(t,a)$ are bounded[6]. D is also bounded due to the additional constraint. Since the system is linear-convex (with the appropriate sign in the constraint for B), one can directly employ the classical "weak-compactness&lower-semicontinuity" approach to prove existence.

If there are two optimal controls $U = (I_0, I, D)$ and $\tilde{U} = (\tilde{I}_0, \tilde{I}, \tilde{D})$, then the control $\hat{U} = 0.5U + 0.5\tilde{U}$ gives the same objective value. On the other hand $\hat{W}(t) < 0.5(W(t) + \tilde{W}(t))$ for some t if (I_0, I) and $(\tilde{I}_0, \tilde{I})$ do not coincide. This implies that $\hat{B}(t) > 0.5(B(t)+\tilde{B}(t))$ for sufficiently large t (while "$\geq$" for all t), which easily contradicts optimality, since additional dividend could be paid. Thus $(I_0, I) = (\tilde{I}_0, \tilde{I})$. The proof of $D = \tilde{D}$ uses the inequality $i < r$. Q.E.D.

In the proofs of the next results we employ suitable control variations. This is because a maximum principle is not available for state constrained problems of the type considered here.

Proof of Lemma 3. For any $0 < \tau_1 < \tau_2$ and for a sufficiently small $h > 0$ and for $\varepsilon > 0$ we define a modified control

$$D_h(t) = \begin{cases} D(t) + \varepsilon & \text{for } t \in [\tau_1, \tau_1 + h], \\ D(t) - e^{i(\tau_2 - \tau_1 - h)}\varepsilon & \text{for } t \in [\tau_2 - h, \tau_2], \\ D(t) & \text{elsewhere.} \end{cases} \tag{70}$$

The controls I_0 and I are not changed. One can easily verify that the corresponding $B_h(\tau_1) = B(\tau_1)$ (which is obvious), $B_h(\tau_2) = B(\tau_2)$, and

$$J_h = J + \frac{\varepsilon}{r}\left[e^{-r\tau_1} - e^{-r(\tau_1+h)}\right]\left[1 - e^{-(r-i)(\tau_2-\tau_1-h)}\right],$$

where J and J_h are the objective values corresponding to controls (I_0, I, D) and (I_0, I, D_h), respectively. Hence, $J_h > J$.

We conclude that a solution containing a positive time interval on which both D and B are strictly positive is not optimal, since variations like (70) with sufficiently small $\varepsilon > 0$ are admissible on such an interval. Q.E.D.

[6]This was proven in a similar situation by Faggian and Gozzi in a draft from 2004, communicated by Gozzi to the fourth author.

Proof of Lemma 4. The solution of (23)–(27) with $B(t)$ fixed equal to zero is known from Feichtinger et al. (2004). The optimal controls are given by the formulas (33). The second claim follows from the expression for the $K(t,a)$ in Feichtinger et al. (2004), which equals $\hat{K}(a)$ for $t \geq a$, in particular for all $t \geq \omega$, and satisfies equation (34).

To prove the last claim it is enough to show that

$$\hat{D} = p\hat{Q} - \hat{W} > 0,$$

where $\hat{Q}$ and $\hat{W}$ correspond to $(\hat{I}_0, \hat{I}, \hat{K})$ via (26) and (27). Then the same inequality will be satisfied also for a "perturbed" initial state value $K_0(a)$ if the perturbation is sufficiently small. The same applies to constraint (31). Indeed (32) implies that $\hat{I}_0 \geq \varepsilon_0$ (for some ε_0), and $\hat{I}(a) > 0$ for all $t \in [0, \omega)$. From (34) we obtain that $\hat{K}(a) \geq \varepsilon_1 > 0$. Then (31) also holds if $K_0(\cdot)$ is close to $\hat{K}(\cdot)$.

Now, let us prove the inequality $\hat{D} > 0$, that is, $p\hat{Q} > \hat{W}$. From (34) we obtain that

$$\hat{K}(a) = e^{-\delta a}\hat{I}_0 + \int_0^a e^{-\delta(a-s)}\hat{I}(s)\,\mathrm{d}s,$$

$$\hat{Q} = f\hat{I}_0 \int_0^\omega e^{-\delta a}\,\mathrm{d}a + f\int_0^\omega \int_0^a e^{-\delta(a-s)}\hat{I}(s)\,\mathrm{d}s\,\mathrm{d}a.$$

Since $r > 0$, we get by changing the order of integration:

$$\hat{Q} > f\hat{I}_0 \int_0^\omega e^{-(r+\delta)a}\,\mathrm{d}a + f\int_0^\omega \int_s^\omega e^{-(r+\delta)(a-s)}\,\mathrm{d}a\hat{I}(s)\,\mathrm{d}s,$$

$$p\hat{Q} > \frac{pf\hat{I}_0}{g(0)} + \int_0^\omega \frac{pf\hat{I}(a)}{g(a)}\,\mathrm{d}a.$$

On the other hand it holds that

$$\hat{W} = \frac{1}{2}\hat{I}_0\left(\frac{pf}{g(0)} + b_0\right) + \frac{1}{2}\int_0^\omega \hat{I}(a)\left(\frac{pf}{g(a)} + b(a)\right)\,\mathrm{d}a.$$

We complete the proof noticing that

$$\frac{pf}{g(0)} > \frac{1}{2}\left(\frac{pf}{g(0)} + b_0\right), \quad \frac{pf}{g(a)} > \frac{1}{2}\left(\frac{pf}{g(a)} + b(a)\right),$$

as a consequence of (32). Q.E.D.

Proof of Proposition 4. 1. If $I(\bar{t}-) \leq 0$, then the first claim follows from $I_0(\bar{t}+) > 0$. Let us consider the case $I_0(\bar{t}-) > 0$. For all sufficiently small $\rho > 0$ the following function is non-negative:

$$\tilde{I}_0(t) = \begin{cases} I_0(t) - \rho & \text{for } t \in [\bar{t} - \rho, \bar{t}], \\ I_0(t) + \gamma(\rho) & \text{for } t \in [\bar{t}, \bar{t} + \rho], \\ I_0(t) & \text{elsewhere}, \end{cases}$$

where the number $\gamma(\rho)$ will be defined later. A priory we require only that $\gamma(\rho) \leq C\rho$, where C is a sufficiently large constant. The investment in old machines is not changed: $\tilde{I} = I$.

Let $\tilde{K}(t,a)$ be the corresponding capital stock, and $\tilde{Q}(t)$ – the corresponding production. Then we define

$$\tilde{D}(t) = \begin{cases} D(t), & \text{for } t \in [0, \bar{t} + \rho], \\ D(t) + p(\tilde{Q}(t) - Q(t)), & \text{for } t > \bar{t} + \rho. \end{cases}$$

Clearly ΔB is increasing on $[\bar{t}-\rho, \bar{t}]$, and we shall determine $\gamma(\rho)$ in such a way that $\Delta B(t) = \tilde{B}(t) - B(t)$ becomes zero at $\bar{t}+\rho$. For $\Delta K(t,a) = \tilde{K}(t,a) - K(t,a)$ and $\Delta Q(t) = \tilde{Q}(t) - Q(t)$ we have

$$\Delta K(t,a) = \begin{cases} 0, & \text{if } t - a \notin [\bar{t} - \rho, \bar{t} + \rho] \\ e^{-\delta a} \Delta I_0(t-a), & \text{if } t - a \in [\bar{t} - \rho, \bar{t} + \rho], \end{cases}$$

$$\Delta Q(t) = \begin{cases} 0, & \text{if } t \notin [\bar{t} - \rho, \bar{t} + \omega + \rho] \\ \int_{[\bar{t}-\rho, \bar{t}+\rho] \cap [t-\omega, t]} e^{-\delta(t-s)} f(s) \Delta I_0(s) \, \mathrm{d}s, & \text{if } t \in [\bar{t} - \rho, \bar{t} + \omega + \rho], \end{cases}$$

$$\Delta Q(t) = \begin{cases} 0, & \text{if } t \notin [\bar{t} - \rho, \bar{t} + \omega + \rho] \\ e^{-\delta(t-\bar{t})} [F\gamma(\rho) - f\rho]\rho + o(\rho^2), & \text{if } t \in [\bar{t} - \rho, \bar{t} + \omega + \rho], \end{cases} \tag{71}$$

where $o(\rho^2)/\rho^2 \to 0$ with $\rho \to 0$.

Since obviously $\Delta Q(t)$ and $\Delta B(t)$ are of order ρ^2 for $t \in [\bar{t} - \rho, \bar{t} + \rho]$, we have

$$\Delta B(t) = -\int_{\bar{t}-\rho}^{t} \left[b_0(\tilde{I}_0(s) - I_0(s)) + \frac{c}{2}(\tilde{I}_0(s) - I_0(s))^2 \right] \mathrm{d}s + o(\rho^2).$$

This implies that

$$\Delta B(\bar{t} + \rho) = \int_{\bar{t}-\rho}^{\bar{t}} \rho[b_0 + cI_0(s)] \, \mathrm{d}s - \int_{\bar{t}}^{\bar{t}+\rho} \gamma(\rho)[b_0 + cI_0(s)] \, \mathrm{d}s + o(\rho^2).$$

In order to ensure that $\Delta B(\bar{t} + \rho) = 0$ we need to chose $\gamma(\rho)$ from the equation

$$\rho[b_0 + cI_0^{-\rho}] - \gamma(\rho)[b_0 + cI_0^{+\rho}] + o(\rho) = 0, \tag{72}$$

where

$$I_0^{-\rho} = \int_{\bar{t}-\rho}^{\bar{t}} I_0(s) \, \mathrm{d}s, \quad I_0^{+\rho} = \int_{\bar{t}}^{\bar{t}+\rho} I_0(s) \, \mathrm{d}s.$$

Clearly the solution $\gamma(\rho)$ satisfies the inequality $\gamma(\rho) \leq C\rho$, if C is chosen sufficiently large, thus the above variation is well defined. Moreover, by the choice of $\tilde{D}$ and $\tilde{B}(\bar{t} + \rho) = 0$ we have $\tilde{B}(t) = B(t)$ for $t \geq \bar{t} + \rho$. It is clear also that $\tilde{K}(t,a) = K(t,a)$ for $t \geq \bar{t} + \omega + \rho$.

Since the control functions $(\tilde{I}_0, \tilde{I}, \tilde{D})$ generate an admissible trajectory, for the objective value $\tilde{J}$ we have $\tilde{J} \le J$, where J is the optimal objective value. We have from (71)

$$0 \ge \tilde{J} - J = \int_{\bar{t}-\rho}^{+\infty} e^{-rt} \Delta D(t)\,\mathrm{d}t = p \int_{\bar{t}+\rho}^{\bar{t}+\omega+\rho} e^{-rt} \Delta Q(t)\,\mathrm{d}t$$

$$= p \int_{\bar{t}+\rho}^{\bar{t}+\omega+\rho} e^{-rt} e^{-\delta(t-\bar{t})} [F\gamma(\rho) - f\rho]\rho\,\mathrm{d}t + o(\rho^2).$$

This implies that

$$\gamma(\rho)F \le \rho f + o(\rho).$$

Combining this with (72) we obtain

$$[b_0 + cI_0^{-\rho}]F \le [b_0 + cI_0^{+\rho}]f + \theta(\rho), \tag{73}$$

(with $\theta(\rho) \to 0$) which implies that

$$I_0^{+\rho} - I_0^{-\rho} \ge \frac{F-f}{f}\left[\frac{b_0}{c} + I_0^{-\rho}\right] + \theta(\rho).$$

We chose a sequence $\rho_k \to 0$ such that the limit of $I(\bar{t} + \rho_k)$ exists and equals the liminf in footnote 5. Taking the limit we obtain the inequality in footnote 5, which proves the first claim of the proposition.

2. To prove the second statement of the proposition we need the following auxiliary result.

Claim. There exists a number $\bar{Q}$ (depending on the functional forms and parameter values of the problem, but independent of F) such that $Q(\bar{t}) \le \bar{Q}$ for every $F \ge f$.

As before, Q is evaluated along the optimal trajectory of the anticipating firm. Let us prove the claim. We have

$$Q(\bar{t}) = \int_0^\omega f(\bar{t}-a)K(\bar{t},a)\,\mathrm{d}a \le f\int_0^\omega K_0(a)\,\mathrm{d}a + f\int_0^{\bar{t}} I_0(t)\,\mathrm{d}t + f\int_0^{\bar{t}}\int_0^\omega I(t,a)\,\mathrm{d}a\,\mathrm{d}t,$$

hence,

$$Q(\bar{t}) \le f(\|K_0\|_1 + \|I_0\|_1 + \|I\|_1, \tag{74}$$

where the L_1-norms are taken on the sets $[0,\omega]$, $[0,\bar{t}\,]$, and $[0,\bar{t}\,] \times [0,\omega]$, respectively. Then from the Cauchy formula for the equation for B we obtain for an appropriate constant C that

$$0 \le B(\bar{t}) \le C\,(B(0) + \|K_0\|_1 + \|I_0\|_1 + \|I\|_1) - \frac{c}{2}\|I_0\|_2^2 - \frac{c}{2}\|I\|_2^2.$$

Since $\|I\|_2^2 \ge \|I\|_1^2/\bar{t}$, the above inequality easily implies that $\|I_0\|_1$ and $\|I\|_1$ are bounded by a constant independent of F. Then (74) implies the claim.

We continue with the proof of the proposition. Let us assume that $B(t) = 0$ in an interval $(\bar{t}, \bar{t} + \varepsilon)$. In this interval we have $W(t) \leq pQ(t)$, so that $b_0 I_0(t) \leq p(\bar{Q} + \theta(t - \bar{t}))$, where $\theta(\varepsilon) \to 0$ with ε. Hence $b_0 I_0^{+\rho} \leq p\bar{Q} + \theta(\rho)$. From (73) we obtain that

$$b_0 F \leq [b_0 + cI_0^{-\rho}]F \leq \left[b_0 + c\frac{p\bar{Q}}{b_0}\right] f + \theta(\rho),$$

which, passing to the limit, implies

$$F/f \leq 1 + \frac{cp\bar{Q}}{b_0^2}.$$

This inequality is violated if F/f is sufficiently large, which completes the proof. Q.E.D.

Proof of Lemma 5. Take two values t and $t' > t$, while assuming without loss of generality, that $\alpha = t' - t < \omega$. Then for an appropriate constant C it holds that

$$|Q(t') - Q(t)| \leq \left| \int_\alpha^\omega f(t' - a)K(t', a)\,\mathrm{d}a - \int_0^{\omega-\alpha} f(t - a)K(t, a)\,\mathrm{d}a \right| + C\alpha.$$

Changing the variables in the first integral we obtain

$$|Q(t') - Q(t)| \leq \left| \int_0^{\omega-\alpha} f(t - a)K(t + \alpha, a + \alpha)\,\mathrm{d}a - \int_0^{\omega-\alpha} f(t - a)K(t, a)\,\mathrm{d}a \right| + C\alpha.$$

$$= \int_0^{\omega-\alpha} |f(t - a)[K(t + \alpha, a + \alpha) - K(t, a)]|\,\mathrm{d}a + C|t' - t|.$$

Now the claim of the lemma follows from the Lipschitz continuity of K along the characteristic lines $(t + s, a + s)$, $s \in \mathbf{R}$. Q.E.D.

Proof of Proposition 5. (i) For given initial values of the state variables $K(0, a) = K_0(a)$ and $B(0) = B_0$, denote by $V(K_0, B_0)$ the corresponding optimal value of the objective function (23). We say that the initial values (K_0^1, B_0^1) are majorized by (K_0^2, B_0^2) if

$$B_0^2 \geq B_0^1, \quad \int_0^\omega K_0^2(a)\,\mathrm{d}a \geq \int_0^\omega K_0^2(a)\,\mathrm{d}a,$$

with at least one of the inequalities being strict, and there is $s \in [0, \omega]$ such that

$$K_0^2(a) \geq K_0^1(a) \ \text{ for } \ a \in [0, s), \quad K_0^2(a) \leq K_0^1(a) \ \text{ for } \ a \in (s, \omega].$$

Since older machines are never more productive than newer ones, it is obvious that $V(K_0^1, B_0^1) < V(K_0^2, B_0^2)$ if (K_0^1, B_0^1) is majorized by (K_0^2, B_0^2).

Our first step will be to prove

Claim 1. There is a constant L (depending on the data of the problem but not on F) such that $I_0(t_2) \geq I_0(t_1) - L(1+F)(t_2 - t_1)$ for every $t_2 > t_1 > \bar{t}$.

Since I_0 is measurable and bounded, according to a theorem by Denjoy (see e.g. Natanson, 1961) almost every point $\tau > 0$ has the following property (called sometimes *approximate continuity*): there is a measurable set $\Omega' \subset [0, \infty)$ such that

$$\lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \text{meas}(\Omega' \cap [\tau - \varepsilon, \tau + \varepsilon]) = 1, \tag{75}$$

and the restriction of I_0 to the set Ω' is continuous.

Let $\tau > \bar{t}$ and $\tau + \alpha$ (with $\alpha > 0$) be two points of approximate continuity of I_0, and let Ω' and Ω'' be the corresponding sets from the definition of this notion. First we shall prove that $I_0(\tau + \alpha) \geq I_0(\tau) - L(1+F)\alpha$. We may assume $I_0(\tau) > 0$, otherwise the last inequality is obvious. Then for every sufficiently small $\varepsilon > 0$ the following is an admissible variation of I_0:

$$\tilde{I}_0(t) = \begin{cases} I_0(t) - \varepsilon & \text{for } t \in [\tau, \tau + \varepsilon] \cap \Omega', \\ I_0(t) + \rho(\varepsilon) & \text{for } t \in [\tau + \alpha - \varepsilon, \tau + \alpha] \cap \Omega'', \\ I_0(t) & \text{elsewhere.} \end{cases}$$

The controls $D(t)$ and I remain the same. The number $\rho(\varepsilon)$ will be chosen in such a way that

$$\int_0^\omega \tilde{K}(\tau + \alpha, a) \, \mathrm{d}a = \int_0^\omega K(\tau + \alpha, a) \, \mathrm{d}a, \tag{76}$$

where $\tilde{K}$ corresponds to $(\tilde{I}_0, I, D)$. Directly from (24) one can obtain that

$$\tilde{K}(\tau + \alpha, a) = K(\tau + \alpha, a) + \begin{cases} -\varepsilon e^{-\delta a} & \text{for } a \in [\alpha - \varepsilon, \alpha] \cap (\tau + \alpha - \Omega'), \\ \rho(\varepsilon) e^{-\delta a} & \text{for } a \in [0, \varepsilon] \cap (\tau + \alpha - \Omega''), \\ 0 & \text{elsewhere.} \end{cases} \tag{77}$$

Integrating we get that (76) holds if and only if

$$\varepsilon \int_{[\alpha - \varepsilon, \alpha] \cap (\tau + \alpha - \Omega')} e^{-\delta a} \, \mathrm{d}a = \rho(\varepsilon) \int_{[0, \varepsilon] \cap (\tau + \alpha - \Omega'')} e^{-\delta a} \, \mathrm{d}a.$$

Replacing the exponents by 1 we obtain the equality

$$\varepsilon \, \text{meas}([\alpha - \varepsilon, \alpha] \cap (\tau + \alpha - \Omega')) + \varepsilon^2 O(\alpha) = \rho(\varepsilon) \text{meas}([0, \varepsilon] \cap (\tau + \alpha - \Omega'')) + \varepsilon^2 O(\varepsilon),$$

where we use the standard notation $O(\alpha)$ for any function converging to zero with linear rate (here this rate is independent of F, since F is not involved in the above equalities). Using (75) for τ and $\tau + \alpha$ we obtain that

$$\rho(\varepsilon) = \varepsilon + o(\varepsilon) + \varepsilon O(\alpha), \tag{78}$$

where $o(\varepsilon)/\varepsilon \to 0$ with $\varepsilon \to 0$. Having in mind (77), this choice of $\rho(\varepsilon)$ ensures that $\tilde{K}(\tau+\alpha,\cdot)$ majorizes $K(\tau+\alpha,\cdot)$. If it happens that $\tilde{B}(\tau+\alpha) > B(\tau+\alpha)$, we would have

$$V(\tilde{K}(\tau+\alpha,\cdot),\tilde{B}(\tau+\alpha)) > V(K(\tau+\alpha,\cdot),B(\tau+\alpha)).$$

Since $D(t)$ was not changed till $\tau+\alpha$, the above inequality contradicts the dynamic programming principle. Therefore, it must hold that

$$\Delta B(\tau+\alpha) = \tilde{B}(\tau+\alpha) - B(\tau+\alpha) \le 0. \tag{79}$$

From the expression for K as a solution of (24), similarly as in (77) (but for time t instead of $\tau+\alpha$) one can obtain that

$$|\Delta Q(t)| = |\tilde{Q}(t) - Q(t)| = \int_0^\omega F|\tilde{K}(t,a) - K(t,a)| \le c_1 F\varepsilon^2,$$

where here and below c_1,, c_2, ... are numbers that are independent of F. Then from the equation for B we derive that for $t \in [\tau,\tau+\alpha]$ it holds that

$$\Delta B(t) = \int_\tau^t e^{i(t-s)} \left\{ p\Delta Q(t) - \left[\left(b_0\tilde{I}_0(s) + \frac{c}{2}\tilde{I}_0(s)^2\right) - \left(b_0 I_0(s) + \frac{c}{2}I_0(s)^2\right)\right]\right\} \,\mathrm{d}s.$$

The integral of the quantity between the brackets is obviously $O(\varepsilon^2)$, as well as $\Delta Q(t)$. Then approximating $e^{i(t-s)} = 1 + O(\alpha)$ for $s \in [\tau,t] \subset [\tau,\tau+\alpha]$ we obtain that

$$\Delta B(\tau+\alpha) \ge \int_{[\tau,\tau+\varepsilon]\cap\Omega'} b_0\varepsilon \,\mathrm{d}s - \int_{[\tau+\alpha-\varepsilon,\tau+\alpha]\cap\Omega''} b_0\rho(\varepsilon)\,\mathrm{d}s$$

$$+ \int_{[\tau,\tau+\varepsilon]\cap\Omega'} c\varepsilon I_0(s)\,\mathrm{d}s - \int_{[\tau+\alpha-\varepsilon,\tau+\alpha]\cap\Omega''} c\rho(\varepsilon) I_0(s)\,\mathrm{d}s. - c_2 F\alpha\varepsilon^2.$$

From (75), (78), and the continuity of I_0 on Ω' and Ω'', we derive that

$$\Delta B(\tau+\alpha) \ge c\varepsilon^2(I_0(\tau) - I_0(\tau+\alpha)) - o(\varepsilon^2) - c_3\alpha\varepsilon^2 - c_4 F\alpha\varepsilon^2.$$

Taking into account (79), deviding by ε^2 and passing to the limit with ε we obtain

$$I_0(\tau) - I_0(\tau+\alpha) \le c_5(1+F)\alpha. \tag{80}$$

Below we assume that I_0 is redefined as

$$I_0(t) = \liminf_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} I_0(s)\,\mathrm{d}s,$$

which changes it only on a set of measure zero. Then for every two numbers t and $t+\alpha$ we have that

$$I_0(t+\alpha) - I_0(t) = \liminf_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\left[\int_{t-\varepsilon}^{t+\varepsilon} I_0(s)\,\mathrm{d}\tau - \int_{t+\alpha-\varepsilon}^{t+\alpha+\varepsilon} I_0(\tau)\,\mathrm{d}\tau\right]$$

$$= \liminf_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} (I_0(\tau) - I_0(\tau + \alpha))\, \mathrm{d}\tau.$$

Here we can use the already obtained estimation (80), since it holds for almost every τ. After integration we obtain for $I_0(t+\alpha) - I_0(t)$ the same inequality as in (80), which proves the claim.

Claim 2. There exist numbers $\bar{B}$ and $\bar{Q}$, which are independent of F such that $B(t) \le \bar{B}$ and $Q(t) \le \bar{Q}$ for every $t \in [0, \bar{t}]$. This was already argued in the proof of Lemma 2.

Now we shall finalize the proof. Assume that $I_0(\bar{t}-) > 0$. Then according to Proposition 4 and Remark 5 we have

$$I_0(\bar{t}+) \ge \frac{b_0}{c} \frac{F - f}{f}.$$

We let F take "large" values and denote $\beta = b_0/2pF$. According to Claim 1 it holds that

$$\int_{\bar{t}}^{\bar{t}+\beta} (I_0(t))^2 \,\mathrm{d}t \ge \int_{\bar{t}}^{\bar{t}+\beta} [I_0(\bar{t}+) - L(1+F)(t - \bar{t})]^2 \,\mathrm{d}t$$

$$= \beta (I_0(\bar{t}+))^2 - I_0(\bar{t}+) L(1+F)\beta^2 + L(1+F)\frac{\beta^3}{3}$$

$$\ge \left(\frac{b_0}{2pF} \frac{b_0(F-f)}{cf} - L(1+F) \frac{b_0^2}{4p^2F^2} \right) I_0(\bar{t}+) - O(1/F^2)$$

$$\ge c_1 I_0(\bar{t}+) - O(1/F^2) \ge c_2 F - c_3,$$

where c_1, c_2, ..., are positive and independent of F (for sufficiently large F).

From (26) we easily obtain that

$$Q(t) \le Q(\bar{t}) + F \int_{\bar{t}}^{t} I_0(s)\,\mathrm{d}s \le \bar{Q} + F \int_{\bar{t}}^{t} I_0(s)\,\mathrm{d}s.$$

Thus from (25) we derive

$$\dot{B}(t) \le iB + p\left(\bar{Q} + F\int_{\bar{t}}^{t} I_0(s)\,\mathrm{d}s\right) - b_0 I_0(t) - \frac{c}{2}(I_0(t))^2, \quad B(\bar{t}) \le \bar{B}.$$

Using the Cauchy formula, we rewrite the inequality $B(\bar{t}+\beta) \ge 0$ as

$$0 \le e^{i\beta}\bar{B} + \int_{\bar{t}}^{\bar{t}+\beta} e^{i(\bar{t}+\beta-t)} \left[p\bar{Q} + pF \int_{\bar{t}}^{\bar{t}+\beta} I_0(s)\,\mathrm{d}s - b_0 I_0(t) - \frac{c}{2}(I_0(s))^2 \right] \mathrm{d}t$$

$$\le e^{i\beta}\bar{B} + p\bar{Q}(1 + O(\beta)) + pF(1+O(\beta)) \int_{\bar{t}}^{\bar{t}+\beta} I_0(t)\,\mathrm{d}t - b_0 \int_{\bar{t}}^{\bar{t}+\beta} I_0(t)\,\mathrm{d}t - \frac{c}{2}\int_{\bar{t}}^{\bar{t}+\beta} (I_0(t))^2\,\mathrm{d}t.$$

The choice of β implies that the sum of the third and the fourth term in the last line is negative for all sufficiently large F. Hence,

$$0 \leq c_4 - (c_2 F - c_3),$$

This, however is false for all sufficiently large F, which is a contradiction, caused by our assumption that $I_0(\bar{t}-) > 0$. This still does not imply the claim (i), since our definition of $I_0(\bar{t}-)$ (see footnote 5) involves liminf instead of lim. However, the quadratic adjustment cost of I_0 easily implies that $\mathrm{liminf}_{h \to 0+} I_0(t) = 0$ yields $\lim_{h \to 0+} I_0(t) = 0$, which proves part (i) of the proposition.

(ii) Assume that F is so large that the claim (i) of Proposition 5 holds.

If the claim of the proposition is false, then there are moments t arbitrarily close to $\bar{t}$ at which $\dot{B}(t)$ exists and is non-positive. Since $D(t) = 0$ whenever $B(t) > 0$ (which holds close to $B(\bar{t})$), we have

$$iB(t) + pQ(t) - b_0 I_0(t) - \frac{c}{2} I_0(t)^2 \leq 0.$$

From $B(\bar{t}) > 0$ we conclude that $I_0(t) \geq \rho > 0$. This contradicts Proposition 5. Q.E.D.

List of Figures

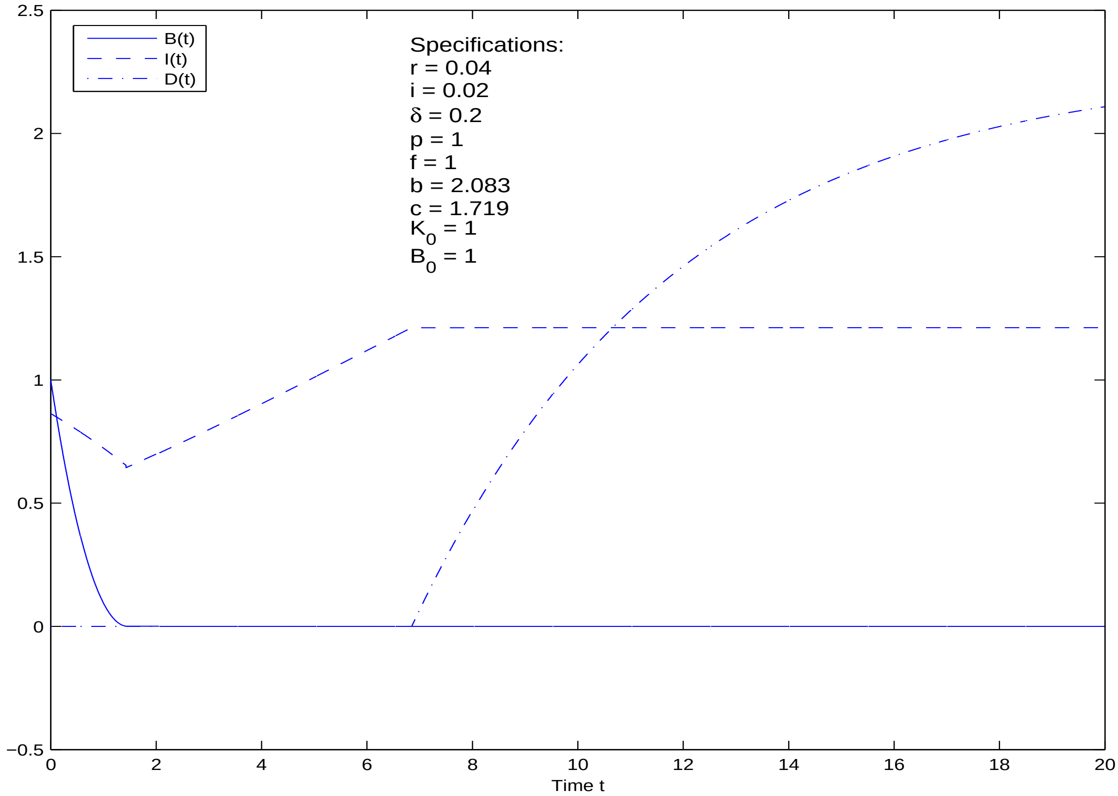

Figure 1: Optimal solution B, I, and D for the investment problem with the given specifications.

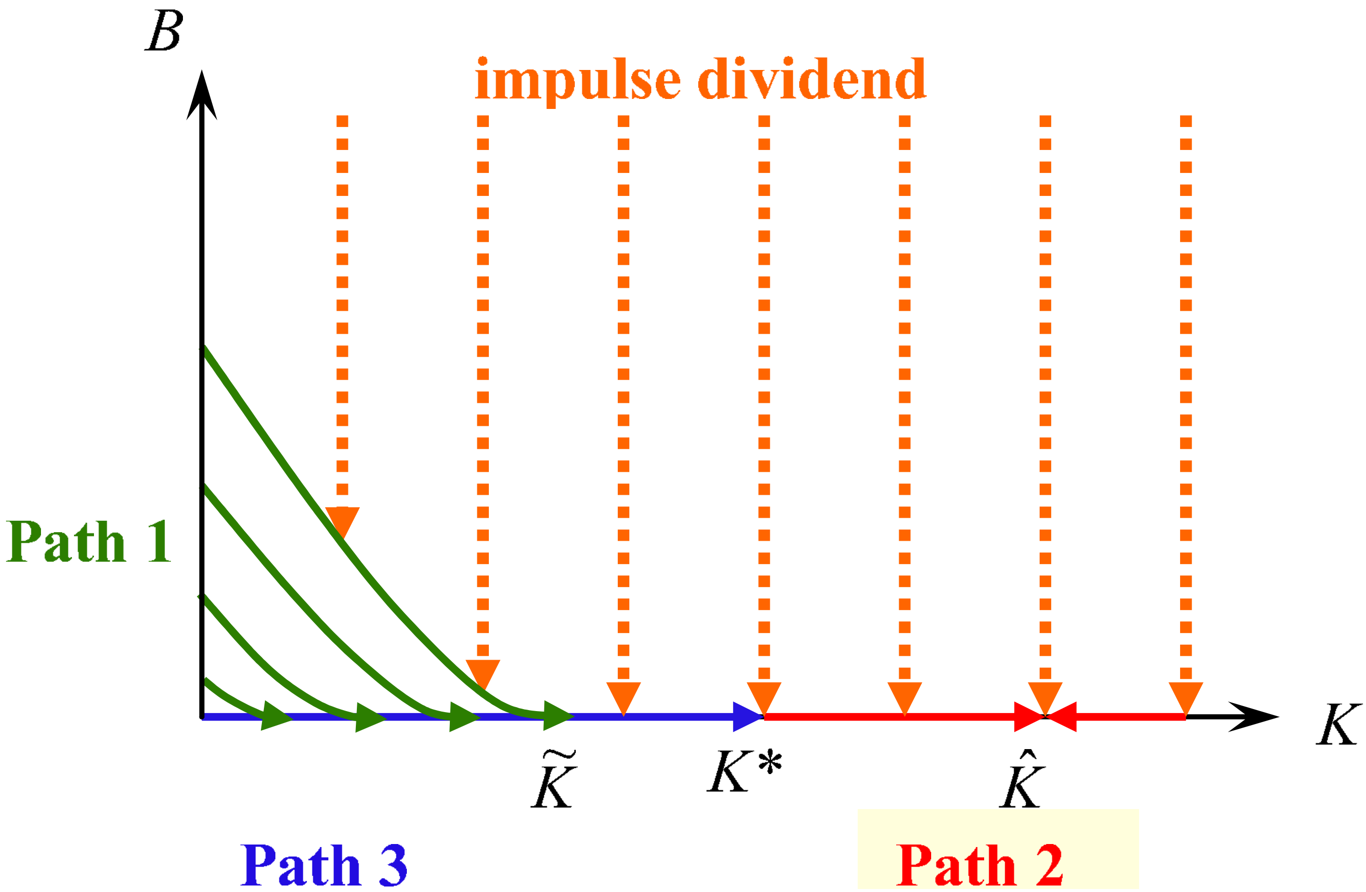

Figure 2: Optimal paths for (K, B) starting from different initial states.