Analogy of the shadow price of population and the reproductive value*

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Abstract

This note shows an analogy between the value of an additional individual (shadow price) for society and the reproductive value within an age-specific social welfare model. This becomes intuitive, when the shadow price is decomposed into two terms. Finally we provide a numerical example for the social welfare model with plots of the crucial terms.

1 Introduction

What is the value of an additional individual for society? What determines this value and how does it change over time and age? These two questions can be dealth with in macro economic social welfare models. The social welfare is defined as the sum of the ulitities of all individuals. In many cases such models do not include a continuous age-structure of the population, but only in the sense of overlapping generations. However, control theoretic methods allow for a continuous age-structure providing the possibility to observe variables both over time and age.

We use such a model to derive the optimal age structure of health investments (reducing mortality) and consumption over time. This question was firstly dealt with in [3], where the

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life span of one individual is considered. As total health expenditures have been accounting for an increasing proportion of GDP, this research was followed by a number of life-cycle models dealing with the same topic in similar ways (see e.g. [9], [10], [1] or [2]). [4] were the first to apply the question to a social welfare model using a discrete dynamic programming approach. As the model includes several drawbacks, we developed a slightly different continuous-time model presented in the following section. Among other things the model provides the answer of the first question stated in this note, i.e. the value of an additional individual for society, which will be refered to as shadow price of population. The plot of this value for one cohort shows some similarity to the reproductive value in the first half of the life span. The reason is revealed if the shadow price is reduced to a direct and an indirect effect. The latter one is analogous to the reproductive value. Firstly introduced by [6], [7] presents the reproductive value in the following way[1]

$$\begin{aligned} v(x) &= \frac{1}{e^{-rx}l(x)} \int_x^\beta e^{-ra} l(a)m(a) \; da \\ &= \int_x^\beta e^{-r(a-x)} \frac{l(a)}{l(x)} m(a) \; da \end{aligned} \tag{1}$$

where $l(a)$ denotes the probability to survive until age a, $m(a)$ the fertility rate of age a, α and β the youngest and oldest age of childbearing respectively and r the discount rate. Thus $\frac{l(a)}{l(x)}$ equals the probability to survive from age x to a (conditional on being alive at age x).

The reproductive value has two interpretations. Fisher regarded the birth of a child as the lending to him of a life. I.e. the newborn has a loan (of 1 unit), which can only be payed back by bearing children. The expected number of children over life equals $\int_\alpha^\beta l(a)m(a) \; da$. As the value of the outstanding loan has to be discounted back, $e^{-r(a-x)}$ is included. Secondly $v(x)$ can be interpreted as the discounted value (measured in units of newborns) of expected future births to a woman aged x.

The rest of the note is organized as follows. In section 2 we present our age-structured social welfare model and show the similarity of the shadow price of population to the reproductive value. Section 3 privides numerical plots of the shadow price of population as well as the direct and indirect effects. Section 4 closes with some final remarks.

2 The Model

The variables of our model evolve in age a and time t, i.e. in two dimensions. The goal is to derive the optimal consumption $c(a,t)$ and health investments $h(a,t)$ such that the social welfare is maximized. The number of individuals aged a at time t is denoted by the state variable $N(a,t)$. In our model $N(a,t)$ is endogenous and reacts on the stock of health

[1]Note that we use the usual demographic notation within this part of the note and another notation (the usual notation for health economics) later on.

investments $X(a,t)$, which is the second state variable. The dynamics of the population is described by the McKendrick partial differential equation [8]

$$N_a + N_t = -\mu(X(a,t))N(a,t) \qquad\qquad N(0,t) = B(t), N(a,0) = N_0 \tag{2}$$

where $\mu(X(a,t))$ denotes the mortality rate for age a at time t depending on the stock of health investments[2] $X(a,t)$. The equation describes the change in the population (age a, time t) for a small (infinitesimal) unit of time, i.e. how $N(a,t)$ changes to $N(a+dt, t+dt)$ (following the 45 degree line in the Lexis diagramm). The change equals the number of deaths in this interval, described by the mortality rate (for this age and time) times the number of individuals who are still alive. N_0 describes the initial age distribution of the population and $B(t)$ the number of newborns at time t, which is endogenously defined by $B(t) = \int_0^\omega \nu(a)N(a,t)\ da$ (where $\nu(a)$ describes the birth rate of an individual aged a).

The stock of health investments $X(a,t)$ accumulates all past health investments. However, we assume that the efficiency of health investments is decreasing at any given point in time. This effect is modeled by the concave health production function $f(h(a,t))$ with $f(0) = 0$, $f_h(h) > 0$ and $f_{hh}(h) < 0$. Thus the change is described by the following partial differential equation

$$X_a + X_t = f(h(a,t)) - \delta X(a,t) \qquad\qquad X(0,t) = 0, X(a,0) = X_0 \tag{3}$$

where δ is the depreciation rate of health investments, as we also assume that health investments are less efficient the older they are. The stock of health investments of newborns is 0, i.e. $X(0,t) = 0$. That does not mean that newborns are unhealthy, but that they (or their parents) have not invested anything in their health. The initial distribution of health is given by $X(a,0) = X_0$.

As in all macro economic models we need some kind of budget constraint. For simplicity (such that the model remains handable) we assume balanced budget in each period, i.e.

$$Y(t) = C(t) + H(t) \tag{4}$$

where $Y(t)$, $C(t)$ and $H(t)$ denote aggregated output, consumption and health investments respectively. The productivity of an a-year old individual at time t is denoted by $p(a,t)$. Thus aggregated output $Y(t)$ of the whole population at time t is defined as

$$Y(t) = \int_0^\omega p(a,t)N(a,t)\ da \tag{5}$$

Analogously $C(t)$ and $H(t)$ are defined as

$$\begin{aligned} C(t) &= \int_0^\omega c(a,t)N(a,t)\ da \\ H(t) &= \int_0^\omega h(a,t)N(a,t)\ da \end{aligned} \tag{6}$$

[2] As it is not important for this note, we will not explain how the mortality rate depends on the stock of health investments.

Note that in our model individuals have in general no balanced budget. Thus intergenerational transfers are possible. Moreover the social optimum with this budget constraint cannot be worse than that with an individual budget constraint (implying (4)), as the space of admissible control trajectories $c(a,t)$ and $h(a,t)$ is greater.

As already mentioned the objective function is the social welfare, which is defined as the sum of the instantaneous utilities of all individuals

$$\int_0^T \int_0^\omega e^{-rt} u(c(a,t)) N(a,t) \; da \; dt \tag{7}$$

$u(c(a,t))$ represents the per capita instantaneous utility, which depends only on consumption and r denotes the discount rate. This is maximized with respect to per capita consumption $c(a,t)$ and per capita health investments $h(a,t)$, which are assumed to be nonnegative $c(a,t) \geq 0$ and $h(a,t) \geq 0$.

Putting all together the social welfare problem can be written as

$$\begin{aligned}
J = \quad & \max_{c,h} \quad \int_0^T \int_0^\omega e^{-rt} u(c(a,t)) N(a,t) \; da \; dt \\
& \text{s.t.} \quad N_a + N_t = -\mu(X(a,t)) N(a,t) \\
& \qquad N(0,t) = B(t) = \int_0^\omega \nu(a) N(a,t) \; da \\
& \qquad N(a,0) = N_0 \\
& \qquad X_a + X_t = f(h(a,t)) - \delta X(a,t) \\
& \qquad X(0,t) = 0, X(a,0) = X_0 \\
& \qquad Y(t) = \int_0^\omega p(a,t) N(a,t) \; da \\
& \qquad C(t) = \int_0^\omega c(a,t) N(a,t) \; da \\
& \qquad H(t) = \int_0^\omega h(a,t) N(a,t) \; da \\
& \text{budget:} \quad Y(t) = C(t) + H(t) \\
& \qquad h(a,t) \geq 0, c(a,t) \geq 0
\end{aligned} \tag{8}$$

Using the theory of age-specific optimal control theory (see [5]) we can derive necessary optimality conditions for this problem. As this is not the scope of this note, this is shifted to the appendix. Thus we only formulate the shadow price of the population $\xi^N(a,t)$, which equals the increase in the social welfare if there is one more individual aged a at time[3] t. $\xi^N(a,t)$ evolves according to (for convenience a and t are skipped)

$$\xi^N_a + \xi^N_t = (r + \mu(X))\xi^N - u(c) - \xi^N(0,t)\nu - \lambda(p - c - h) \tag{9}$$

[3] More precisely $\xi^N(a,t) = \frac{dJ}{dN(a,t)}$

Together with the condition $\xi^N(\omega, t) = 0$ the partial differential equation can be solved and leads to

$$\xi^N(a,t) = \int_a^\omega \big(u(c) + \xi^N(0, t-a+s)\nu(s) + \lambda(p-c-h)\big) e^{-r(s-a) - \int_a^s \mu(X)\ ds'}\ ds \qquad (10)$$

where λ denotes the shadow price of the budget constraint, which is the increase in social welfare for a small relaxation of the budget constraint. As already mentioned this can be decomposed into a direct and an indirect effect, i.e.

$$\begin{aligned} \xi^N(a,t) &= \int_a^\omega e^{-r(s-a)} \big(u(c) + \lambda(p-c-h)\big) e^{-\int_a^s \mu(X)\ ds'}\ ds + \\ &\quad + \int_a^\omega e^{-r(s-a)} \xi^N(0, t-a+s)\nu(s) e^{-\int_a^s \mu(X)\ ds'}\ ds \end{aligned} \qquad (11)$$

The first integral is the direct part. It comprises the discounted flow of expected future utility of the individual and the discounted value of the individual's future net contribution to social welfare. Here, $e^{-\int_a^s \mu(X)\ ds'}$ represents the probability of survival from age a to age s. Using demographic notation this equals $\frac{l(s)}{l(a)}$. $u(c)$ denotes the instantaneous utility of consumption at age a and time t and $\lambda(p-c-h)$ (positive or negative depending on the level of productivity) describes the net contribution to social welfare by an individual of age a at time t ($p-c-h > 0$: the individual adds to social welfare as its contribution to production outweighs its consumption and health investment, $p-c-h < 0$: the other way around). Finally, $e^{-r(s-a)}$ discounts the values back to time t. Therefore the direct effect describes the additional social welfare (for the individual itself and for others) that is generated by preserving the life of one individual.

The second integral is the indirect effect. Analogously to the direct effect we have the discount factor and the conditional probability of survival. $\nu(s)$ denotes the fertility of an s-year old individual. Thus the factor $\xi^N(0, t-a+s)$ (denoting the additional social welfare of an additional newborn at time $t-a+s$) is the only difference to the reproductive value. In (1) this value equals 1. This difference arises from the fact that (1) measures the reproductive value in amounts of individuals, i.e. each individual has value 1. Our indirect effect is expressed in additional units of utility and thus multiplied by $\xi^N(0, t-a+s)$. Consequently it measures the expected social welfare generated by the descendants of an additional individual.

In the following section we provide a numerical example of the above social welfare model. As we only focus on the shadow price of population within this note, we will only present a plot of this value, as well as seperate plots of the direct and indirect effect.

3 Numerical Results

For the numerical calculation we used the following functional specification

$$
\begin{aligned}
u(c(a,t)) &= b + \frac{c(a,t)^{1-\sigma}}{1-\sigma} \\
f(h(a,t)) &= \log\big(1 + h(a,t)\big) \\
\mu(X(a,t)) &= \tilde{\mu}(a)\big(1 + X(a,t)\big)^{-\alpha}
\end{aligned} \qquad (12)
$$

where $b = 5$, $\sigma = 2$ and $\alpha = 0.5$. For the depreciation of the health stock and for the time preference rate we used $\delta = 0.1$ and $r = 0.03$ respectively. The maximal life-span ω is set to 110. $\tilde{\mu}(a)$ is the age-specific base mortality rate, which is effective when nothing is invested in health.

The age-specific productivity profile has been taken from [11]. For the mortality and fertility data we used the human mortality data base for the years 1990-2000.

After calculating the optimization problem we obtain the shadow price of the population $\xi^N(a,t)$. The separated effects are plotted in figure 1 (x-axis: age, y-axis: units of utility). The direct effect equals the discounted aggregated social welfare induced by his presence. It is decreasing over time, which reflects the diminishing expected remaining lifespan and thus the diminishing expected value of future consumption and production. Furthermore, the social value of an individual drops to the extent that its productivity decreases with age. The profile of the indirect effect is completely analogous to the reproductive value, only multiplied by $\xi^N(0, t-a+s)$ within the integral. The reason for the shape is the following. The fertility rate is positive between approximately 18 and 42 and zero (or very low) otherwise. Thus in the first years no newborns are expected and the indirect effect rises according to the discount and survival rate. The discounted value of expected future births rise, as they come closer. After that period, in the ages with positive fertility rates, the indirect effect decreases, as the number of expected future births decreases. The older individuals are, the fewer newborns they can be expected to bear. When the fertility is zero again, this effect is zero, as no newborns can be expected.

Both effects together sum up to the shadow price of population $\xi^N(a,t)$, what can easily be seen from the shape of the plots.

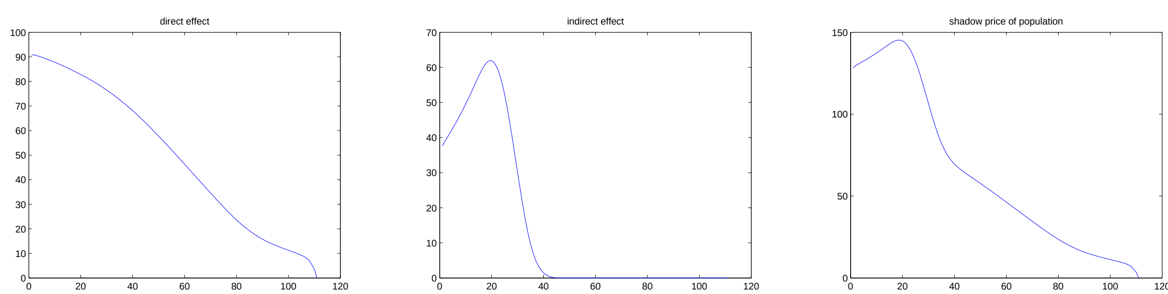

Figure 1: Direct (left) and indirect (middle) effect and the whole shadow price of population $\xi^N(a,t)$ (right) for one cohort (x-axis: age, y-axis: units of utility)

4 Conclusion

The aim of the note is to show the analogy between the shadow price of population (applied to the context of health and welfare economics) and the reproductive value. The analogy becomes very intuitive, when the shadow price is seperated in the direct and the indirect part, which seems to be a more general expression of the reproductive value. More specifically, the shadow price of the population as derived by us embraces both the conventional economic value of a life (expressed in utility terms), i.e. private utility + net contribution to the societal budget, and its reproductive value. In contrast to the purely demographic concept, future descendents, too, are valued in terms of their utility and net contribution to the societal budget. This allows for an complete explanation of the shadow price. The numerical example further illustrates that the theoretical considerations are correct.

It can be suspected that this analogy holds in a social welfare model, whenever the evolution of population is endogenously included in the model.

A Appendix

The Lagrangian of the social welfare problem (8) reads as follows[4]

$$\begin{aligned}\mathcal{L} &= u(c)N - \xi^N \mu(X)N + \xi^X(f(h) - \delta X) + \eta^B \nu N + \eta^Y pN + \eta^C cN + \eta^H hN + \\ &\quad + \lambda(Y - C - H) + \bar{\lambda} h \end{aligned} \tag{13}$$

Applying age-specific control theory (see [5]) we obtain the following adjoint system

$$\begin{aligned}\xi^N_a + \xi^N_t &= (r + \mu(X))\xi^N - u(c) - \eta^B \nu - \eta^Y p - \eta^C c - \eta^H h \\ \xi^X_a + \xi^X_t &= (r + \delta)\xi^X + \xi^N \mu_X(X)N \\ \eta^B &= \xi^N(0, t) \\ \eta^Y &= \lambda \\ \eta^C &= -\lambda \\ \eta^H &= -\lambda \end{aligned} \tag{14}$$

together with the endpoint conditions

$$\xi^N(a, T) = \xi^X(a, T) = 0 \qquad \text{and} \qquad \xi^N(\omega, t) = \xi^X(\omega, t) = 0 \tag{15}$$

Collecting the terms we obtain (9). Finally, we can derive the necessary first order conditions for the controls $c(a, t)$ and $h(a, t)$

[4]Note that we do not need an Lagrangian multiplier for condition $c \geq 0$, as we implicitly assume $\lim_{c \to 0+} u_c = +\infty$.

$$
\begin{aligned}
\mathcal{L}_c &= u_c(c)N - \lambda N + \bar{\lambda} = 0 \\
\mathcal{L}_h &= \xi^X f_h(h) - \lambda N = 0
\end{aligned} \tag{16}
$$

as well as the complemtary slackness condition, i.e. $\bar{\lambda}h = 0$.

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