Age specific dynamic labor demand and human capital investment *†

Alexia Prskawetz‡ Vladimir M. Veliov§

Abstract

This paper studies the optimal age-specific labor demand and human capital investment at the firm level under conditions of changes in age-specific adjustment costs of hiring/firing. For this purpose we extend the standard dynamic labor demand model by introducing "age" as a second dynamic variable and distinguish between two types of workers: "low skilled" and "high skilled". For the case of a linear revenue and production function we prove that firms do not anticipate the change in adjustment costs in their optimal decisions. This result no longer holds if a nonlinear revenue or production function is considered.

*This research was partly financed by the Austrian Science Foundation (FWF) under grant No P18161-N13, and by the Austrian National Bank (ÖNB) under grant No 11621.

†We are grateful for comments and suggestions on an earlier draft by Raouf Boucekkine, Gustav Feichtinger, Peter Kort, Ingrid Kubin, Thomas Lindh, Julien Pratt and Gernot Tragler. In particular we would like to thank Max Halvarsson for providing age-structured employment structure for Sweden.

‡Vienna Institute of Demography, Prinz Eugenstr. 8-10, 1040 Vienna, Austria (alexia.fuernkranz-prskawetz@oeaw.ac.at)

§Institute of Mathematical Methods in Economics (research group on Operations Research and Nonlinear Dynamical Systems), Vienna University of Technology, Argentinierstrasse 8, A-1040 Vienna, Austria (vveliov@eos.tuwien.ac.at), and Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria.

1 Introduction

The balance among the age groups in the labor force is expected to change markedly over the next years. In the majority of industrialized countries we will have a reversal of the proportions of the labor force in the youngest and oldest age groups over the coming decades. The question is then *how will firms in their hiring and firing strategy react to the expected change of the labor supply that is caused by demographic change*. At the same time, since a large portion of human capital accumulation in the form of training and on-the-job learning takes place inside firms (see [4] and [23]) it is of interest *how firms will react in terms of their age-specific human capital investment under conditions of labor force aging.*[1] Our paper is intended to raise and to address the following questions: Under which conditions will it be optimal for firms to hire more older workers? How will the optimal age structure of the labor force look like? Will firms increase their investment into firm-specific human capital in accordance with the optimal age structure of hiring/firing? Are changes in age-specific costs of hiring/firing also anticipated in the optimal decisions at the firm level?

We investigate the problem of *optimal employment and training policy at the firm level* allowing for heterogeneity of workers with respect to their age and firm-specific human capital and considering the changing age structure of the labor supply.

The theoretical framework we choose is a classical dynamic labor demand model ([24]). To capture the heterogeneity of workers by age and human capital we introduce "age" as a second dynamic variable and distinguish two types of workers: unskilled versus skilled workers. We furthermore introduce age-specific adjustments costs of hiring/firing and human capital investment and age-specific wage schedules. The specific form of the adjustment costs of hiring/firing we postulate may represent the tightness of the labor market for specific age groups of the labor force[2].

Since hiring costs essentially account for recruitment (advertising, searching, screening) and training costs while firing costs essentially consist in severance payments, administrative constraints and possible reorganization costs, a link between adjustment costs and the age structure of the potential working age population seems natural. Recalling the search-matching models in labor economics (cf. [28]) a key assumption is that firms cannot hire employees instantaneously, but must search (through a costly and time-consuming process) in the pool of unemployed workers. The probability that a vacancy is matched

[1] We only consider firm-specific human capital investment assuming that firms are not willing to invest in general training when labor markets are competitive.

[2] Given the structure of the adjustment costs, we then concentrate only on the demand side of the labor market.

with an unemployed worker will then depend on the tightness of the labor market as represented by the ratio of vacancies to unemployed. Obviously, the lower this ratio the more likely a match will occur and this may be related to lower adjustment costs of hiring workers. Since unemployment rates are known to be age-dependent (generally decreasing with age), a change in the demographic composition of the workforce will have a direct effect on the overall unemployment rate and hence the probability of a job match (assuming no change in vacancies). Moreover, as recently illustrated in [16], young and old workers may be characterized by different separation rates and productivity, thereby indirectly affecting the equilibrium unemployment rate[3]. As indicated in [16] the altered age composition may either increase or decrease the equilibrium unemployment rate depending on specific parameter values of the model. Hence, the consequences of an aging of the labor supply, on age-specific adjustment costs are not clear a priori. We therefore allow for a general form of age-specific adjustment costs and present analytical results how the age structure of the optimal labor demand will depend on quite general parametric variations in the adjustment cost function.

That the employment structure at the firm level indeed follows a rather peculiar age shape is supported by empirical data. In Figure 1a we plot the mean age structure of employees for selected industries in the mining and manufacturing sector in Sweden for 1996[4]. The structure of employees shows a clear hump-shaped age pattern. The aging of the employee structure becomes clear if we consider the evolution of the mean age over time (Figure 1b). As we will show in this paper, by including age structure into a standard dynamic labor demand model we are able to model the empirically observed age structure of employees at the firm level and its evolution over time.

While the age structure has been introduced into the neoclassical theory of optimal investment in several recent contributions (e.g., [3], [10]), we are only aware of one paper (cf. [7]) that augmented the classical theory of labor demand by modelling the age structure of the workers. As outlined in [7] there are clear differences between the theory of capital and the theory of labor demand. These include the fact that (a) a firm accumulates the capital stock that it buys in the capital goods market, while the firm rents labor services at current wage rates; (b) political decisions may influence the span of working life time; (c) a large part of labor adjustment does not arise for technical reasons but because of legal regulations. Our paper adds to the last point. We argue that adjustment costs will

[3]A different approach has been discussed in [31] where a theory is developed that can explain the negative correlation between the youth share of the working age population and the youth unemployment rate that stands in contrasts to the literature of an adverse effect of cohort size on youth unemployment. The key argument in [31] is that the presence of young workers generates a more fluid labor market which in turn stimulates vacancy creation.

[4]We base our figures on a panel of employer-employee matched data from Statistics Sweden covering the 1985-1996 period.

Figure 1.a

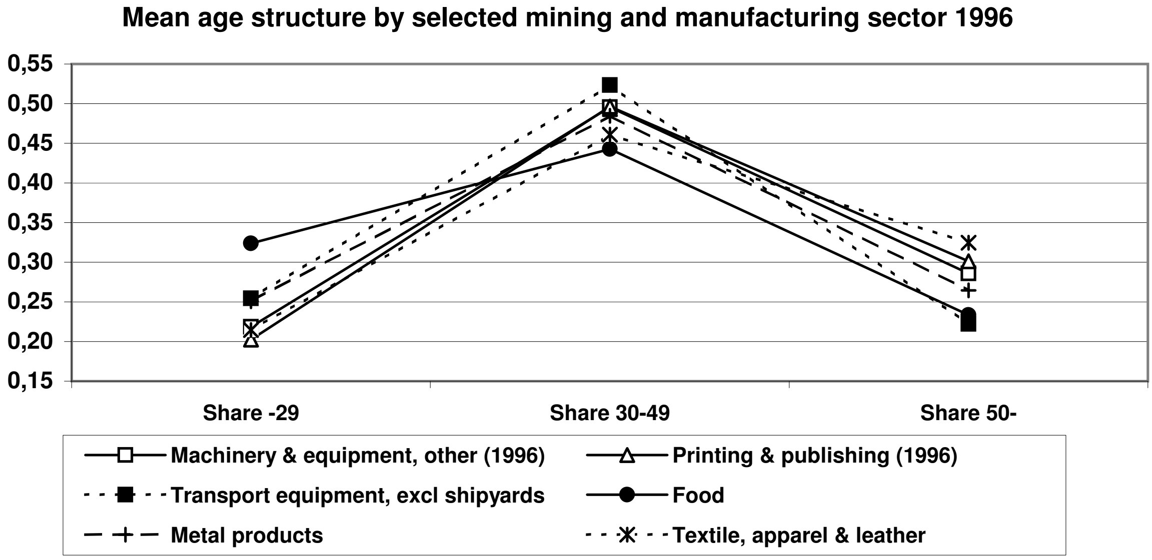
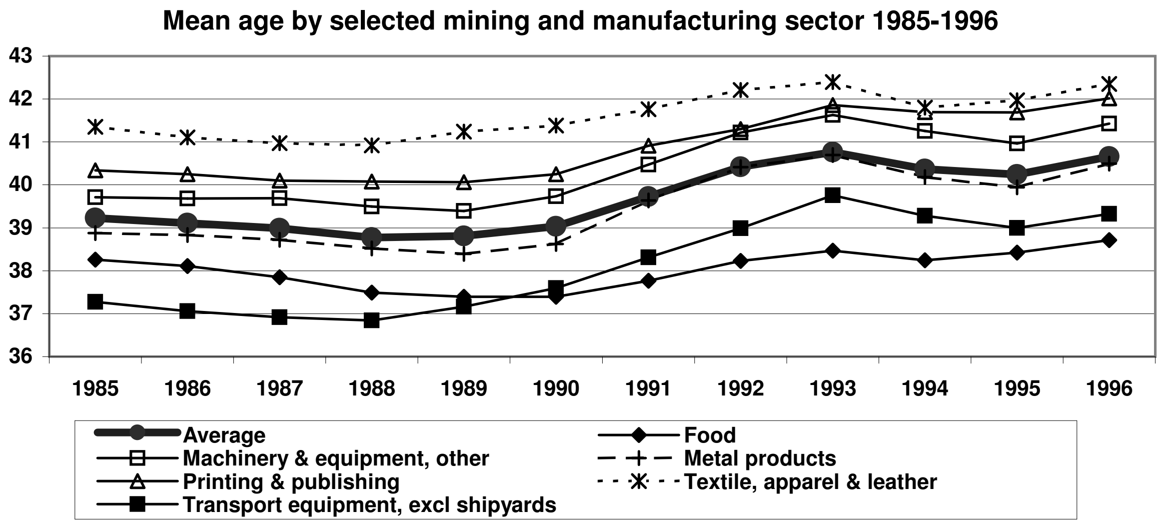


Figure 1.b



also reflect the tightness of the labor market as caused by demographic shifts in the anticipated labor supply. In addition to the optimal age-specific hiring/firing at the firm level (as in [7]), we also allow for an optimal investment in age-specific human capital. To solve the continuous-time dynamic optimization model we apply the maximum principle for systems with distributed parameters as developed by [11].

The structure of the paper is as follows. In the next section we present the theoretical framework of our study. Optimality conditions and the optimal steady state are presented in Section 3 and Section 4. We show that the optimal steady state is reached in one generation if the firm's revenue function is linear. Section 5 is devoted to the case of a linear revenue and a linear production function. First we obtain certain relations between the optimal level of hiring/firing (and the human capital investment) for different adjustment cost functions of hiring/firing. Then we present a general proposition for the sensitivity of the optimal mean age of hiring with respect to parametric changes of the adjustment cost function. We apply this proposition to establish some qualitative consequences for the optimal mean age of hiring, of changes in the magnitude, the slope, or the shape of the adjustment cost function with respect to age. We conclude this section with a key property of optimal policies in case of a linear production and linear revenue function: the non-anticipation property. Changes of the adjustment costs are not incorporated into the firms' optimal decisions prior to the point in time when these changes occur. The validity of our analytical results and in particular the non-anticipation property are illustrated (through numerical simulations) in the final subsection of Section 5. In Section 6 we extend our framework to allow for nonlinear production and revenue functions. We show that most of our analytical results also hold in this more general case (though analytical results are not provided in this case). However, in contrast to the case of linear production and revenue functions, the firm anticipates a change in adjustment costs and reacts in advance to the change by adjusting its optimal hiring/firing and human capital investment policy. The final section (Section 7) concludes the paper. The benchmark data specifications are given in Appendix 1. Some more technical proofs are collected in Appendix 2. Appendix 3 presents an approach for identifying from scarce data the quit rates of workers used in the model.

2 The model

Below t denotes time and s denotes the "active" age of an individual: $s = a - a^0$, where a is the biological age of the individual, and a^0 is the age at which the individual first started to work[5]. Further we assume that all individuals start working at the same age

[5] Often we shall refer to the active age merely as to "age".

a^0 and all retire at age $a^0 + \omega$. Following [13] we distinguish the workers also by their skills, considering for simplicity two levels of qualification—unskilled and skilled workers. We denote by $L(t,s)$ the amount of unskilled workers (or put differently, workers without firm-specific human capital) of age s being employed by the firm at time t, and by $H(t,s)$—the amount of skilled workers that emanate from the firms' investment in (firm-specific) human capital. Depending on the age of the workers, productivity of skilled and unskilled workers will change. Productivity of skilled workers exceeds productivity of unskilled workers at each age. We assume that the production technology depends only upon labor and it is additive in the age groups of workers[6]:

$$Y(t) = \int_0^\omega F(s, L(t,s), H(t,s))\,\mathrm{d}s,$$

where $F(s,L,H)$ constitutes the output resulting from L unskilled and H skilled workers of age s, and $Y(t)$ is the output at time t resulting from labor compositions $L(t,\cdot)$ and $H(t,\cdot)$.

The output of size Y gives a revenue $R(Y)$. In the case of perfect competition $R(Y) = pY$, where p is the price of a unit of a output, assumed further equal to one. In general, $R(Y)$ is an increasing concave function.

The firm can alter employment by either hiring with age-specific intensity $u(t,s) > 0$, or by age-specific dismissal, $u(t,s) < 0$. All new entrants into the firm are classified as unskilled workers, i.e. workers that do not yet possess firm-specific human capital. Moreover, in order not to complicate the model too much, we assume further that the firm does not fire skilled workers.[7] In addition, exogenous turnover of rates $\delta(s) \geq 0$ and $\nu(s) \geq 0$ (for unskilled and skilled workers, respectively) can take place (cf. Kraft [19]). It represents the spontaneous and costless attrition of employees through quits. Early retirements (before age ω) can also be included as components of these rates.

Upgrading of unskilled workers into skilled workers takes place at a rate $(\beta + v(t,s))\gamma(s)$. Here $v(t,s)$ represents the costly investment rate into firm-specific human capital at time t and for workers of age s, β represents the costless learning by doing.[8] The function $\gamma(s)$

[6] A similar set up is presented in [24],p.478. We neglect capital stock assuming that it is exogenous or predetermined and all other factors are optimally deployed. An alternative explanation is given in [25], p.9 "Although capital is not introduced explicitly in the problem for ease of notation, plants should be thought as using both capital and labor. However if capital is either not costly to adjust, or if its adjustment costs are additively separable from those for labor (there are no interrelated adjustment costs), then one would still obtain the same FOCs for labor."

[7] The study by Salvanes [30] for Norway indicates that there is indeed a decline in job destruction with educational level.

[8] Note, that $\beta > 0$ guarantees that even in case of zero investment rate into human capital upgrading, part of the initial low skilled workers will acquire firm-specific human capital and become high skilled workers.

reflects the age-dependence of learning abilities of the workers.

The equations for the stock of unskilled and skilled workers are therefore

$$\begin{aligned} L_t + L_s &= -\delta(s)L(t,s) + u(t,s) - (\beta + v(t,s))\gamma(s)L(t,s), \quad && L(0,s) = L_0(s),\ L(t,0) = 0, \\ H_t + H_s &= -\nu(s)H(t,s) + (\beta + v(t,s))\gamma(s)L(t,s), && H(0,s) = H_0(s),\ H(t,0) = 0. \end{aligned}$$

The left hand side, $L_t + L_s = \lim_{h\to 0}(L(t+h, s+h) - L(t,s))/h$ represents the change in one unit of time of the unskilled labor that is of age s at time t. This change is composed of voluntary quits at the rate $\delta(s)$, hiring, $u(t,s)$, and upgrading of unskilled to skilled workers at the rate $(\beta + v(t,s))\gamma(s)$. Similarly, the left hand side, $H_t + H_s$ represents the change in one unit of time of the skilled labor that is of age s at time t. This change is composed of voluntary quits at the rate $\nu(s)$ and inflow at the rate $(\beta + v(t,s))\gamma(s)$ of workers whose skill has been upgraded. The initial conditions for unskilled and skilled labor, $L_0(s)$ and $H_0(s)$, are given. The hiring/firing intensity $u(t,s)$, as well as the upgrading of workers $v(t,s)$ are considered as control variables.

The firm faces convex adjustment costs $C(t,s,u)$, for hiring and dismissal of employees, increasing for $u > 0$ and decreasing for $u < 0$, with $C(s,0) = C_u(s,0) = 0$. The cost of upgrading the human capital of $v(t,s)\gamma(s)L$ workers of age s is $D(v(t,s))(L + l_0)$ (here l_0 is a fixed cost component, independent of the number of people under training). Moreover, the firm pays wages $w^L(s)$ to unskilled and wages $w^H(s)$ to skilled workers employed.[9]

The formal problem of the firm is then to choose the age-specific schedule of hiring and firing, $u(t,s)$, together with the optimal age-specific investment rate, $v(t,s)$, in upgrading human capital to maximize profits. That is, the firm chooses at each instant of time an age dependent in/out-flow of labor as well as an age dependent investment in human capital to maximize profits. Discounting the future with a rate $r > 0$ we come up with the following dynamic optimization problem with state variables $L(t,s)$, $H(t,s)$, and control variables $u(t,s)$ and $v(t,s)$:

$$\begin{aligned} \max \int_0^\infty & e^{-rt} \{R(Y(t)) \qquad\qquad (1) \\ & - \int_0^\omega \left[w^L(s)L(t,s) + w^H(s)H(t,s) + C(t,s,u(t,s)) + D(v(t,s))(L(t,s) + l_0)\right] \mathrm{d}s \Big\}\, \mathrm{d}t \end{aligned}$$

[9]As noted in [24], p. 478, the firm could in principle use its wage as a short run instrument of employment policy. However we ignore this possibility in the current version of the paper.

subject to

$$L_t + L_s = -\delta(s)L(t,s) + u(t,s) - \gamma(s)(\beta + v(t,s))L(t,s), \tag{2}$$
$$L(0,s) = L_0(s), \quad L(t,0) = 0.$$
$$H_t + H_s = -\nu(s)H(t,s) + \gamma(s)(\beta + v(t,s))L(t,s), \tag{3}$$
$$H(0,s) = H_0(s), \quad H(t,0) = 0,$$
$$Y(t) = \int_0^\omega F(s, L(t,s), H(t,s))\,\mathrm{d}s \tag{4}$$
$$v(t,s) \geq 0, \tag{5}$$
$$L(t,s) \geq 0, \quad H(t,s) \geq 0. \tag{6}$$

The last three constraints are to prevent negative amounts of labor or negative amounts of training.

3 Optimality conditions

We assume that all exogenous functions are non-negative and piece-wise continuous in t and s, that the derivatives that appear later on exist and are continuous, that C and D are monotone increasing in u and respectively v, that the second derivatives $C_{uu}(t,s,\cdot)$ and $D_{vv}(\cdot)$ are strictly positive (for all t, s), $D(+\infty) = +\infty$, and that the functions R and F are strictly increasing and concave (not necessarily strictly). Moreover, $l_0 \geq 0$ and $r > 0$.

Notice that the state constraint $H(t,s) \geq 0$ is automatically fulfilled due to the control constraint $v(t,s) \geq 0$, therefore it can be ignored. In most of the considerations below we assume that the constraint $L(t,s) \geq 0$ is also automatically fulfilled, which is the case if $w^L(s) \leq R'(Y)F_L(s,L,H)$ and $w^H(s) \leq R'(Y)F_H(s,L,H)$ for all (reasonable) positive L, H and Y (the marginal revenue from a worker of any age exceeds the wage). In some of the results, however, as well as in all numerical experiments we take into consideration the state constraints (6).

Lemma 1 *For problem (1)–(6) the maximum in (1) is finite and there is an admissible control pair (u,v) for which it is attained (that is, an optimal control). Any optimal control (u,v) is essentially bounded*[10].

[10]In addition, the optimal u and v are Lipschitz continuous, as it follows from the proposition below.

A sketch of the proof is given in Appendix 2.

To obtain necessary optimality conditions one could apply the maximum principle for age-structured systems obtained in Brokate [5] or in the recent paper by Feichtinger et al. [11]. However these, and all to the authors known results that are applicable to systems such as (1)–(5) concern finite horizon problems. Thanks to the stability of the system it is possible to obtain here a transversality condition (see (7) below) that complements the usual boundary conditions (cf. [11])[11].

Proposition 1 *Let (u, v, L, H, Y) be an optimal solution to the problem (1)–(5). Then the adjoint system*

$$\begin{aligned}
\xi_t + \xi_s &= (r + \delta(s))\xi + \gamma(s)(\beta + v(t,s))(\xi - \eta) - R'(Y)F'_L(s, L, H) + w^L(s) + D(v(t,s)), \\
&\qquad\qquad \xi(t, \omega) = 0, \\
\eta_t + \eta_s &= (r + \nu(s))\eta - R'(Y)F'_H(s, L, H) + w^H(s), \\
&\qquad\qquad \eta(t, \omega) = 0,
\end{aligned}$$

with the additional conditions

$$\limsup_{t\to\infty} \left[\|\xi(t,\cdot)\|_{L_\infty([0,\omega])} + \|\eta(t,\cdot)\|_{L_\infty([0,\omega])} \right] < +\infty, \tag{7}$$

has a unique solution (ξ, η), and the following relations hold for almost all (t, s):

$$u(t,s) = C_u^{-1}(t, s, \xi(t,s)), \tag{8}$$

$$v(t,s) = \max\left\{ 0, D_v^{-1}\left(\frac{(\eta(t,s) - \xi(t,s))\gamma(s)L(t,s)}{L(t,s) + l_0} \right) \right\}, \tag{9}$$

where $C_u^{-1}(t, s, \cdot)$ is the inverse of the derivative C_u with respect to u, and $D_v^{-1}(\cdot)$ is the inverse of the derivative D_v with respect to v.

As a byproduct, one can calculate the gradient of the objective function with respect to the control functions as follows:

$$\begin{aligned}
\frac{\mathrm{d}J(u)}{\mathrm{d}u}(t,s) &= \xi(t,s) - C_u(t, s, u(t,s)), \\
\frac{\mathrm{d}J(v)}{\mathrm{d}v}(t,s) &= (\eta(t,s) - \xi(t,s))\gamma(s)L(t,s) - D_v(v(t,s))(L(t,s) + l_0).
\end{aligned}$$

[11] Although we shall not make use of the transversality conditions in the analysis below (since in the cases we analyze, the adjoint system has a unique solution even without the transversality conditions), these conditions play an important role for the justification of the numerical approach we use. However, this issue will not be discussed in this paper.

These formulas provide the base of an efficient gradient-type numerical method for solving the problem[12].

4 Optimal steady-state

Formally, the equations for the optimal steady-state $(\bar{u}(s), \bar{v}(s), \bar{L}(s), \bar{H}(s), \bar{Y})$ and the corresponding co-state $(\bar{\xi}(s), \bar{\eta}(s))$ can be obtained assuming time-invariance in the primal and adjoined equations. This procedure gives the system (with the dot meaning differentiation in s)

$$
\begin{aligned}
\dot{\bar{L}} &= -\delta(s)\bar{L} + \bar{u} - \gamma(s)(\beta + \bar{v})\bar{L}, \quad \bar{L}(0) = 0,\\
\dot{\bar{H}} &= -\nu(s)\bar{H} + \gamma(s)(\beta + \bar{v})\bar{L}, \quad \bar{H}(0) = 0,\\
Y &= \int_0^\omega F(s, \bar{L}(s), \bar{H}(s))\,\mathrm{d}s,\\
\dot{\bar{\xi}} &= (r + \delta(s))\bar{\xi} + \gamma(s)(\beta + \bar{v})(\bar{\xi} - \bar{\eta}) - R'(Y)F'_L(s, \bar{L}, \bar{H}) + w^L(s) + D(\bar{v}), \quad \bar{\xi}(\omega) = 0,\\
\dot{\bar{\eta}} &= (r + \nu(s))\bar{\eta} - R'(Y)F'_H(s, \bar{L}, \bar{H}) + w^H(s), \quad \bar{\eta}(\omega) = 0,
\end{aligned}
$$

with u and v expressed in the state-co-state feedback form

$$
\begin{aligned}
u(s, \bar{\xi}) &= C_u^{-1}(s, \bar{\xi}),\\
v(\bar{L}, \bar{\xi}, \bar{\eta}) &= \max\left\{0, D_v^{-1}\left(\frac{(\bar{\eta} - \bar{\xi})\gamma(s)\bar{L}}{\bar{L} + l_0}\right)\right\}.
\end{aligned}
$$

The above equations represent the maximum principle for an appropriate optimal control problem for an ordinary differential system, and similarly as in Lemma 1 one can prove existence of a unique solution of this problem.

Proposition 2 *Assume that $R(Y) = Y$. Then for any nonnegative initial data $L_0(s)$ and $H_0(s)$ an optimal solution of problem (1)–(5) reaches the optimal steady state in time at most ω.*

Notice that the above property does not require linearity of the production function. If the production function is also linear and training is not present ($v = 0$ and $\beta = 0$), this result

[12] A description of a class of gradient projection methods for solving general optimal control problems for age-structured systems is given in the forthcoming paper [33].

is obtained in [7]. We mention also that in the case of a nonlinear revenue function $R(Y)$ the optimal steady state is reached asymptotically, but not in finite time. This principle difference is caused by the aggregated quantity Y which brings a nonlocal feature of the dynamics. In the case of a linear $R(Y)$ involvement of Y could be avoided, since Y can be included directly in the objective function. In this case the distributed optimal control that we consider splits into a continuum of ordinary optimal control problems (one for each cohort of workers). This, actually, is shown in the proof of Proposition 2, given in Appendix 2.

5 Linear revenue and production functions

We consider first the case of a linear revenue and a linear production function

$$R(Y) = Y, \quad F(s, L, H) = \pi^L(s)L + \pi^H(s)H \tag{10}$$

where $\pi^L(s)$ and $\pi^H(s)$ denote age-specific productivity of low and high skilled workers.

We establish a number of qualitative features of the optimal hiring policies among which the most important is the role that the adjustment cost function plays for the age structure of the employees. We also establish conditions for the so-called non-anticipation behavior of optimal policies at the firm level. In this case, the current policy of the firm is not influenced by future changes in the adjustment cost function.

5.1 The role of the adjustment costs

In this subsection we study the dependence of the optimal solution of problem (1)–(5) on the magnitude and the shape of the adjustment cost function. For simplicity we assume in most of the statements that the adjustment cost function $C(t, s, u)$ is quadratic[13]:

$$C(t, s, u) = \frac{1}{2}c(t, s)u^2. \tag{11}$$

We call the multiplier $c(t, s)$ the *adjustment factor* and always suppose it to be strictly positive and piece-wise continuous with respect to t and s.

[13]We are aware that those standard assumptions are rejected by empirical evidence (cf. [15] and [29]) but assume symmetric, strictly convex adjustment costs to yield analytic results.

The following two propositions are generalizations of results shown in [7] that relate the dynamics of hiring/firing and employment to changes in the adjustment cost of hiring/firing. In the case of $\beta + v(t,s) = 0$ and $H_0(s) = 0$ we have $H(t,s) = 0$ and our model reduces to the one investigated in [7]. However, even in this case our results extend those in [7] in several directions. In particular, while the results in [7] only refer to the steady state, our propositions also hold for the transient behavior.

Proposition 3 *Assume that the training rate $v(t,s)$ is either exogenous, or the training is a control variable with no fixed costs, i.e. $l_0 = 0$. Then (in the case of endogenous v) the optimal investment rate in human capital, $v(t,s)$, is independent of the adjustment cost function, $C(t,s,u)$. If, in addition, $C(t,s,u)$ has the quadratic form (11), then the following statements hold.*

(i) Dividing the adjustment factor $c(t,s)$ by a constant leads to multiplication by the same constant of the optimal hiring/firing policy.

(ii) Dividing the adjustment factor $c(t,s)$ by a constant leads to multiplication by the same constant of the optimal stock of skilled and unskilled workers for $t > \omega$;

(iii) If $c(t,s)$ and $\tilde{c}(t,s)$ are two adjustment factors and (u,v) and $(\tilde{u},\tilde{v})$ are the corresponding optimal control functions, then for every t and s

$$u(t,s)c(t,s) = \tilde{u}(t,s)\tilde{c}(t,s), \qquad v(t,s) = \tilde{v}(t,s).$$

Proof. In the present situation the adjoint system reads as (we skip the argument (t,s))

$$\xi_t + \xi_s = (r + \delta(s))\xi + \gamma(s)(\beta + v)(\xi - \eta) - \pi^L(s) + w^L(s) + D(v), \qquad \xi(t,\omega) = 0, \tag{12}$$

$$\eta_t + \eta_s = (r + \nu(s))\eta - \pi^H(s) + w^H(s), \qquad \eta(t,\omega) = 0, \tag{13}$$

and in the case $l_0 = 0$ the expression for the optimal training rate is

$$v(t,s) = \max\{0, D_v^{-1}(\gamma(s)(\eta(t,s) - \xi(t,s)))\}. \tag{14}$$

The key observation in the proof is that in the case of exogenous v, and also in the case of $l_0 = 0$, the adjoint system has a closed form (independent of the state and the control variables). The boundary conditions determine a unique solution, since the system splits into a family of ODEs along the characteristic lines. Obviously this solution is independent of the function C. This implies the claim of the proposition concerning v.

If the adjustment cost function has the quadratic form (11), then the optimal hiring/firing is determined by

$$u(t,s) = \frac{\xi(t,s)}{c(t,s)}, \tag{15}$$

which directly implies (i) and (iii).

To prove (ii) we consider the primal system (2)–(3) and observe that (with the optimal or exogenous v) the first equation splits into ODEs along every characteristic line, which are linear in L and u. If we take $t > \omega$ and $s \in [0, \omega]$, the corresponding characteristic through (t,s) starts at $(t-s,0)$, and since $t-s>0$ we have $L(t-s,0)=0$. Then the solution $L(t,s)$ is a linear functional of u. Similar consideration of (3) shows that also $H(t,s)$ is a linear functional of u. This, together with (i), proves (ii). Q.E.D.

Proposition 3 (i) and (ii) imply that a proportional change in the adjustment costs at every age translates into a proportional change of hiring/firing at every age which also translates (after at most one generation) into a proportional change in employment. The last result (iii) characterizes the optimal hiring/firing policy if age-specific adjustment factor changes. Intuitively, the condition stated in (iii) indicates that the higher is the adjustment factor for a given age, the lower is hiring/firing at this age.

Although changes in the adjustment cost function lead to changes in the number of unskilled workers, according to the first claim of the proposition the optimal investment rate into human capital remains unaffected, as long as the training does not require fixed costs. The reason is twofold. First, the per capita cost of a training rate v applied to a certain number of workers, L, does not depend on L (thanks to $l_0 = 0$). Second, the net marginal revenue produced by an unskilled/skilled worker is independent of the number of unskilled/skilled workers (due to the linearity of the production and revenue functions). Thus the decision for the training rate at any given time and age is independent of the current values of L and H.

Similar as in [7], definite implications of the age-structure of the employment can only be derived under further restrictive assumptions on the adjustment cost of hiring/firing and further parameters of the model.

Proposition 4 *Consider the problem without training ($v(t,s)=0$). Assume the following: $C(t,s,u) = C(s,u)$ is independent of t, twice differentiable, $C_{uu}(s,u) \geq c_1 > 0$, and $C_{us}(s,u) \geq 0$; δ and ν are age-independent and $\nu \leq \delta$ (the quit rate of the skilled workers is lower than that of the unskilled ones); $\gamma(s)$, $\pi^L(s) - w^L(s)$ and $\pi^H(s) - w^H(s)$ are decreasing; for every s*

$$\pi^H(s) - w^H(s) \geq \pi^L(s) - w^L(s) > 0. \tag{16}$$

Then the optimal hiring control $u(t,s)$ is always nonnegative[14]*, and for every cohort emanating from some moment t, the hiring of this cohort decreases with time:*

$$s \longrightarrow u(t+s,s) \quad \text{strictly decreases with} \quad s.$$

The proof is given in Appendix 2.

We mention that the assumption $C_{us}(s,u) \geq 0$ in the above proposition is fulfilled in the case of a quadratic adjustment cost (11) if and only if the adjustment factor $c(s)$ is non-decreasing. If this is not the case, also if δ is not constant (namely, monotone decreasing) the monotonicity of hiring with age may disappear.

The proposition implies that at the steady-state the optimal hiring $u(s)$ is decreasing with age. A special case of this observation (without heterogeneity of labor and with all data age-independent) is obtained in Proposition 4 in [7]. Notice, that the above results indicate that firms always hire more younger than older workers.

If all data are age-independent and $\nu = \delta$, then the total amount of labor, $L(s) + H(s)$, at the steady-state is a single-picked function (cf. Proposition 4 (b) in [7])[15].

So far, we have studied the dynamics of hiring/firing and employment if adjustment costs of hiring/firing change either proportionality at all ages or age-specific. Furthermore, we have shown that firms will always hire more younger than older workers given that the surplus between age-specific productivity and wages decreases over age. We next investigate the change in adjustment costs more thoroughly by distinguishing between a change in the magnitude versus the shape of the adjustment costs.[16] We are interested in the results of these changes on the mean age of hiring. We assume the adjustment costs quadratic as in (11), and include in the adjustment factor a real parameter σ: $c(t,s) = c(t,s,\sigma)$. This representation of adjustment costs allows a general investigation of the mean age of hiring as it will depend on the parameter σ. The parameter σ could either capture a change in the magnitude or the slope of the adjustment costs.

The mean age of hirings, $A(t;\sigma)$ (we indicate its dependence on the parameter σ) is defined as

$$A(t;\sigma) = \frac{\int_0^\omega s u(t,s)\,\mathrm{d}s}{\int_0^\omega u(t,s)\,\mathrm{d}s},$$

[14]In particular, the state constraint $L \geq 0$ is automatically satisfied.

[15]Indeed, summing up (2) and (3) we obtain the same equation as in [7] and the result follows as in [7].

[16]One might assume that a change in the absolute number of the labor supply will be reflected in the magnitude of the costs while a change in the age structure of the labor supply will be reflected in the slope of the adjustment costs.

where u is the optimal hiring policy corresponding to parameter value σ. To avoid the ambiguity of this notion when firing also takes place for some ages we assume below that the optimal $u(t,s)$ corresponding to σ from some set of parameters of interest (say $(\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$) is non-negative for all ages. This property holds, for example, under the conditions of Proposition 4.

Proposition 5 *Consider problem (1)–(5) with either exogenous training rate v, or with v being a control and $l_0 = 0$. Assume also $u(t,s) \geq 0$ for the optimal u corresponding to any $\sigma \in (\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$. Let the adjustment factor $c(t,s,\sigma)$ be non-negative and continuously differentiable with respect to σ at $\sigma = \sigma_0$, and let c and c_σ be piece-wise continuous in t and s. The following properties hold true for any fixed t:*

(i) if the function

$$[0,\omega] \ni s \longrightarrow \kappa(s) = \frac{c_\sigma(t,s,\sigma_0)}{c(t,s,\sigma_0)} \tag{17}$$

is monotone increasing then the average age $A(t;\sigma)$ is decreasing with σ around σ_0;

(ii) if the function κ in (17) is monotone decreasing then the average age $A(t;\sigma)$ is increasing around σ_0;

The proof is given in Appendix 2.

Note, that expression (17) gives the rate of change of the adjustment cost factor with respect to the parameter σ at every age s. Proposition 5 yields that an increase (respectively decrease) in this rate over age s implies a decrease (respectively increase) in the average age of hiring as a function of the parameter σ. In the following corollary we apply the general results as given in Proposition 5 to characterize the change in the mean age of hiring if we assume a change in the magnitude, slope or shape of the adjustment cost function.

Corollary 1 *Under the conditions of Proposition 5 let us consider an adjustment factor $c(s)$ at a fixed time t.*

(i) A shift of $c(s)$ by a constant σ (negative or positive), that is $c(s,\sigma) = c(s) + \sigma$, has the following effect on the average age of recruitment, $A(\sigma)$:
(i.a) $A(\sigma)$ increases with σ if $c(s)$ is increasing in s;
(i.b) $A(\sigma)$ decreases with σ if $c(s)$ is decreasing in s.

(ii) Consider a constant change of the slope of $c(s)$, that is $c(s,\sigma) = c(s) + \sigma s$.
(ii.a) If $c(s) \geq sc'(s)$ for all s (this is fulfilled for any concave function $c(s)$ with $c(0) \geq 0$), then the average age $A(\sigma)$ decreases with σ.
(ii.b) If $c(s) \leq sc'(s)$ for all s, then the average age $A(\sigma)$ increases with σ.

(iii) Consider a linear adjustment function $c(s) = c_0 + c_1 s$ and a nonlinear modification $c(s,\sigma) = c(s) + \sigma\alpha(s)$, with $\alpha(0) = 0$, $\alpha'(0) = 0$. Then the average age $A(\sigma)$ decreases with σ if α is convex, and increases with σ if α is concave.

Proof. We need only to consider the function $\kappa(s) = c_\sigma(s)/c(s)$ in each of the three cases.

(i) Here $\kappa(s) = 1/c(s)$, which is increasing/decreasing whenever c is decreasing/increasing.

(ii) Here $\kappa(s) = s/c(s)$, therefore

$$\kappa'(s) = \frac{c(s) - sc'(s)}{c(s)^2}$$

and the claims follow from the proposition. Moreover, concavity of c implies

$$0 \leq c(0) \leq c(s) - sc'(s).$$

(iii) We have $\kappa(s) = \alpha(s)/c(s)$, hence

$$\kappa'(s) = \frac{\alpha'(s)c(s) - \alpha(s)c'(s)}{c(s)^2}.$$

Since c is linear we have

$$(\alpha'(s)c(s) - \alpha(s)c'(s))' = \alpha''(s)c(s),$$

which has a constant sign if α is convex or concave. Since $\kappa'(0) = 0$, κ has the same constant sign. Q.E.D.

The above proposition and corollary give useful information of how changes in the magnitude, slope and shape of the adjustment factor influences the average age of recruitment. As numerical experiments show properties (i) and (ii) in Corollary 1 still hold also in the case of a non-linear revenue function.

5.2 The non-anticipation property

In this subsection we analyze what would be the effect of changes in adjustment costs (e.g., caused by future demographic changes) on the firm's optimal policy of hiring/firing and human capital investment today[17].

In general, we consider two adjustment cost functions $C(t,s,u)$ and $\tilde{C}(t,s,u)$, such that $\tilde{C}(t,s,u) = C(t,s,u)$ for $t \leq \bar{t}$ in order to compare the optimal hiring and training polices (u,v) and $(\tilde{u},\tilde{v})$, respectively, of two identical firms with different expectations for the adjustment costs after time $\bar{t}$. If the two firms chose equal policies till $\bar{t}$ this would mean that the future change does not influence the behavior of the firm now: the *non-anticipation property* takes place. Otherwise anticipation effects occur, where originally identical firms behave differently due to different forecasts of the future labor market.

Proposition 6 *Consider the problem (1)–(5) under the conditions listed at the beginning of Section 3 (fulfilled for each of the functions C and $\tilde{C}$), with the additional specification (10). Let the training rate v be either exogenous, or is a control and $l_0 = 0$. Then $u(t,s) = \tilde{u}(t,s)$ for all s and $t \leq \bar{t}$, and $v(t,s) = \tilde{v}(t,s)$ (if v is exogenous this holds by definition) for all s and t.*

The proof uses the same argument as in Proposition 3.

Proof. The adjoint system (12), (13) together with the expression (9) for the v has a closed form, and the boundary conditions in (12), (13) define a unique solution. Obviously this solution does not depend on C. Since C is not involved also in the formula (9) for v, the latter is independent of C. The claim for u follows from (8). Q.E.D.

We mention that the above non-anticipation property is not a typical phenomenon. Even for the model (1)–(5) with the linearity requirement (10), the function C is the only exogenous data with respect to which the non-anticipation property takes place. Moreover, as it will be seen in the next section, the property disappears as a result of any nonlinearity in the production or in the revenue function. The non-anticipation property disappears even in the case of a linear production and a linear revenue function if $l_0 > 0$. The reason is that if $l_0 > 0$, then the adjoint system (12), (13) does not have a closed form, since the right-hand side depends on the state L (see (9)). Since this system is solved backwards

[17] Our results in the previous section support the close connection between the age structure of hiring/firing and the adjustment costs of hiring/firing.

in time and age, the solution at every time t depends on future values of L, therefore, on future values of u. The latter, however, depend on (future values) of C (see (8)). Therefore a future change of C would cause a change of ξ for earlier times, and—due to (8)—a change of u at earlier times.

On the other hand, the non-anticipation property has an important meaning. If it takes place for some of the parameters of a given problem, then to obtain the current value of the optimal solution one does not need any information about the future behavior of this parameter.

In the context of this paper, the non-anticipation property with respect to the adjustment cost function implies that in the case of a linear production and revenue function the optimal hiring and training policies would not be corrupted by wrong predictions about the adjustment costs (or demographic trends). This, however, is no longer true for a non-linear production or revenue function (Section 6), or in the case of substantial amount of fixed training costs.

Before we proceed to the case of a nonlinear production function or nonlinear revenue function we illustrate the propositions stated so far by numerical simulations.

5.3 Numerical analysis

In this subsection we illustrate the theoretical results obtained above, and complement them with some numerical observations.

Recalling that adjustment costs of hiring/firing incorporate both institutional factors and costs of search and initial training of new employees, these costs will also reflect the scarcity of specific groups in the labor market. Thus we assume that the slope and/or the magnitude of $c(\cdot)$ reflects not only institutional conditions, but also the age-specific availability of labor in the labor market. We are interested in the effect of those changes on the optimal age-specific hiring/firing and training rates at the firm level.

We consider three different scenarios, (C1)–(C3), for the adjustment factor $c(t, s)$, assuming that the adjustment cost has the quadratic form given by (11):

$$(C1): \ c_1(t,s) = 5;$$

$$(C2): \ c_2(t,s) = \begin{cases} c_1(t,s) & \text{for } t \le 50, \\ 5 - 0.08s & \text{for } t > 50; \end{cases}$$

$$(C3): \ c_3(t,s) = \begin{cases} c_1(t,s) & \text{for } t \le 50, \\ 6.8 - 0.08s & \text{for } t > 50. \end{cases}$$

That is, in all scenarios the adjustment factor is the same till $t = \bar{t} = 50$, while it differs for the three scenarios thereafter, i.e. for $t > 50$. In the first alternative case (C2) the adjustment factor decreases with age while in case (C3) there is additionally an increase in the level of the adjustment costs. All other parameters and functional forms are summarized in Appendix 1. The three adjustment factors are plotted in Figure 2 for times $t > \bar{t}$.

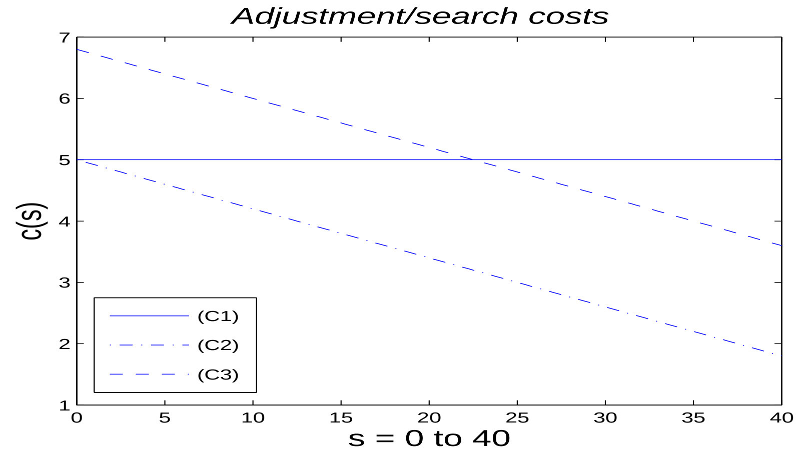


Figure 2: Three scenarios of adjustment costs: solid line in (C1), dash-dotted line in (C2), and dashed line in (C3).

We mention that for numerical well-posedness we take $l_0 > 0$ in the benchmark case (see Appendix 1). This is to avoid the paradoxical situation arising if $l_0 = 0$, where infinitely intensive training of nobody costs nothing (notice that $L(t, 0) = 0$). However, the value of l_0 is relatively small compared with the values of L for most of the ages in the benchmark case—therefore, the numerical results resemble the theoretical ones for $l_0 = 0$. Some of the numerical results are discussed below.

The following facts follow from the theoretical results obtained in the previous sections for $l_0 = 0$.

Optimal hiring

(i) According to Proposition 6 the optimal hiring policies coincide on $[0, \bar{t}]$ (the non-anticipation property).

(ii) According to the first claim of Proposition 3 the training rate $v(t, s)$ is the same for all the scenarios. This need not be the case in presence of fixed costs for training ($l_0 > 0$).

(iii) According to Proposition 3 (iii) there are more hirings at all ages in scenario (C2) than in (C1) and (C3) for $t > \bar{t}$. According to the same claim for scenario (C1) there are more hirings of young workers and less hirings of old workers than in scenario (C3).

Optimal age of hiring (cf. Figure 3)

(iv) According to Corollary 1 (ii.a) the average age of hiring is higher for (C2) than for (C1). Correspondingly, the average age of the unskilled workers in (C2) and (C3) increases considerably after time $\bar{t}$. From Corollary 1 (i.b) it follows that the average age of hiring is higher for (C2) than for (C3). Theoretically it is not clear what is the relation between the average age of hirings for scenarios (C1) and (C3). In the benchmark case it happens that it is lower for (C1).

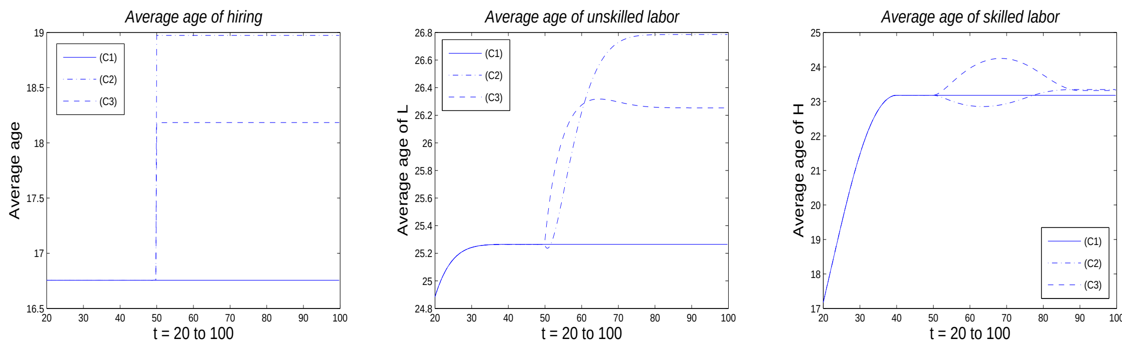


Figure 3: Average age of hiring (left), average age of unskilled labor (middle), and average age of skilled labor (right) in the three scenarios: solid line in (C1), dash-dotted line in (C2), and dashed line in (C3).

Note that Figure 3 also shows that the dynamics of the average age of the unskilled and skilled workers are more complicated than that of the total hirings. Also note that in the long run, the average age of the skilled workers is less sensitive than that of the average age of the unskilled ones. In the transition period the changes are comparable. The reason is that the training mainly applies to workers at lower ages (cf. Figure 4) and in the long run the average age of the skilled workers is determined by the age structure of the training rate rather than the change in the inflow of unskilled workers. Note that the age profile of the training rate does not change after time $\bar{t}$, as mentioned in point (ii) above. The small deviations in the training rate (Figure 4) are due to the positive value of the fixed cost of training, l_0, taken in the benchmark case for numerical well-posedness. This figure also indicates that the assumption of zero fixed costs of training is essential in propositions 3, 5 and 6.

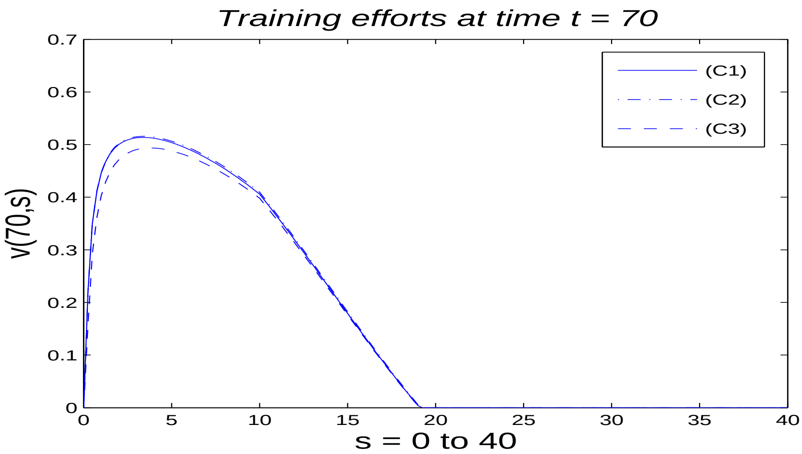


Figure 4: Training rate at time $t = 70$, $v(70, \cdot)$, in the three scenarios: solid line in (C1), dash-dotted line in (C2), and dashed line in (C3).

Figure 5 reveals the effect of the adjustment costs (demographic factors) on profit. Clearly in the case where the labor force is larger in size ((C2) as compared to (C1)) the profits of the firm will be higher. It is remarkable, however, that during the time period $[50, 55]$ of the transition period the instantaneous profits in the case (C3) are the highest (Figure 5). This reveals another aspect of the so-called *Solow effect* (Cutler et al. [8]), in which a temporary profit component arises due to a shrinkage in the labor force that reduces hiring costs (rather than the reduced investment costs in physical capital during population aging as evident in the Solow model).

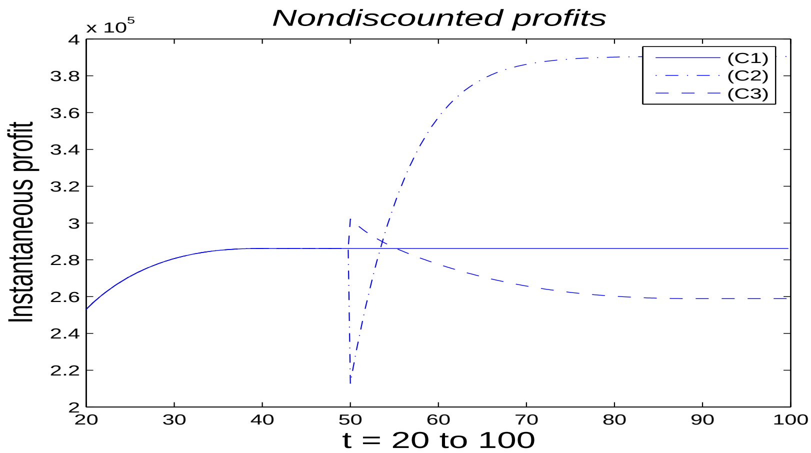


Figure 5: Experiment 1: Instantaneous non-discounted profits in the three scenarios: solid line in (C1), dash-dotted line in (C2), and dashed line in (C3).

6 The case of a nonlinear revenue or production function

In this section we show that non-linearity in either F or R changes qualitatively the transition behavior of the firm in case of a change in the labor adjustment costs, namely inasmuch as the firm reacts to the change in the labor market before the change has even taken place. Put differently, the firm will anticipate the change and incorporate this knowledge in the optimal decisions regarding its hiring/firing policy as well its human capital investment. It was proven in the previous section that such an anticipation effect does not appear in the case of a linear production function and linear revenue function.

In this experiment the production function is assumed to have the form (18) (see Appendix 1) with $\beta_L = \beta_H = 0.7$. That is, we postulate a production function with diminishing returns to scale in each input. We consider two scenarios for the adjustment factor:

$$\begin{aligned}
&(C1): \; c_1(t,s) = 5; \\
&(C2): \; c_2(t,s) = \begin{cases} c_1(t,s) & \text{for } t \leq 50, \\ 13 - 0.2s & \text{for } t > 50. \end{cases}
\end{aligned}$$

In case (C2) the adjustment costs are for all ages higher than in case (C1). The slope of $c(s)$ in (C2) is negative. Figure 6 shows that total hiring drops down immediately after $t = 50$, while the average age of hiring jumps up (Figure 7, left). The average age of the unskilled and of the skilled workers responds correspondingly (Figure 7, middle and right). Thus the same pattern of dependence of the optimal solution on the magnitude and the slope of the adjustment cost coefficient $c(s)$, which was proven in Section 5.1 in the linear case, takes place also for the present model with a nonlinear production function. What is remarkable in all these figures is the anticipation effect: the firm increases its total hiring more than 10 years before the shock moment $t = 50$ (Figure 6) and thus shifts the age distribution of hiring to younger ages, resulting in a lower average age of hiring prior to $t = 50$ (Figure 7, left).

Figure 8 (left) shows the total training rate, $\int_0^\omega v(t,s)\,\mathrm{d}s$. The higher values for (C2) before $t = 50$ are due to more intensive hiring of young workers in $[40, 50]$, and the fact that the training is more productive in young ages. The drop of the training rate after $t = 50$ is due to fact that most of the workers employed in the period of intensive hiring before $t = 50$ leave the age interval for which training is productive, while hiring after this point is at much lower level. Later the training rate stabilizes at a value that is higher than in the benchmark case (C2). Figure 8 (right) depicts the age profile of the optimal training rate at time $t = 70$ (after stabilization). There is a considerable shift to older ages in case (C2), which corresponds to the higher average age of workers.

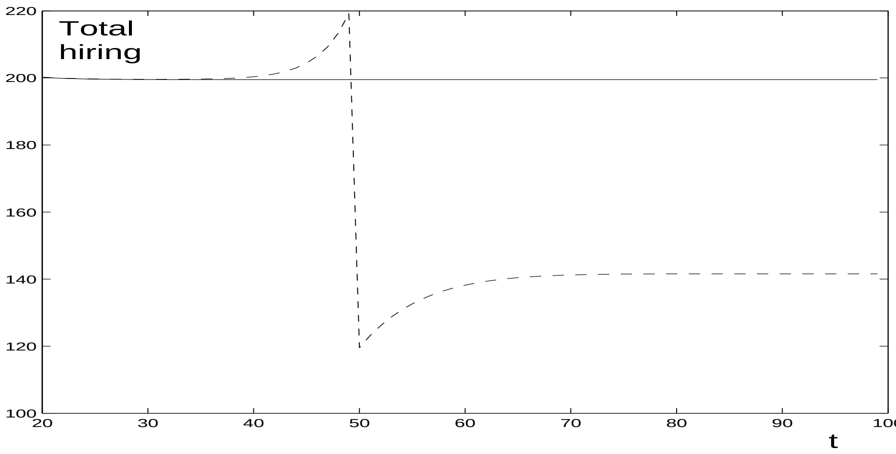


Figure 6: Total hiring in the two scenarios: solid line in (C1), dashed line in (C2).

It is remarkable that the average level of skills of the workers is not higher in (C2) even after the stabilization. Figure 9 (left) shows that the fraction of skilled workers after year 73 becomes smaller for (C2) than for (C1) despite the higher training rate. The reason is that this effort is applied to relatively fewer workers, due to the considerably higher age of workers in case (C2). The transitional increase of the level of skills after $t = 50$ is a result mainly of the higher training rate before $t = 50$ and the lower hiring level after that moment (so that the inflow of low skilled workers is smaller).

The instantaneous profits (9, right) follow a similar pattern as the fraction of skilled workers, but the positive transitional effect in (C2) is shorter than for the level of skills, since higher adjustment costs and lower productivity due to aging start to dominate at about $t = 55$. Before the shock hits the firm, the anticipation effect implies more total hirings and higher rates of skill upgrading, thereby reducing the instantaneous profits.

Experiments with a nonlinear revenue function $R(y)$ give similar results. We shall mention that in this case the system does not necessarily reach its optimal steady state after ω years of time-invariance, as it was proven to be the case with a linear revenue function.

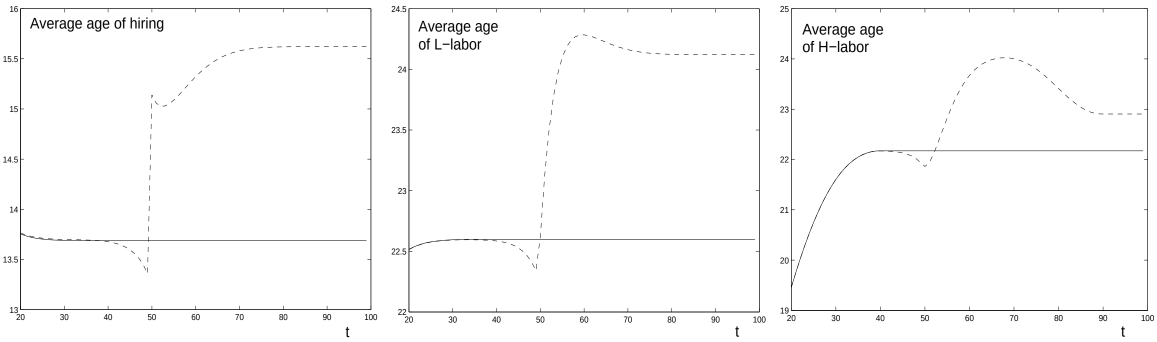


Figure 7: Average age of hiring (left), average age of unskilled labor (middle), and average age of skilled labor (right) in the two scenarios: solid line in (C1), dashed line in (C2).

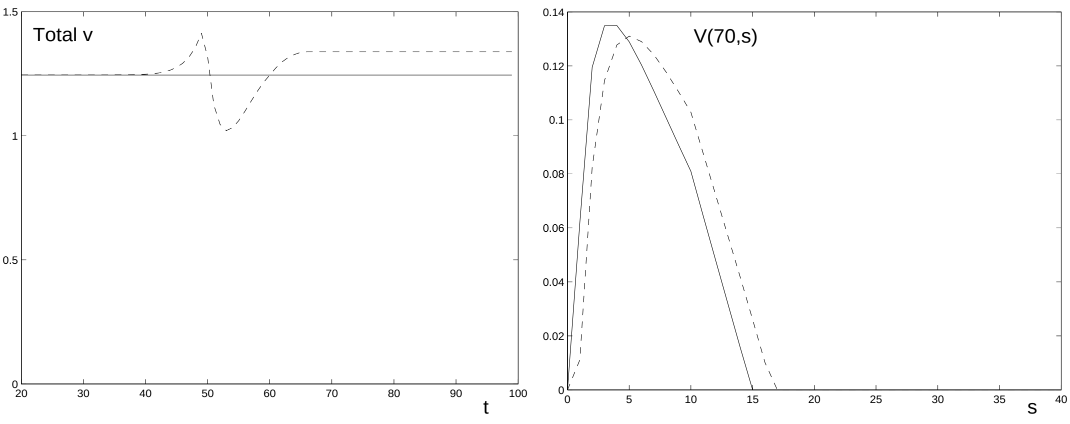


Figure 8: Total training rate v (left), and training rate at time $t = 60$, $v(60, \cdot)$ (right) in the two scenarios: solid line in (C1), dashed line in (C2).

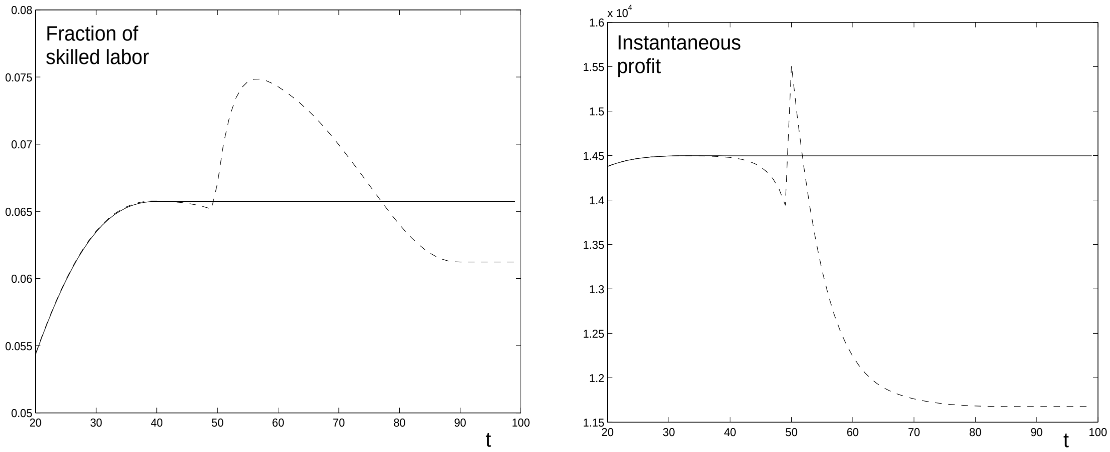


Figure 9: Fraction of skilled labor $H/(L+H)$ (left), and instantaneous non-discounted profits (right) in the two scenarios: solid line in (C1), dashed line in (C2).

7 Conclusion

The aim of the paper is to introduce age structure into a classical model of dynamic labor demand and to allow for human capital upgrading at the firm level. To capture the heterogeneity of workers by age and human capital we introduce “age” as a second dynamic variable and distinguish between two types of workers: unskilled versus skilled workers.

While the age structure has been introduced into the neoclassical theory of optimal investment in several recent contributions, the introduction of age into the classic dynamic labor demand model constitutes a novel approach. As empirical data show (cf. Figure 1), the employment structure follows a clear hump-shaped age pattern at the firm level and the age pattern itself is rather dynamic indicating an ageng of the labor force. Within the framework of our proposed model we show that these stylized facts may be the result of an optimal age-specific labor demand at the firm level.

We introduce analytical results in case of a linear revenue and production function at the firm level. In particular, we establish a number of qualitative features of the optimal hiring policies and the mean age of hiring as it will depend on the age structure of the adjustment costs. Moreover, we show that for the case of linear revenue and production functions, the optimal policy is not influenced by changes in the adjustment cost functions prior to the point in time when the changes have been introduced. We term this fact the non-anticipation property (similar to a finding in the capital investment literature, cf. [12]). Our results are not only valid for the stationary case but hold for the transient behavior as well. We also show that optimal investment in human capital remains unaffected by changes in the adjustment costs as long as we postulate that human capital investment does not require fixed costs. Under further assumptions (e.g., the surplus between age-specific productivity and wages decreases over age) we can prove that firms always hire more younger than older workers, a phenomenon clearly supported by empirical data and the theoretical literature.

Once we allow for nonlinear revenue and production functions the non-anticipation property breaks down. Firms then adjust their optimal labor demand well in advance of the time when adjustment costs are going to change. Many of the results obtained are quite intuitive: e.g., an increase in adjustment costs at every age leads to a temporary increase in hiring, lowering the mean age of hiring and increasing the human capital investment already several years before the change takes place. Once the change in adjustment costs of labor has taken place, firms will react as predicted by our analytical results in case of the linear production and revenue function.

While our results are promising, further analysis requires various extensions. Our production function only includes labor as input, an obvious extension of our framework is to allow for physical capital in addition to human capital. So far we assume that low-skilled workers (and high-skilled workers, respectively) of different ages are perfectly substitutable. An extension of the model could also allow for different elasticities of substitution between workers of different age. The most interesting ultimate aim may be a double vintage structure (of capital and labor). Within such a framework one could allow for the fact that workers of different age are often associated with capital of different vintage. A further extension of the model could assume a more general adjustment cost function, allowing for asymmetry and fixed costs. Obviously empirical estimations on adjustment costs that are age-specific are missing and should be a further topic of research. Another promising extension of our framework could be the introduction of job duration as a further dimension of heterogeneity of labor, in addition to age. Our model constitutes only a partial equilibrium analysis and an extension to a general equilibrium model would constitute an important next step. Obviously such an approach would require to merge the firm-level labor demand model with the macro-level economic framework that includes endogenous wage setting.

Notwithstanding these extensions, our framework should be regarded as a first important attempt to introduce age structure into models of dynamic labor demand. It is to be hoped that the vintage of labor demand will gain a similar role in economic theory as the promising and growing literature of vintage capital investment.

Appendix 1: Benchmark data specification

In the numerical experiments below we assume that F is additive separable in the two skill levels:

$$F(s, L, H) = \pi^L(s)\frac{L^{\beta_L}}{\beta_L} + \pi^H(s)\frac{H^{\beta_H}}{\beta_H}, \tag{18}$$

where the $\pi_L(s)$ (resp. $\pi_H(s)$) is the age-specific efficiency of unskilled (resp. skilled) workers[18].

The following data specifications are used as benchmark settings in the numerical experiments:

$\omega = 40$ – age of retirement minus initial working age a^0;
$L_0(s) = 0.8(200s/\omega^2)$ – initially employed unskilled workers;
$H_0(s) = 0.2(200s/\omega^2)$ – initially employed skilled workers;
$\delta(s) = \nu(s) = b_0 - b_1(s + a^0 - a_0)$ – quit rates, where
$a^0 = 20$, $a_0 = 15$, and $b_0 = 0.18$, $b_1 = 0.004$ are identified as described in Appendix 3 (retirements are not included in these numbers – all workers retire at age ω);
$\gamma(s) = \max\{\gamma_0(1 - (s - 10)/\omega), \gamma_0\}$ – efficiency of learning at age s, where
$\gamma_0 = 0.1386$ is the efficiency of learning at age a^0;
$\beta = 0$;
$w_L(s) = 0.8w(s)$, $w_H(s) = 1.6w(s)$, where $w(s)$ are aggregated data for the age-specific wages; from data for France, 1998 (Figure 10);
$C(s, u) = 0.5 * c(s)u^2$ – adjustment cost of hiring/firing[19], where

[18]The optimization considerations below involve the derivative of F with respect to L and H, which does not exist at zero if $\beta_L < 1$ or $\beta_H < 1$. At the optimum, however, $L(s)$ and $H(s)$ are never zero, which justifies the differentiation along an optimal trajectory. In the numerical solution, where differentiation is involved also outside the image of L and H, we use the regularization

$$F(s, L, H) = \pi^L(s)\frac{(L+\alpha)^{\beta_L} - \alpha}{\beta_L} + \pi^H(s)\frac{(H+\alpha)^{\beta_H} - \alpha}{\beta_H},$$

where α is a small positive number.

[19]A more realistic cost function has the form

$$C(s, u) = \begin{cases} \frac{1}{2}c^+(s)u^2 & \text{if } u \geq 0 \\ \frac{1}{2}c^-(s)u^2 & \text{if } u < 0, \end{cases}$$

where the c^- and c^+ correspond to firing and hiring costs, respectively. This function $C(s, u)$ is differentiable, but not twice differentiable. Our numerical solver works with such C (both theoretically and practically), but in some of the theoretical results it is convenient to assume the existence of the second derivative C_{uu} in order to avoid the technical complications. Therefore in the benchmark case we take $c^-(s) = c^+(s) = c(s)$.

$c(s) = c_0 + c_1 s$, $c_0 = 5$, $c_1 = 0$;
$D(v) = v(d_0 + \frac{d_1}{2} v)$ – cost of learning effort, with
$d_0 = 10$, $d_1 = 50$;
$l_0 = 10$ – size-independent teaching cost;
$F(s, L, H)$ – given by (18) with
$\beta_L = 1$, $\beta_H = 1$;
$\pi^L(s) = c_L \exp(\frac{q_1^2}{(s-m_L)^2 - q_2^2})$, $\quad \pi^H(s) = c_H \exp(\frac{q_1^2}{(s-m_H)^2 - q_2^2})$ are age-specific productivities of unskilled and skilled workers (see Figure 11),
$c_L = 500$, $c_H = 1000$ – scaling factors,
$m_L = 13$, $m_H = 20$ – age of maximal productivity,
$q_1 = 100$, $q_2 = 60$ – parameters identified from data;
$R(Y) = Y$;
$r = 0.03$ – discount rate.

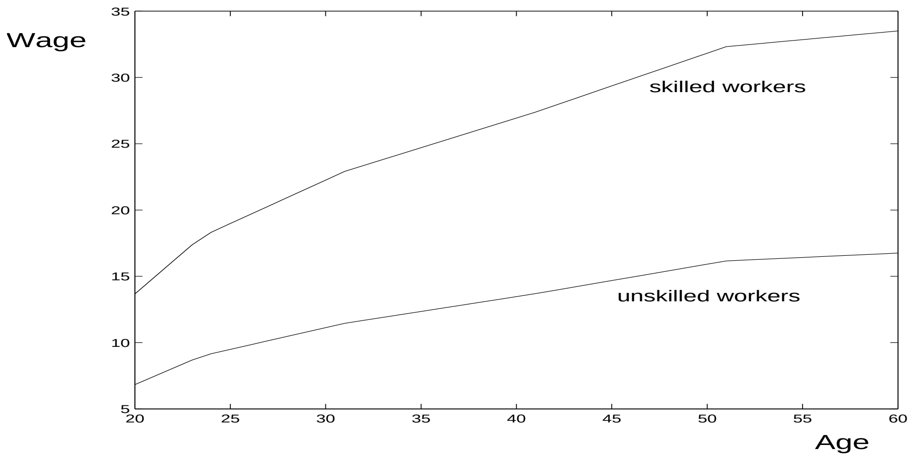


Figure 10: Age-specific wages (in Euro per year) for skilled and unskilled workers. Source: OECD, salary records of enterprises, net annual earnings of full-time, full-year workers, France, 1998.

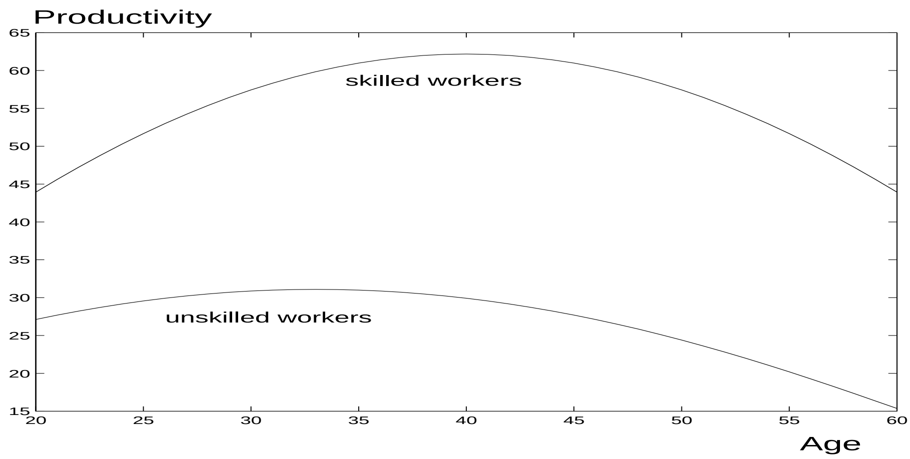


Figure 11: Age-specific productivity of skilled and unskilled workers. Source: [32] and own calculations.

Appendix 2: Proofs

Proof of Lemma 1. We shall only sketch the idea. First of all, one can prove that if the L_∞-norm of a control u is large enough (or infinite), then one can modify u to a control with a finite and smaller L_∞-norm which gives a larger value to the objective function. Hence $\|u\|_{L\infty}$ is bounded. This holds due to the finite horizon in the age dimension (all hired individuals remain employed no longer than ω time units), due to the concavity of the functions R and F (which does not allow for more than a linear growth of production with respect to labor), and due to the strong convexity of C (higher than linear increase of the cost of hiring). The strict proof, however, requires a substantial work, is not short, and will be published elsewhere in a more general setting[20].

Second, we notice that the same applies also to the control v, provided that $l_0 > 0$. The vague argument (which can be strictly justified) is that if $v(t, s) > N$ on some set Ω of measure $\mu > 0$ with sufficiently large N, then when v is modified as zero on the set Ω, the lost profit can be estimated from above by $e^{-t_1 r}\mu C_L l_0 C_F C_R$ (where C_L is a bound for the optimal L—it exists, since u can be assumed bounded according to the above argument, C_F is a bound on the derivative of F in H, C_R is a bound of R', $[t_1, t_2]$ is an interval containing the set Ω), while the gain from the lower cost would be at least $e^{-t_2 r}\mu D(N) l_0$. Since $t_2 - t_1$ can be assumed independent of N, $l_0 > 0$, and $D(+\infty) = +\infty$, the gain dominates for all sufficiently large N. This implies boundedness of v in the case $l_0 > 0$.

From the above two properties it follows that all the elements of any maximizing sequence (u_k, v_k) are contained in a sufficiently large ball in the space L_∞.

Third, since the equations are linear with respect to u and v, and the integrand in (1) is convex in these variables, and since we have already proven uniform boundedness of any maximizing sequence, to prove existence one can apply the standard approach of choosing an L_2-weakly convergent maximizing subsequence and then use the weak lower semicontinuity of the integral in (1).

The case $l_0 = 0$ requires separate consideration, since in this case the abjective function could fail to be strongly convex in v if $L(t, a) = 0$ for some t and a.

Consider a sequence $l_0^\varepsilon \longrightarrow 0$, and let $(u^\varepsilon, v^\varepsilon, L^\varepsilon, H^\varepsilon)$ be a solution of the problem corresponding to l_0^ε, existence of which is proven above. According to the first part of the proof u^ε is bounded uniformly in ε, therefore L^ε and H^ε are also uniformly bounded, as well

[20] The proof is based on ideas from private communications of the second author with S. Faggian and F. Gozzi, 2004.

as Y^ε. From the adjoint equation for η^ε (in Proposition 1) and the boundary condition $\eta^\varepsilon(t,\omega) = 0$ it follows that also η^ε is uniformly bounded. Since $L/(L + l_0^\varepsilon) \le 1$, D_v^{-1} is monotone increasing and $D(0) = 0$, one can use (9) to estimate v from below by ξ (needed only for negative ξ). Using the adjoint equation for ξ in Proposition 1 and the inequality $vD_v(v) \ge D(v)$ (which follows from the convexity of D and $D(0) = 0$) one can prove that ξ^ε and v^ε are also bounded uniformly in ε. Then Proposition 1 implies also equi-continuity of u^ε and v^ε. The Arzela-Ascoli theorem gives existence of a limit (in the uniform metric) of a subsequence of $(u^\varepsilon, v^\varepsilon)$, and it is easy to show by contradiction that the limit is an optimal control pair for the problem with $\varepsilon = 0$. Q.E.D.

Proof of Proposition 2. First of all we notice that in the case $R(Y) = Y$ the integral relation (4) need not be involved in the model. The term $\int_0^\omega F(s, L(t,s), H(t,s))\,\mathrm{d}s$ can be included directly in the objective function. Then the problem takes the followin g general form:

$$J(w) = \int_0^\infty e^{-rt} \int_0^\omega g(s, x(t,s), w(t,s))\,\mathrm{d}s\,\mathrm{d}t \quad \longrightarrow \quad \max \tag{19}$$

$$x_t + x_a = f(s, x, w), \quad x(0,s) = x_0(s), \;\; x(t,0) = 0, \tag{20}$$

$$w(t,s) \in W \tag{21}$$

where x and w are state and control vectors, W is a closed and convex, f and g are functions of corresponding dimensions. In our case $x = (L, K)$, $w = (u, v)$, etc.

For a given control–trajectory pair (w, x) define the functions

$$z[\tau](s) = x(\tau + s, s), \quad y[\sigma](t) = x(t, \sigma + t) \tag{22}$$

$$q[\tau](s) = w(\tau + s, s), \quad p[\sigma](t) = w(t, \sigma + t), \tag{23}$$

where $\tau \ge 0$, $s, \sigma \in [0, \omega]$, $t \in [0, \omega - \sigma]$.

Changing the order of integration in the second integral in the right-hand side we represent

$$\int_0^\infty \int_0^\omega \ldots\,\mathrm{d}s\,\mathrm{d}t = \int_0^\omega \int_t^\omega \ldots\,\mathrm{d}s\,\mathrm{d}t + \int_0^\omega \int_s^\infty \ldots\,\mathrm{d}t\,\mathrm{d}s.$$

Changing the variables $s = \sigma + t$ in the first integral in the right-hand side, and $t = \tau + s$ in the second integral, then changing the order of integration we obtain

$$J(w) = \int_0^\omega J_0[\sigma](p[\sigma], y[\sigma])\,\mathrm{d}\sigma + \int_0^\infty e^{-r\tau} J_1[\tau](q[\sigma], z[\sigma])\,\mathrm{d}\tau,$$

where

$$J_0[\sigma](p[\sigma], y[\sigma]) = \int_0^\sigma e^{-rt} g(\sigma + t, y[\sigma](t), p[\sigma](t))\, \mathrm{d}t, \tag{24}$$

$$J_1[\tau](q[\sigma], z[\sigma]) = \int_0^\omega e^{-rs} g(s, z[\tau](s), q[\tau](s))\, \mathrm{d}s. \tag{25}$$

Moreover, for every $\sigma, \tau \in [0, \omega]$ the functions $y[\sigma]$ and $z[\tau]$ satisfy the equations

$$\frac{\mathrm{d}y[\sigma](t)}{\mathrm{d}t} = g(\sigma + t, y[\sigma](t), p[\sigma](t)), \quad y[\sigma](0) = x_0(\sigma), \tag{26}$$

$$\frac{\mathrm{d}z[\tau](s)}{\mathrm{d}s} = g(s, z[\tau](s), q[\tau](s)), \quad z[\tau](0) = 0. \tag{27}$$

Then the optimization problem for $J(w)$ splits into two parametric families of problems. The first family is parameterized by $\sigma \in [0, \omega]$ and consists of the problems

$$\text{maximize } J_0[\sigma](p, y) \quad \text{subject to (26) and } \; p(t) \in W. \tag{28}$$

The second family is parameterized by $\tau \geq 0$ and consists of the problems

$$\text{maximize } J_1[\tau](q, z) \quad \text{subject to (27) and } \; q(t) \in W. \tag{29}$$

If (w, x) is an optimal solution of the original problem $\min J(w)$, then the traces $(p[\sigma](\cdot), y[\sigma](\cdot))$ and $(q[\sigma](\cdot), z[\sigma](\cdot))$ must be optimal solutions of the corresponding problem (28) or (29) and visa versa.

Now notice that all problems from the family (29) coincide. Therefore a solution (q, z) of one of them solves (29) for all $\tau \geq 0$. Then (q, z) together with a solution $(p[\sigma], y[\sigma])$ of (28) define a solution (x, w) to the original problem (19), (20), (21) by (22), (23). Clearly, for $t \geq \omega$ it holds that

$$x(t, s) = x((t - s) + s, s) = z[t - s](s) = z(s) = z[\omega - s](s) = x(\omega, s),$$

which means that x is at a steady state for $t \geq \omega$. Q.E.D.

Proof of Proposition 4. Using (16) and (13) we obtain

$$\eta_t + \eta_s = (r + \nu)\eta - \pi^H(s) + w^H(s) \leq (r + \nu)\eta,$$

which together with the boundary condition $\eta(t, \omega) = 0$ implies $\eta(t, s) \geq 0$.

Subtracting (12) form (13) and using that $\eta \geq 0$ and $\nu \leq \delta$ we obtain

$$(\eta - \xi)_t + (\eta - \xi)_s \leq (r + \delta)(\eta - \xi) + \beta\gamma(s)(\eta - \xi) \\ -(\pi^H(s) - w^H(s) - \pi^L(s) + w^L(s)), \quad (\eta - \xi)(t, \omega) = 0.$$

Since the last term is non-negative, this implies that $\eta(t, s) - \xi(t, s) \geq 0$.

Using (8) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} u(t + s, s) = u_t + u_s = (C_u^{-1})_s(s, \xi(t, s)) + (C_u^{-1})_\xi(s, \xi(t, s))(\xi_t + \xi_s). \tag{30}$$

Since

$$C_u(s, C_u^{-1}(s, \xi)) = \xi \quad \forall \xi \geq 0,$$

differentiating by ξ and s we obtain

$$C_{uu}(s, C_u^{-1}(s, \xi)).(C_u^{-1})_\xi(s, \xi) = 1, \quad C_{us}(s, C_u^{-1}(s, \xi)) + C_{uu}(s, C_u^{-1}(s, \xi))(C_u^{-1})_s(s, \xi) = 0.$$

Then the assumptions about C imply that

$$(C_u^{-1})_\xi > 0, \quad (C_u^{-1})_s \leq 0.$$

It remains to prove that $\xi_t + \xi_s < 0$. Indeed, this implies the first claim of the proposition due to the end-point conditions $\xi(t, \omega) = 0$, and the second claim, according to (30).

From (12) we have

$$\xi_t + \xi_s = (r + \delta)\xi - g(t, s), \quad \xi(t, \omega) = 0,$$

where

$$g(t, s) = \beta\gamma(s)(\eta - \xi) + \pi^L(s) - w^L(s).$$

For every $\tau \geq 0$ we consider the characteristic line $(\tau + s, s)$, $s \in [0, \omega]$ and define $\varphi(s) = \xi(\tau + s, s)$. Then $\varphi'(s) = (\xi_t + \xi_s)(\tau + s, s)$ and we have to prove that $\varphi'(s) \leq 0$. Obviously φ satisfies the equation

$$\varphi'(s) = (r + \delta)\varphi - h(s), \quad \varphi(\omega) = 0,$$

where

$$h(s) = \beta\gamma(s)(\eta(\tau + s, s) - \xi(\tau + s, s)) + \pi^L(s) - w^L(s)$$

An easy exercise with the Cauchy formula shows that if h is nonnegative and decreasing, then φ is also decreasing (strictly, if strictly positive near $t = \omega$). In our case $h(s) \geq 0$ due to (16) and $\eta - \xi \geq 0$. Moreover, $\gamma(s)$ and $\pi^L(s) - w^L(s)$ are assumed to be decreasing. Therefore, it remains to prove that $\psi(s) = \eta(\tau + s, s) - \xi(\tau + s, s)$ is monotone decreasing.

By the same argument as for φ we prove that $s \longrightarrow \eta(\tau+s,s)$ is monotone decreasing. Subtracting (12) form (13) we obtain

$$\psi' = [(r+\delta)+\beta\gamma](\eta-\xi) - (\pi^H(s) - w^H(s) - \pi^L(s) + w^L(s)) - (\delta-\nu)\eta$$

Then using for a third time the above mentioned consequence of the Cauchy formula and the assumed inequalities we obtain that ψ is decreasing, which completes the proof. Q.E.D.

Proof of Proposition 5. Since the time t is fixed we omit it in the proof. Using (15) we represent the average age of hiring at time t for a given parameter value σ as

$$A(\sigma) = \frac{\int_0^\omega s\frac{\xi(t,s)}{c(s,\sigma)}\,\mathrm{d}s}{\int_0^\omega \frac{\xi(t,s)}{c(s,\sigma)}\,\mathrm{d}s}.$$

In the beginning of the proof of Proposition 3 it is argued that the adjoint state ξ is independent of σ. Then the derivative of A is

$$A_\sigma = \frac{-\int_0^\omega sc_\sigma(s,\sigma)\frac{\xi(s)}{c(s,\sigma)^2}\,\mathrm{d}s\int_0^\omega \frac{\xi(s)}{c(s,\sigma)}\,\mathrm{d}s + \int_0^\omega s\frac{\xi(s)}{c(s,\sigma)}\,\mathrm{d}s\int_0^\omega c_\sigma(s,\sigma)\frac{\xi(s)}{c(s,\sigma)^2}\,\mathrm{d}s}{\left(\int_0^\omega \frac{\xi(s)}{c(s,\sigma)}\,\mathrm{d}s\right)^2}.$$

Therefore

$$\text{sign } A_\sigma = \text{sign } \left\{\int_0^\omega s\varphi(s,\sigma)\,\mathrm{d}s\int_0^\omega \frac{c_\sigma(s,\sigma)}{c(s,\sigma)}\varphi(s,\sigma)\,\mathrm{d}s - \int_0^\omega s\frac{c_\sigma(s,\sigma)}{c(s,\sigma)}\varphi(s,\sigma)\,\mathrm{d}s\right\},$$

where

$$\varphi(s,\sigma) = \frac{\frac{\xi(s)}{c(s,\sigma)}}{\int_0^\omega \frac{\xi(s)}{c(s,\sigma)}\,\mathrm{d}s}$$

is a probability density. Since $\sigma=\sigma_0$ is fixed we skip it in the notations below. We obtain that A_σ has the same sign as

$$\Delta = \int_0^\omega s\varphi(s)\,\mathrm{d}s\int_0^\omega \kappa(s)\varphi(s)\,\mathrm{d}s - \int_0^\omega s\kappa(s)\varphi(s)\,\mathrm{d}s, \tag{31}$$

with

$$\kappa(s) = \frac{c_\sigma(s)}{c(s)}.$$

The key role in the proof is played by the next lemma.

Lemma 2 *Let φ be a probability density on $[0, \omega]$ (that is, a measurable, non-negative function with the integral on $[0, \omega]$ equal to 1). Let $\kappa : [0, \omega] \mapsto (-\infty, +\infty)$ be absolutely continuous. The following is true:*

(i) if κ is increasing, then the quantity Δ in (31) is nonpositive;

(ii) if κ is decreasing, then the Δ is non-negative;

In both cases the sign of Δ is definite if the increase (or decrease, respectively) of κ is strict.

Proof of the lemma. Since γ is absolutely continuous, one can represent

$$\kappa(s) = \kappa(0) + \int_0^s \dot{\kappa}(\tau)\,\mathrm{d}\tau.$$

Obviously

$$\Delta = \int_0^\omega s\varphi(s)\,\mathrm{d}s \int_0^\omega \int_0^s \dot{\kappa}(\tau)\,\mathrm{d}\tau\,\varphi(s)\,\mathrm{d}s - \int_0^\omega s \int_0^s \dot{\kappa}(\tau)\,\mathrm{d}\tau\,\varphi(s)\,\mathrm{d}s.$$

Changing the order of integration we have

$$\Delta = \int_0^\omega s\varphi(s)\,\mathrm{d}s \int_0^\omega \int_\tau^\omega \varphi(s)\,\mathrm{d}s\,\dot{\kappa}(\tau)\,\mathrm{d}\tau - \int_0^\omega \int_\tau^\omega s\varphi(s)\,\mathrm{d}s\,\dot{\kappa}(\tau)\,\mathrm{d}\tau.$$

Since $\dot{\kappa}$ is assumed of constant sign we have

$$\text{sign } \Delta = \text{sign } (\dot{\kappa})\ .\ \text{sign } \left(\int_0^\omega s\varphi(s)\,\mathrm{d}s \int_\tau^\omega \varphi(s)\,\mathrm{d}s - \int_\tau^\omega s\varphi(s)\,\mathrm{d}s \right),$$

provided that the last term in the right-hand side has a constant sign for all τ. Let us denote this term by $\zeta(\tau)$. We have

$$\dot{\zeta}(\tau) = \varphi(\tau) \left(\tau - \int_0^\omega s\varphi(s)\,\mathrm{d}s \right).$$

Thus we have that the function ζ is non-increasing for $\tau \leq \int_0^\omega s\varphi(s)\,\mathrm{d}s \in (0, \omega)$, then it is non-decreasing till ω. Since obviously $\zeta(0) = \zeta(\omega) = 0$, we conclude that ζ is everywhere non-positive, hence sign $\Delta = -$sign $(\dot{\kappa})$ and the first two claims of the lemma are proven. The last claim can be checked by examining the above proof in the case of a strictly increasing (decreasing) κ. Q.E.D.

The claim of the proposition is a direct consequence of the above lemma. Q.E.D.

Appendix 3: Approximating the quit rate from scarce data

In this appendix we propose an approach for evaluating the age-specific quit rate of employees, given the average tenure in several age groups. We use it to identify the quit rate $\delta(s)$ in our model having information about the average tenure in three age groups (see below).

The data available are of the following type: the average tenure of each of several age groups $[a_1, a_2]$, $[a_2, a_3]$, ..., $[a_k, a_{k+1}]$ is given: $d_1, \dots, d_k$.

Suppose that $\bar{\delta}(a)$ is the quit rate, that it depends only on age, and that the working population in any age between a_1 and a_{k+1} has the same size[21], $\bar{M}$, and there is no working population of age below a_1. Also suppose that the workers that quit are immediately employed again.

Let a_0 be the initial age of employment (which in a given data source may be different from the initial age of employment a^0 used in our model). Denote by $M(a)$ the number of employees of age a and tenure $a - a_0$ (that is, employees who have never changed the job), and by $N(a, \tau)$ —the number of employees of age a and tenure $\tau < a - a_0$. According to the above assumptions, the following equations hold:

$$\dot{M} = -\bar{\delta}(a)M, \quad M(a_0) = \bar{M}, \;\; a \geq 0,$$

$$N_a + N_\tau = -\bar{\delta}(a)N(a, \tau), \quad N(a, 0) = \bar{\delta}(a)\bar{M}, \;\; a \geq 0, \; \tau \in [0, a).$$

One can verify that the unique solution is

$$M(a) = \bar{M}e^{-\int_{a_0}^{a} \bar{\delta}(\theta)\, \mathrm{d}\theta},$$

$$N(a, \tau) = \bar{M}\bar{\delta}(a - \tau)e^{-\int_{a-\tau}^{a} \bar{\delta}(\theta)\, \mathrm{d}\theta}.$$

Integrating by parts, one can calculate the average tenure at age a as

$$\frac{(a - a_0)M(a) + \int_0^{a-a_0} \tau N(a, \tau)\, \mathrm{d}\tau}{\bar{M}} \int_0^{a-a_0} e^{-\int_{a-\tau}^{a} \bar{\delta}(\theta)\, \mathrm{d}\theta}\, \mathrm{d}a.$$

Hence, the average tenure for ages in the interval $[a_i, a_{i+1}]$ is

$$A[a_i, a_{i+1}] = \frac{1}{a_{i+1} - a_i} \int_{a_i}^{a_{i+1}} \int_0^{a_0} e^{-\int_{a-\tau}^{a} \bar{\delta}(\theta)\, \mathrm{d}\theta}\, \mathrm{d}a.$$

[21] According to our interpretation of the quit rate, retirement is not counted as quitting. Therefore the constant size supposition is not that unrealistic, since the retired individuals can be regarded as still working. This is consistent with the nature of the data d_i, since retirements do not affect the average tenure of the first $k - 1$ age groups, and only slightly influence the average tenure of the last group.

Therefore, $\bar{\delta}$ satisfies the equations

$$A[a_i, a_{i+1}] = d_i, \quad i = 1, \dots, k. \tag{32}$$

Then one can try to obtain a solution $\delta(\cdot)$ of the above system in a class of functions depending on k free parameters, or even depending on less parameters, if least squares or other fitting is applied.

For the data we have $k = 3$, therefore $\bar{\delta}$ could be sought as a quadratic function, but we chose to use a linear approximation

$$\bar{\delta}(a) = \beta_1(a - a_0) + \beta_0,$$

obtained by the least square interpretation of (32). The available data are $a_0 = a_1 = 15$, $a_2 = 25$, $a_3 = 45$, $a_4 = 60$, $d_1 = 1.6$, $d_2 = 9.0$, $d_3 = 17.5$, and the numerical solution of (32) gives $\beta_0 = 0.18$, $\beta_1 = -0.004$. This gives the specification

$$\delta(s) = -0.004(s + a^0 - a_0) + 0.18.$$

References

[1] J.M. Abowd and F. Kramarz (2003) The costs of hiring and separations. *Labour Economics*, **10**:499–530.

[2] D. Acemoglu and J.-S. Pischke (1998) Why do firms train? Theory and Evidence. *The Quarterly Journal of Economics*, **113**(1):78–118.

[3] E. Barucci and F. Gozzi (2001) Technology Adoption and Accumulation in a Vintage-Capital Model. *Journal of Economics*, **74**:1–38.

[4] G. Becker (1964) *Human Capital*. The University of Chicago Press, Chicago.

[5] M. Brokate (1985) Pontryagin's principle for control problems in age-dependent population dynamics. *J. Math. Biology*, **23**:75–101.

[6] D. Carlson, A. Haurie, and A. Leizarowitz (1991) *Infinite Horizon Optimal Control*, Springer-Verlag, Berlin.

[7] T. Christiaans (2003) Aging in a neoclassical theory of labor demand. *Discussion Paper* No. 112-03, University of Siegen.

[8] D.M. Cutler, J.M. Poterba, L.M. Sheiner, and L.H. Summers (1990) An Aging Society: Opportunity or Challenge? *Brookings Papers on Economic Activity*, **1**: 1–74.

[9] G. Feichtinger and R.F. Hartl (1986) *Optimale Kontrolle Oekonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften*, de Gruyter, Berlin.

[10] G. Feichtinger, R.F. Hartl, P.M. Kort, and V.M. Veliov. Environmental Policy, the Porter Hypothesis and the Composition of Capital. *Journal of Environmental Economics and Management*, **50**(2):434–446, 2005.

[11] G. Feichtinger, G. Tragler, and V.M. Veliov (2003) Optimality conditions for age-structured control systems. *J. Math. Anal. Appl.*, **288**(1):47–68.

[12] G. Feichtinger, R.F. Hartl, P.M. Kort and V.M. Veliov (2005) Financially constrained capital investments: the effects of disembodied and embodied technological progress. (To appear.)

[13] S.-I. Fukuda and R.F. Owen (2004) Firm-specific human capital, sunk costs and macroeconomic adjustment in Japan: who bears the burden. *Working Paper.*

[14] J. Gould (1968) Adjustment costs in the theory of investment of the firm. *Review of Economic Studies*, **35**:47–56.

[15] D. Hamermesh (1989) Labor Demand and the Structure of Adjustment Costs. *American Economic Review*, **79**:675-689.

[16] P. Hetze and C. Ochsen (2005) How aging of the labor force affects equilibrium unemployment. *Thünen-Series of Applied Economic Theory.* Working Paper No. 57, University Rostock.

[17] C. Holt, F. Modigliani, J. Muth, and H. Simon (1960) *Planning production inventories, and work force.* Prentice-Hall.

[18] F. Jaramillo, F. Schiantarelli, and A. Sembenelli (1993) Are adjustment costs for labor asymmetric? An econometric test on panel data for Italy. *The Review of Economics and Statistics*, **75**(4):640-648.

[19] K. Kraft (1997) Hiring and Dismissal Costs in Theory and Practice: A Comparison of Institutional Constraints and Employment Adjustment Patterns in Six OECD Countries. *Kyklos*, **50**:341-368.

[20] E.P. Lazear (1988) Ajdusting to an aging labor force. *NBER Working Paper Series*, No. 2802.

[21] D. Laing, T. Palivos, and P. Wang (2003) The economics of 'new blood'. *Journal of Economic Theory*, **112**:106–156.

[22] C. Mathieu and Y. Nicolas (2002) Adjustment costs in labor demand: a comparison between France and the Czech Republic. Mimeo.

[23] J. Mincer (1974) *Schooling, experience, and earnings*. Columbia University Press, New York.

[24] S.J. Nickell (1986) Dynamic models of labour demand. *Handbook of Labour Economics*, pp. 473-522.

[25] O. A. Nilsen, K.G. Salvanes, and F. Schiantarelli (2003) Employment Changes, the Structure of Adjustment Costs, and Plant Size. *IZA Discussion Paper* No. 920.

[26] J. Pack, H. Buck, E. Kistler, H.G. Mendius, M. Morschhäuser, and H. Wolff (1999) Future Report Demographic Change. `http://www.demotrans.de/documents/Future.pdf`

[27] J.-S. Pischke (2004) Labor market institutions, wages, and investment, *IZA Discussion Paper* No. 1268.

[28] C.A. Pissarides (2002) *The Economics of Search, Encyclopedia of the Social and Behavioral Sciences*. Elsevier.

[29] G.A. Pfann and B. Verspagen (1989) The structure of adjustment costs for labour in the Dutch manufacturing sector. *Economics Letters*, **29**:365-371.

[30] K.G. Salvanes (1999) Employment policies at the plant level: Job and worker flows for heterogeneous labour and heterogeneous plants in Norway. Discussion Paper 6/99, Norwegian School of Economics and Business Administration.

[31] R. Shimer (2001) The impact of young workers on the aggregate labor market. *Quarterly Journal of Economics* **116**(3):969–1007.

[32] V. Skirbekk (2003) Population Aging and Productivity. *Options*, IIASA, No 9-10.

[33] J. Chrissoverghi, G. Tragler, and V.M. Veliov. Numerical Simulation and Optimal Control of Age- and Size-Structured Systems. (forthcoming)

[34] M.L. Weitzman (2003) *Income, Wealth, and the Maximum Principle*. Harvard University Press, Cambridge, Massachusetts.